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Counting cycles in graphs with small excess

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Abstract. The excess of a k-regular graph G of girth g is defined to be the difference between G's order and the well-known Moore bound, and k-regular graphs of girth g and minimal excess are called (k, g)-cages. Despite the fact that the Moore bound is widely believed to be a poor predictor of the order of cages, meaningful improvements are hard to come by. We present a number of formulas for counting cycles of lengths close to the girth in k-regular graphs of girth g and excess not exceeding 3. Based on these formulas, we attempt to exclude the existence of graphs with small excess for infinite families of degree-girth pairs. Perhaps surprisingly, we observe that counting cycles does not exclude too many families, an observation made previously in the setting of strongly regular graphs by Vašek Chvátal.

1. Introduction

We use the term (k, g)-*graph* to denote a (finite, simple) k-regular graph of girth g. A (k, g)-*cage* is a smallest k-regular graph of girth g; its order is denoted by $n(k, g)$. The existence of (k, g)-graphs for any degree/girth pair (k, g) with $k \geq 2$ and $g \geq 3$ has been known since the 1960's [10, 24], but the orders $n(k, g)$ have been determined only for very limited sets of parameters (k, g) [13].

The so-called *Moore bound* is a well-known lower bound on the order $n(k, g)$ of k-valent cages of girth g. The precise form of the bound depends on the parity of g:

$$(1)\quad n(k,g) \geq M(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & , \quad g \text{ odd} \\ 2(1 + (k-1) + \cdots + (k-1)^{(g-2)/2}) & , \quad g \text{ even} \end{cases}$$

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Graphs whose orders equal the Moore bound are called *Moore graphs* and are very rare. They are known to exist if and only if

$k = 2$ and $g \geq 3$: cycles;

$g = 3$ and $k \geq 2$: complete graphs;

$g = 4$ and $k \geq 2$: complete bipartite graphs;

$g = 5$ and $k = 2, 3, 7$: the 5-cycle, the Petersen graph, the Hoffman-Singleton graph;

$g = 6, 8$, or 12: symmetric generalized n-gons of order $k - 1$;

with the existence of the $(57, 5)$-graph or order matching the Moore bound still unresolved [4, 9, 13].

Outside the above cases for which the classification of the parameter pairs of the Moore graphs asserts the existence of a graph whose order matches the Moore bound, the obvious lower bound for the order of a (k, g)-cage is the value of the Moore bound plus one, $M(k, g) + 1$, when k is even, and the value of the Moore bound plus two, $M(k, g) + 2$, when k is odd (with the second statement following from the fact that odd degree regular graphs must be of even order, and the fact that both Moore bounds are even for odd k). The difference between the order of a (k, g)-graph and the Moore bound $M(k, g)$ is called the *excess* of the graph, and it is almost universally believed that the excess of the majority of cages is significantly bigger than 2. No unified opinion appears to exist on whether the excess of cages can be arbitrarily large. Inspecting the lists of the best (smallest) known (k, g)-graphs in [13] quickly reveals a significant gap between the orders of the best known graphs and the orders predicted by the Moore bound; with the excess quickly becoming a multiple of the Moore bound. This makes our inability to obtain more meaningful improvements on the Moore bound even more frustrating.

In [7], Brown showed that $n(k, 5)$ is never equal to $M(k, 5) + 1$. For girth 7, Eroh and Schwenk [11] showed the non-existence of k-regular graphs of girth 7 and order $M(k, 7) + 1$. Note that in this case, the McGee graph achieves the lower bound $M(3, 7) + 2$, hence is a cage. The case of k-regular cages of odd girth and excess 1 has been completed by Bannai and Ito [5] who have shown (using spectral methods) that no k-regular graphs of order $M(k, g) + 1$ exist for any odd $g \geq 5$. For excess 2, Kovács has shown that no graphs of excess 2, girth 5, and odd degree k which is not of the form $\ell^2 + \ell + 3$ or $\ell^2 + \ell - 1$, where ℓ is a positive integer, exist [17]. Eroh and Schwenk [11] showed that $n(k, 5)$ is not equal to $M(k, 5) + 2$ for $5 \leq k \leq 11$. Most recent results concerning odd girth and excess 2 are due to Garbe [15]. He showed the non-existence of graphs of excess 2 for parameters $(k, 9)$ with $k \equiv 1 \pmod 2$; $(k, 13)$ with $k \equiv 0 \pmod 5$, $k \equiv 5 \pmod 7$, $k \equiv 4 \pmod{11}$ and $k \equiv 2, 5, 10, 12 \pmod{13}$; $(k, 17)$ with $k \equiv 3 \pmod 5$, $k \equiv 0, 2 \pmod 7$, $k \equiv 9 \pmod{11}$ and $k \equiv 6 \pmod{13}$; $(k, 21)$ with $k \equiv 7, 10 \pmod{11}$; $(k, 25)$ with $k \equiv 2, 3, 4 \pmod 5$, $k \equiv 2, 6 \pmod 7$, $k \equiv 2, 6, 8 \pmod{11}$ and $k \equiv 4, 7 \pmod{13}$; $(k, 29)$ with $k \equiv 2 \pmod 5$ and $k \equiv 0, 8, 11 \pmod{13}$. Furthermore, he showed that there are no excess 2 graphs in the families of $(3, 2s + 1)$-graphs with $s \equiv 0 \pmod 4$, $(5, 2s + 1)$-graphs with $s \equiv 2 \pmod 4$, $(7, 2s + 1)$-graphs with $s \equiv 0 \pmod 2$, and $(9, 2s + 1)$-graphs with $s \equiv 0 \pmod 4$.

All that is known for even girth is summarized in the following two theorems of Biggs and Ito.

Theorem 1.1 ([6])**.** *Let G be a (k, g)-cage of girth $g = 2m \geq 6$ and excess e. If $e \leq k - 2$, then e is even and G is bipartite of diameter $m + 1$.*

Let $D(k, 2)$ be the incidence graph of a symmetric $(v, k, 2)$-design.

Theorem 1.2 ([6])**.** *Let G be a a (k, g)-cage of girth $g = 2m \geq 6$ and excess 2. Then $g = 6$, G is a double-cover of $D(k, 2)$, and k is not congruent to* 5 *or* 7 (mod 8)*.*

Note that no examples of even girth cages of order $M(k, g) + 1$ have been found. In fact, Theorem 1.1 excludes the possibility of odd excess for the vast majority of even girth cages. Combining Theorem 1.1 with counting cycles, we show that no even girth (k, g)-graphs with $k \geq 3, g \geq 6$, and order $M(k, g) + 1$ exist (Corollary 3.3). Together with the result of Bannai and Ito [4], this means that the only (k, g)-graphs of order $M(k, g) + 1$ can possibly be graphs of girths 3 and 4. Based on these observations, we obtain a complete classification of the parameters (k, g) for which there exists a (k, g)-graph of order $M(k, g) + 1$ (Theorem 3.4).

Excess larger than 1 is even harder to deal with. We limit ourselves to excesses 2 and 3, and present formulas for counting cycles of lengths close to the girth for the majority of parameter pairs (k, g). We then attempt to use these exact counts to argue the non-existence of (k, g)-graphs of excess $e \leq 3$ for infinite families of parameters (k, g). Even though the original motivation for our approach is based on a method employed in [16] for the case of small vertex-transitive graphs of given degree and girth, it turns out that an almost identical approach appears already in the 1971 paper of Friedman [14] who employed counting cycles to show that Moore graphs for certain parameter pairs (k, g) cannot exist (the paper appeared prior to the completion of the classification of Moore graphs). When compared to [14], our paper deals with a wider range of cycle lengths, includes graphs of even girth, which were not considered by Friedman, and considers excesses greater than 0. Two other papers dealing with counting cycles in Moore graphs of even girth (or more precisely, counting k-gons in projective planes which correspond to $2k$-cycles in their corresponding Levi graphs) have been brought to our attention by one of our referees [18, 25].

Overall, our paper is meant to be a comprehensive treatise of the strengths and weaknesses of using counting cycles for determining parameter pairs (k, g) for which (k, g)-graphs of excess in the range $0, 1, 2, 3$ do not exist. For that reason, and for the sake of completeness, we occasionally include results that have been previously proved by other techniques. Another reason for that is that we aim to develop our technique from the very beginning and to demonstrate its versatility.

Despite the fact that this approach appears to loose its strength with the increase of the excess considered, [3] contains infinite families of parameters (k, g) for which counting cycles allowed the authors to show that the excess of (k, g)-graphs from these families must exceed 4. Even though these are very special families of parameters, due to the fact that no parallel results considering graphs of excess greater than 4 exist, the results contained in [3] are the first of their kind.

In addition to determining the order of the smallest (k, g)-graph, one may be interested in determining the *entire spectrum of orders of (k, g)-graphs for a given parameter pair (k, g)*. The original article on the existence of (k, g)-graphs [10] already contains the observation that given any pair of parameters (k,g), $k,g \geq 3$, a k-regular graph of girth g and order $2m$ exists for every $m \geq 2\sum_{t=1}^{g-2}(k-1)^t$. As odd-degree regular graphs cannot have an odd number of vertices, this result shows that, in the case of odd k, the spectrum of orders of (k,g)-graphs contains all sufficiently large possible orders. As for the remaining even orders in case of even k, a paper of Sachs [24] establishes the existence of such graphs for certain multiples of g, and could be most likely extended to prove the existence of (k,g)-graphs with even k for all (i.e., odd and even) sufficiently large orders. The results presented in our paper can be viewed as an attempt to determine the other side of the spectrum – the possible orders of (k,g)-graphs that do not differ from the Moore bound by more than 3.

The main conclusion of our investigation is the (surprising?) observation that counting cycles does not exclude the existence of many parameter families even when graphs of only a small excess are considered. The situation is somewhat similar to that of the existence of strongly regular graphs. A (v,k,λ,μ)-*strongly regular graph* G is a k-regular graph of order v in which every pair of adjacent vertices belongs to λ triangles and every pair of non-adjacent vertices is connected via μ paths of length 2. It is easy to see that a Moore graph of diameter 2 must necessarily be a strongly regular graph. Thus, the existence of the 'unresolved' $(57,5)$-Moore graph is also a question of the existence of the corresponding strongly regular graph. In [8], Chvátal observed (and provided a precise argument for his observation) that counting cycles cannot exclude the existence of strongly regular graphs whose non-existence could not be established using arguments based on spectral properties of their adjacency matrices. Even though the results obtained in this article strongly suggest the existence of similar results with regard to (k,g)-graphs of orders close to the Moore bound, making such statements more precise would require developing spectral methods for these families of graphs. To the best of our knowledge, nobody has made much progress with respect to spectral theory of graphs of excess larger than 2.

In addition to obtaining a number of results concerning the numbers of cycles of lengths close to the girth for graphs with excess $0 \leq e \leq 3$, we map the situation for each of these excesses and apply the obtained results to exclude as many families of pairs as possible.

2. Counting cycles in Moore graphs

Vertex-transitive graphs constitute a significant part of the known cages and the smallest known (k,g)-graphs. The role of vertex-transitivity in the Cage Problem is still poorly understood, and in some cases the order of the smallest vertex-transitive (k,g)-graph exceeds the order of the smallest (k,g)-graph by a significant amount (for example, while the order of the smallest $(3,11)$-graph is 112, the order of the smallest vertex-transitive $(3,11)$-graph is 192 [23]).

The number of cycles of length c passing through a given vertex in a vertex-transitive graph is easily seen to be independent of the choice of the vertex. A similar observation can be made for cycles of length c close to the girth in Moore graphs (we will make this observation more precise), but no such result holds for cages in general. The key observation of the forthcoming section is that even though the numbers of cycles of the same length passing through vertices of a (general) cage may differ from vertex to vertex, these numbers must be the same for all vertices of the Moore graphs. This is, in a way, a curious observation – the vast majority of the known Moore graphs are vertex-transitive – and so the causality is not all that clear in this case. Are the vertex-transitive Moore graphs highly symmetric because the numbers of cycles through each vertex is the same or are these numbers the same because these Moore graphs have to be vertex-transitive? The vertex-transitivity of Moore graphs does not appear to follow from their combinatorial properties and proving that the numbers of small cycles are the same everywhere does not require vertex-transitivity. These connections are particularly interesting with regard to the existence of the $(57, 5)$-Moore graph which has been proved (if it exists) to not be vertex-transitive and to have a small automorphism group [2, 20, 19]. As we will show in this section, if it exists, it has to have the property that the numbers of cycles of length close to 5 must be the same for each of its vertices. Thus, if such a graph exists, it must have an unusual structure: it has to be regular with respect to the number of cycles through each of its vertices, but it cannot be vertex-transitive.

For any given vertex $v \in V(G)$ and integer $c \geq 3$, let $\mathbf{c}_G(v, c)$ denote the *number of cycles* of length c in G passing through v. The following lemma is obvious:

Lemma 2.1. *Let G be a graph and $c \geq 3$. The sum*

$$\sum_{v \in V(G)} \mathbf{c}_G(v, c)$$

is divisible by c.

The next lemma illustrates the key calculations used in this paper. As mentioned in the introduction, the first two results of this lemma have already been proved by Friedman [14]. The third result concerning cycles of length $g + 2$ has been stated by Friedman for 3-regular graphs only and was stated in the form $\mathbf{c}_G(v, g+2) = 0$, with the proof omitted. The fourth result is new.

Lemma 2.2. *Let G be (k, g)-Moore graph of* odd girth*. Then the following hold for all $v \in V(G)$:*

1. $\mathbf{c}_G(v, g) = \dfrac{k}{2}(k-1)^{(g-1)/2}$,
2. $\mathbf{c}_G(v, g+1) = \dfrac{k(k-2)}{2}(k-1)^{(g-1)/2}$,
3. $\mathbf{c}_G(v, g+2) = \dfrac{k(k-2)(k-3)}{2}(k-1)^{(g-1)/2}$,
4. $\mathbf{c}_G(v, g+3) = \dfrac{k(k-2)(k^2-4k+5)}{2}(k-1)^{(g-1)/2}$.

Proof. Let v be an arbitrary vertex of G, and let $N_i(v) = \{u \in V(G) \mid d_G(v, u) = i\}$. The following properties follow easily from the properties of the Moore graphs:

1. $N_i(v) \cap N_j(v) = \emptyset$, for all $0 \leq i \neq j \leq \dfrac{g-1}{2}$;
2. $|N_0(v)| = 1, |N_1(v)| = k$ and $|N_i(v)| = k(k-1)^{i-1}$, for all $0 \leq i \leq \dfrac{g-1}{2}$;
3. $V(G) = \displaystyle\bigcup_{0 \leq i \leq (g-1)/2} N_i(v)$;

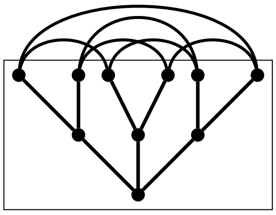

FIGURE 1. The Moore tree and horizontal edges in the $(3, 5)$-cage, Petersen graph.

4. each vertex $u \in N_{(g-1)/2}(v)$ is adjacent to exactly one vertex in $N_{(g-3)/2}(v)$ and $(k-1)$ vertices in $N_{(g-1)/2}(v)$ (we will call the edges connecting the vertices from $N_{(g-1)/2}(v)$ *horizontal*).

We call the tree obtained from G by removing the horizontal edges *the Moore tree of* G (as its vertices give us the Moore bound), and note that this tree consists of k branches rooted at v (see Figure 1). Each vertex $u \in N_{(g-1)/2}(v)$ is adjacent through a horizontal edge to each of the $(k-1)$ branches distinct from its own.

Consider now a cycle of length g passing through v. The fact that G is a Moore graph implies that any such cycle has to consist of two disjoint paths starting at v of length $(g-1)/2$ connecting v to $u_1, u_2 \in N_{(g-1)/2}(v)$ and one horizontal edge connecting u_1 to u_2. As the two $(g-1)/2$-paths from v to u_1 and u_2 are uniquely determined by the vertices u_1 and u_2, there is a one-to-one correspondence between the horizontal edges connecting the vertices in $N_{(g-1)/2}(v)$ and the cycles of length g through v. Therefore,

$$\mathbf{c}_G(v,g) = |N_{(g-1)/2}(v)|(k-1)/2 = k(k-1)^{(g-3)/2} \cdot \frac{(k-1)}{2} = \frac{k}{2}\,(k-1)^{(g-1)/2}\,,$$

as claimed in the first part of our lemma.

Consider next a cycle of length $g+1$ passing through v. It again has to include two vertex-disjoint paths (sharing no other vertex than v) of length $(g-1)/2$ connecting v to a pair of vertices $u_1, u_2 \in N_{(g-1)/2}(v)$ and a 2-path connecting u_1 to u_2. Note that no path of length 2 between two vertices in $N_{(g-1)/2}(v)$ can use other than horizontal edges. Moreover, no such path connects two vertices u_1 and u_2 with the property that the shortest path between v and u_1 and the shortest path between v and u_2 share more than the vertex v: if the two paths shared more than just v, say a vertex w from $N_i(v)$ for some $0 < i \leq (g-1)/2$, this would cause $d_G(w, u_1)$ and $d_G(w, u_2)$ both to be smaller than $(g-1)/2$ and would give rise to a cycle containing w, u_1 and u_2 of length $d_G(w, u_1) + d_G(w, u_2) + 2 \leq 2(g-3)/2 + 2 < g$, a contradiction. Consequently, the number of 2-paths connecting two vertices in $N_{(g-1)/2}(v)$ belonging to different branches with respect to v is equal to $\mathbf{c}_G(v, g+1)$. As the number of such 2-paths starting at a fixed vertex $u \in N_{(g-1)/2}(v)$ is equal to $(k-1)(k-2)$ (there are only $(k-2)$ 'unused' horizontal edges adjacent to the neighbor of u chosen in the first step), we obtain the desired bound

$$\mathbf{c}_G(v, g+1) = |N_{(g-1)/2}(v)|(k-1)(k-2)/2 =$$

$$= k(k-1)^{(g-3)/2} \cdot \frac{(k-1)(k-2)}{2} = \frac{k(k-2)}{2}\,(k-1)^{(g-1)/2}\,.$$

For any cycle of length $g+2$ passing through v, the two sub-paths of this cycle of length $(g-1)/2$ connecting v to a pair of vertices $u_1, u_2 \in N_{(g-1)/2}(v)$ have to be connected by a path of length 3. It is still the case that any such 3-path has to consist exclusively of horizontal edges: to be a part of a cycle through u_1, any such path has to start with a horizontal edge (there is only one non-horizontal edge adjacent to u_1 and it had already been used to get to u_1): if we followed this initial horizontal edge by a pair of non-horizontal edges finishing in u_2, we would use up our only non-horizontal edge connecting u_2 to v and if we followed this initial horizontal edge by a single non-horizontal edge, we would end up in $N_{(g-3)/2}(v)$ and would have no horizontal edge to complete the path. Thus, distinct cycles of length $g+2$ through v determine 3-paths consisting of horizontal edges. While there are exactly $(k-1)(k-2)(k-2)$ paths of length 3 starting at any $u_1 \in N_{(g-1)/2}(v)$ that consist entirely of horizontal edges, not all such paths give rise to a $(g+2)$-cycle through v: only those of these 3-paths give rise to a $(g+2)$-cycle through v that connect vertices $u_1, u_2 \in N_{(g-1)/2}(v)$ with the property that the shortest path between v and u_1 and the shortest path between v and u_2 share no more than the vertex v. Therefore, when choosing the third horizontal edge to complete a 3-path giving rise a $(g-2)$-cycle through v, we have to avoid using the one horizontal edge terminating at a vertex whose shortest path to v shares more than v with the shortest path connecting v to u_1 (each $u \in N_{(g-1)/2}(v)$ has exactly one such neighbor). This forces the number of horizontal 3-paths starting at u_1 and corresponding to $(g+2)$-cycles through v to be equal to $(k-1)(k-2)(k-3)$ and the total number of such paths to be equal to

$$|N_{(g-1)/2}(v)|(k-1)(k-2)(k-3)/2 =$$

$$= k(k-1)^{(g-3)/2} \cdot \frac{(k-1)(k-2)(k-3)}{2} = \frac{k(k-2)(k-3)}{2}\,(k-1)^{(g-1)/2}\,,$$

and the result follows.

Finally consider a cycle through v of length $g+3$. Unlike the above cases, this time we have two types of cycles to consider. Both types have to consist of two paths of length $(g-1)/2$ sharing no other vertices but v and connecting v to $u_1, u_2 \in N_{(g-1)/2}(v)$, but they differ in the way u_1 and u_2 are connected. The first type is just like the cases considered above: u_1 and u_2 are connected via a 4-path comprised of horizontal edges. However, the number of such paths has to be calculated more carefully than in the previous case and splits into two calculations. While it is still true that for the first edge of the path we can choose any of the $(k-1)$ horizontal edges of u_1, and similarly, we have $(k-2)$ choices for the second and $(k-2)$ choices for the third edge, the number of choices for the fourth edge differs according to the branch of the Moore tree of G to which the other end of the third edge belongs: If the third edge ends in the branch of u_1 (which happens exactly once), we can choose any of its $(k-2)$ remaining horizontal edges to complete the path (as we will always end up in a branch different from the branch of u_1). When choosing any of the horizontal edges not terminating in the branch of u_1 limits our choice of the fourth edge to those not terminating in the branch of u_1. Thus, the number of length $(g+3)$-cycles of this first type is

$$|N_{(g-1)/2}(v)| \cdot \frac{1}{2} \cdot [(k-1)(k-2)\cdot 1 \cdot (k-2) + (k-1)(k-2)(k-3)(k-3)] =$$

$$= k(k-1)^{(g-3)/2}\ \frac{(k-1)(k-2)(k^2-5k+7)}{2} =$$

$$= \frac{k(k-2)(k^2-5k+7)}{2}\ (k-1)^{(g-1)/2}\ ,$$

For the rest of the cycles of length $g+3$, the path connecting u_1 and u_2 does not have to consist entirely of horizontal edges. While it is still the case that the first and fourth edge of the path have to be horizontal (as we use the only non-horizontal edges adjacent to u_1 and u_2 in the paths connecting them to v), the second edge can dip down into $N_{(g-3)/2}(v)$ and the third edge then needs to come back into $N_{(g-1)/2}(v)$. As we only have one non-horizontal edge for the dip-down and $(k-2)$ edges to come back up, and the last (horizontal) edge cannot connect to the branch of u_1 (so we have $(k-2)$ edges to choose from for the last horizontal edge), the total count of the $(g+3)$-cycles of this second type comes to:

$$|N_{(g-1)/2}(v)|(k-1)(k-2)^2/2 =$$

$$= k(k-1)^{(g-3)/2} \cdot \frac{(k-1)(k-2)^2}{2} = \frac{k(k-2)^2}{2}\ (k-1)^{(g-1)/2}\ ,$$

Adding the numbers of the two different types of cycles of length $(g+3)$ yields the final claim of the lemma. *q.e.d.*

Combining Lemmas 2.1 and 2.2 yields:

Corollary 2.3. *Let G be a (k,g)-Moore graph of* odd girth. *Then the following hold:*

1. $g \mid M(k,g) \cdot \frac{k}{2}(k-1)^{(g-1)/2}$, *for all $v \in V(G)$;*
2. $(g+1) \mid M(k,g) \cdot \frac{k(k-2)}{2}(k-1)^{(g-1)/2}$, *for all $v \in V(G)$;*
3. $(g+2) \mid M(k,g) \cdot \frac{k(k-2)(k-3)}{2}(k-1)^{(g-1)/2}$, *for all $v \in V(G)$;*
4. $(g+3) \mid M(k,g) \cdot \frac{k(k-2)(k^2-4k+5)}{2}(k-1)^{(g-1)/2}$, *for all $v \in V(G)$.*

Even though the parameter pairs (k,g) of the Moore graphs are classified and all but one pairs (k,g) for which (k,g)-Moore graphs exist are known, the main argument of the classification uses linear algebra of adjacency matrices and integral eigenvalues, and no 'purely graph theoretical' proof is known. The corollary we just proved provides an alternate way to exclude specific pairs. Given a pair (k,g), it is easy to calculate the four values listed in the corollary, and then check their divisibility by the corresponding cycle lengths. If either of the four tests fails, no Moore graph with parameters (k,g) exists. Considering, for example, the smallest degrees for which k-regular graphs of girth 5 fail to exist, we obtain the following table:

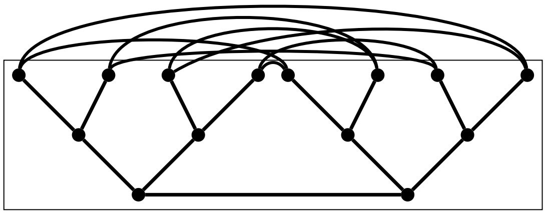

FIGURE 2. The Moore tree and horizontal edges in the $(3, 6)$-cage, Heawood graph.

(k, g)	$\mathbf{c}_G(v, g)$	$\mathbf{c}_G(v, g+1)$	$\mathbf{c}_G(v, g+2)$	$\mathbf{c}_G(v, g+3)$
(4,5)	**306**	612	**612**	**3,060**
(5,5)	1,040	3,120	**6,240**	31,200
(6,5)	2,775	11,100	**33,300**	**188,700**
(8,5)	12,740	76,440	382,200	2,828,280
(9,5)	**23,616**	165,312	991,872	8,265,600
(10,5)	40,905	327,240	2,290,680	21,270,600

where the bold faced numbers fail to meet the divisibility requirements. Out of the first six potential candidates, four are correctly excluded by our test. It is easy to see why this test can never provide us with complete lists of excluded pairs: once $2k$ or $(k+1)$ are divisible by all four cycle lengths $g, g+1, g+2, g+3$, none of the tests can exclude the pair (k, g). The most interesting case is of course the pair $(57, 5)$ – the only pair of parameters for which the existence of a Moore graph is unresolved. In agreement with the results of Chvátal [8], all four of the corresponding numbers pass the divisibility tests, and we fail to resolve this last open case.

For the remainder of this section, we develop results similar to the above for Moore graphs of *even* girth. Instead of counting the number of cycles through a fixed vertex, we switch to counting the *number of cycles through a fixed edge.* The reader familiar with the proof of the Moore bound for even girth graphs can visualize the edge to be the one we start developing the Moore tree from (i.e., each terminal vertex of the edge is the root of a tree of depth $(g-2)/2$ and the leaves of these two trees are to be connected by edges that complete g-cycles, see Figure 2).

For any given edge $e \in E(G)$ and integer $c \geq 3$, let $\overline{\mathbf{c}}_G(e, c)$ denote the *number of cycles* of length c in G containing e. The proofs of the following lemmas and corollary follow along the same lines as the above proofs and use the facts that the number of edges in a (k, g)-Moore graph is $(M(k, g) \cdot k)/2$ and a cycle of length c contains c distinct edges. While it is easy to see that all Moore graphs of even girth are bipartite, and hence contain no cycles of odd length, the non-existence results for cycles of lengths $g+1$ and $g+3$ can be obtained without using this observation.

Lemma 2.4. *Let G be a graph and $c \geq 3$. The sum*

$$\sum_{e \in E(G)} \overline{\mathbf{c}}_G(e, c)$$

is divisible by c.

Lemma 2.5. *Let G be (k, g)-Moore graph of* even girth. *Then the following hold for all $e \in E(G)$:*

1. $\overline{\mathbf{c}}_G(e, g) = (k-1)^{g/2}$,
2. $\overline{\mathbf{c}}_G(e, g+1) = 0$,
3. $\overline{\mathbf{c}}_G(e, g+2) = (k-1)^{g/2}(k-2)^2$,
4. $\overline{\mathbf{c}}_G(e, g+3) = 0$.

In view of the widely believed conjecture that all even girth cages (not just the Moore graphs) must be bipartite (see, for example, [13]), the above lemma together with similar lemmas in the forthcoming sections as well as Theorem 1.1 may be seen supporting this conjecture.

Corollary 2.6. *Let G be a (k, g)-Moore graph of* even girth. *Then the following hold:*

1. $g \mid M(k, g) \cdot \dfrac{k}{2} \cdot (k-1)^{g/2}$,
2. $(g+2) \mid M(k, g) \cdot \dfrac{k}{2} \cdot (k-1)^{g/2}(k-2)^2$.

Recall that the existence of Moore graphs of girths $6, 8$ and 12 is equivalent to the existence of projective planes, and symmetric generalized quadrangles and generalized hexagons, respectively. Thus, excluding cages for girths $6, 8$ or 12 would have the potential of proving the non-existence of corresponding generalized polygons whose existence is in question. Unfortunately, not a single pair $(k, 6), (k, 8)$ or $(k, 12)$ is excluded by the above corollary for any $3 \leq k \leq 2000$. On the other hand, the above divisibility criteria exclude for example 800 of the pairs $(k, 10)$, 1143 pairs $(k, 14)$, and all the way to 1809 pairs $(k, 38)$ from the range $3 \leq k \leq 2000$.

We conclude the section with one more observation that might be known as 'folklore'. The reason we decided to mention this result is that we will return with similar arguments in the forthcoming sections as well.

Lemma 2.7. *If $k \geq 3$ and $g \geq 3$ is odd, such that a (k, g)-Moore graph exists, then there is a $(k-1, g)$-graph of order $k(k-1)^{(g-3)/2}$.*

If $k \geq 3$ and $g \geq 4$ is even, such that a (k, g)-Moore graph exists, then there is a $(k-1, g)$-graph of order $2(k-1)^{(g-2)/2}$.

Proof. In both cases, the graph is obtained by considering the graph induced in the corresponding Moore graph by the leaves of any of its Moore trees. *q.e.d.*

3. Graphs of excess 1

In this section we focus on (k, g)-graphs of orders exceeding the Moore bound by 1. It is easy to check that for odd k, both Moore bounds $M(k, g)$ in (1) are even, and thus, $M(k, g) + 1$ is odd, and as such, cannot be the order of a regular graph of odd degree. Hence, no odd degree regular graphs of order $M(k, g) + 1$ exist. Also, based on the result of Bannai and Ito [4], no such graphs exist for odd girth $g \geq 5$, and based on Theorem 1.2 of Biggs and Ito, no such graphs exist for even girth $g \geq 8$ or even girth $g \geq 6$ and $k \geq 4$. The above results together yield the non-existence of (k, g)-graphs of excess 1 for all $g \geq 5$ and $k \geq 3$. In what follows, we employ counting cycles to reprove some of the results concerning graphs of even girth (and to demonstrate our techniques again) and complete the classification of (k, g)-graphs of order $M(k, g) + 1$.

Other than Moore graphs, cages do not necessarily have the property $\mathbf{c}_G(v, c) = \mathbf{c}_G(v', c)$ or $\overline{\mathbf{c}}_G(e, c) = \overline{\mathbf{c}}_G(e', c)$, for all $v, v' \in V(G), e, e' \in E(G)$, and $c \geq 3$. Nevertheless, in case of excess 1, these numbers do have to be equal for all edges of the graph.

Lemma 3.1. *Let $k, g \geq 4$ be even integers, and G be (k, g)-graph of order $M(k, g)+1$. Then,*

1. $\overline{\mathbf{c}}_G(e, g) = (k-1)^{g/2} - \dfrac{k}{2}$, *for all $e \in E(G)$,*
2. $\overline{\mathbf{c}}_G(e, g+1) = \dfrac{k^2}{4}$, *for all $e \in E(G)$.*

Proof. Let $k, g \geq 4$ be even, $|V(G)| = M(k, g) + 1$, and $e = \{u, v\}$ be any edge of G. The key observation to proving these results is to note that all (k, g)-graphs with the above parameters and excess 1 are of the same structure. Namely, they consist of the (k, g)-Moore tree 'rooted' at e and one extra vertex w of distance $(g-2)/2+1$ from both u and v that is attached via $k/2$ edges to the subtree attached to u and via the same number of edges to the subtree attached to v. This observation follows easily by understanding the way the Moore tree might be completed into a Moore graph: All the $(k-1)^{g/2}$ edges emanating from the the subtree attached to u must be paired with the $(k-1)^{g/2}$ edges emanating from the the subtree attached to v, and thus, if the number of edges connecting w to the two branches differed between them, we would be left with some 'dangling' edges that could not be attached to anything.

The rest of the proof follows the lines of the proofs from the previous section. The number of g-cycles must be equal to the number of 'horizontal' edges connecting the two branches and all such cycles must avoid the new vertex w. The $(g+1)$-cycles must use the new vertex w, and thus each $(g+1)$-cycle must use one of the $k/2$ left edges followed by one of the $k/2$ right edges attached to w. *q.e.d.*

Corollary 3.2. *Let $k, g \geq 4$ be even integers, and G be (k, g)-graph of order $M(k, g) + 1$. Then,*

1. $g \mid [M(k, g) + 1] \cdot \dfrac{k}{2} \cdot \left[(k-1)^{g/2} - \dfrac{k}{2}\right]$,
2. $(g+1) \mid [M(k, g) + 1] \cdot \dfrac{k^3}{8}$.

Once again, the above test cannot possibly exclude all pairs (k, g) for a fixed degree k: Once $k/2$ is divisible by both g and $g+1$, both values satisfy the divisibility requirement and one cannot exclude the possibility $n(k, g) = M(k, g) + 1$. Thus, we cannot obtain the same kind of general result as the result of Bannai and Ito for odd g from Corollary 3.2 alone. However, as bipartite graphs do not contain odd length cycles, Lemma 3.1 that asserts the existence of a non-zero number of cycles of odd length $g + 1$ together with Theorem 1.1 that asserts the bipartiteness of these graphs yield:

Corollary 3.3. *If $k \geq 4$ and $g \geq 6$ are both even, the order of any (k, g)-graph G that is not a Moore graph is greater than $M(k, g) + 1$.*

We close this section with a complete classification of parameter pairs (k, g) for which there exists a (k, g)-graph of order $M(k, g) + 1$.

Theorem 3.4. *Let $k \geq 2$ and $g \geq 3$. A (k, g)-graph of order $M(k, g) + 1$ exists if and only if $k \geq 4$ is even and $g = 3$.*

Proof. As stated in the introduction for this section, due to parity reasons, if k is odd, no (k, g)-graphs of order $M(k, g) + 1$ exist.

If $k = 2$, $M(2, g) = g$, and any $(2, g)$-graph is a system of disjoint cycles of length at least g, with at least one of the cycles of length exactly g. Therefore, there is no $(2, g)$-graph of order $M(2, g) + 1 = g + 1$. (If one were willing to allow for the $(g+1)$-cycle to be considered a $(2, g)$-graph, even though it does not contain a cycle of length g, then a $(2, g)$-graph would exist for all $g \geq 3$.)

If $k \geq 4$, the results of Bannai and Ito assert the non-existence of (k, g)- graphs of order $M(k, g) + 1$ for all odd girths $g \geq 5$. The above corollary (as well as the results of Biggs and Ito [6]) excludes the existence of such graphs for all even girths $g \geq 6$. Thus, the only possible pairs that might admit the existence of a (k, g)-graph of order $M(k, g) + 1$ are the pairs $(k, 3)$, $(k, 4)$, $k \geq 4$ even.

We first show the existence of such graphs for all pairs $(k, 3)$, $k \geq 4$ and even. It is easy to see that, for any odd $n \geq 5$, the graph $K_{n+1} - ((n+1)/2)K_2$. (the complete graph minus a perfect matching) is an $(n-1, 3)$-graph of order $M(n-1, 3) + 1$, and hence a $(k, 3)$-graph of order $M(k, 3) + 1$ exists for every even $k \geq 4$.

We conclude our proof by arguing that no $(k, 4)$-graphs, $k \geq 4$, of order $M(k, 4)+1$ exist. We proceed by contradiction. Let G be a $(k, 4)$-graph of order $M(k, 4) + 1$, $k \geq 4$ even. Let $e = \{u, v\}$ be any edge of G. Then G contains the Moore tree rooted at e containing two subtrees attached to the end-points of e, with the subtrees in this case simply being $(k-1)$-stars (recall that $g = 4$). It also contains an extra vertex w of degree k. All the edges adjacent to w must terminate in the leaf sets of the Moore tree (which are simply neighbors of u or v), and since the number of edges terminating in each leaf set must be the same (so that we can balance out the edges between the two leaf sets), $k/2$ edges connect w to one leaf set and $k/2$ to the other. Let $v' \neq v$ be a neighbor of u that is also a neighbor of w. There are still $k-2$ edges connecting v' to the neighbors of v different from u. Since w is adjacent to $k/2$ of them, and $(k-2) + k/2 > k-1$ for $k \geq 4$, there must exist a neighbor of v, say u', connected to both v' and w. However, the vertices u', w, v' form a triangle, and G cannot be of girth 4; a contradiction. *q.e.d.*

The above classification yields the interesting observation that no Moore (k, g)-graph of girth greater than 3 can be extended into a k-regular graph of the same girth by adding a single vertex.

4. Graphs of excess 2

We focus first on graphs of odd girth g. Every vertex u of a (k, g)-graph G of odd girth has to be the root of a Moore tree on $M(k, g)$ vertices. The 'additional' excess vertices must lie outside this Moore tree and must be of distance at least $(g+1)/2$ from u. Thus, for each vertex u of G, the excess vertices are determined by their distance from u being larger than $(g-1)/2$, and we will call them the *excess*

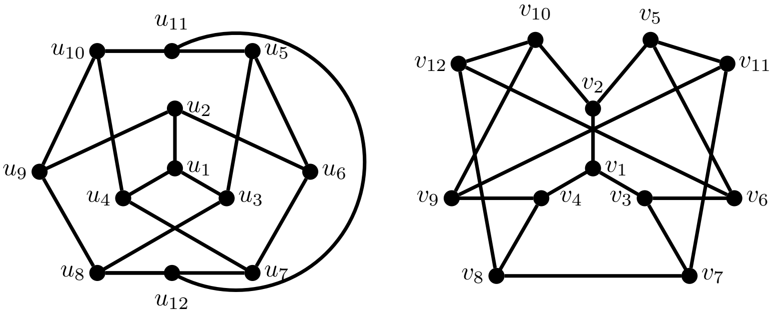

FIGURE 3. The two $(3, 5)$-graphs of order 12.

vertices with respect to u and will denote the set of these vertices by X_u. In case of excess 2, each vertex u of G corresponds to two excess vertices, say $X_u = \{w_1, w_2\}$. We claim that both vertices must be of distance $(g+1)/2$ from u. If this was not the case, and for example w_1 was of distance larger than $(g+1)/2$ from u, all of its edges would have to connect w_1 to vertices of distance at least $(g+1)/2$ from u, but there are at most two such vertices, and w_1 is one of them. Hence, both w_1 and w_2 are adjacent to the leaves of the Moore tree rooted at u, and there are only two possibilities to consider: either w_1 and w_2 are adjacent or they are not. Unlike the case of Moore graphs, G may contain vertices of both kinds: those whose excess vertices are adjacent and those whose are not. The following two examples exhibit several cases of such co-existence (in rather small graphs).

Both examples can be constructed from the Petersen graph by deleting edges and adding two vertices. The first example starts from the Petersen graph viewed as a 6-cycle with the $(3, 5)$-Moore tree attached in its center. Choosing any two opposing edges of the 6-cycle, subdividing them by introducing a new vertex into each, and subsequently joining the two new vertices via an edge, results in a $(3, 5)$-graph of order 12 (see Figure 3 left). The second example is constructed from the Petersen graph by removing every other edge of the outer 6-cycle and attaching two extra vertices to the vertices of degree 2 (see Figure 3 right). Both graphs are of order $12 = M(3, 5) + 2$, and there exist no other $(3, 5)$-graphs of order 12. This last claim is easy to verify either via a careful consideration of the possibilities or via a computer search [12].

Examining the two examples from Figure 3 yields several interesting observations. First, the graph on the left contains both vertices whose excess set consists of two adjacent vertices (the vertices u_1, u_2, u_{11}, u_{12}) as well as vertices whose excess points are of distance 2 (all other vertices). On the other hand, the graph on the right contains vertices whose excess vertices are of distance 3 (the vertices v_1, v_{11}, v_{12}), as well as vertices whose excess sets are formed by adjacent vertices (all other vertices).

The next example of a graph of girth 5 and excess 2 is Robertson's graph of degree 4 and order 19 which is the unique $(4, 5)$-cage [13]. This graph exhibits vertices of all three types. It contains 4 vertices whose excess vertices are of distance 1, 12 vertices with excess vertices of distance 2, and 3 vertices with excess vertices of distance 3.

Based on the above observations, we will call the vertices of G whose excess vertices are of distance 1 *vertices of type* $d1$, vertices whose excess vertices are of distance 2 *vertices of type* $d2$, and all other vertices *vertices of type* $d3$. The number of cycles passing through a vertex u of G depends of its type. We have the following.

Lemma 4.1. *Let G be (k,g)-graph of* odd *girth $g \geq 5$ and excess 2.*

1. *If $u \in V(G)$ is of type $d1$, then*
   (*a*) $\mathbf{c}_G(u,g) = \frac{k}{2}(k-1)^{(g-1)/2} - k + 1$;
   (*b*) $\mathbf{c}_G(u,g+1) = 2\binom{k-1}{2} + (k(k-1)^{(g-3)/2} - 2(k-1))\binom{k-1}{2} + +2(k-1)\binom{k-2}{2}$.
2. *If $u \in V(G)$ is of type $d2$, then*
   (*a*) $\mathbf{c}_G(u,g) = \frac{k}{2}(k-1)^{(g-1)/2} - k$;
   (*b*) $\mathbf{c}_G(u,g+1) = 2\binom{k}{2} + (k(k-1)^{(g-3)/2} - 2k + 1)\binom{k-1}{2} + +2(k-1)\binom{k-2}{2} + \binom{k-3}{2}$.
3. *If $u \in V(G)$ is of type $d3$, then*
   (*a*) $\mathbf{c}_G(u,g) = \frac{k}{2}(k-1)^{(g-1)/2} - k$;
   (*b*) $\mathbf{c}_G(u,g+1) = 2\binom{k}{2} + (k(k-1)^{(g-3)/2} - 2k)\binom{k-1}{2} + 2k\binom{k-2}{2}$.

Proof. As we have observed several times already, every g-cycle containing u has to contain a unique horizontal edge connecting two leaves of the Moore tree rooted at u, and each such edge determines its own g-cycle. Hence $\mathbf{c}_G(u,g)$ is equal to the number of horizontal edges. If u is of type $d2$ or $d3$, all edges incident to w_1 and w_2 are adjacent to the leaves of the Moore tree, which diminishes the number of horizontal edges by k (there are $2k$ edges from the leaves of the tree to X_u that replace k of the horizontal edges). Hence, the total number of horizontal edges is $(k/2)(k-1)^{(g-1)/2} - k$. By the same kind of argument, if u is of type $d1$, $\mathbf{c}_G(u,g) = (k/2)(k-1)^{(g-1)/2} - k + 1$.

Consider next the $(g+1)$-cycles. For all types of vertices, these cycles include one (but not both) or none of the excess vertices. If they include an excess vertex, they also include two edges incident to this vertex. These two edges determine the rest of the cycle uniquely, as their other endpoints are connected to u via unique $(g-1)/2$-paths. It follows that there are $2\binom{k}{2}$ such cycles for u's of types $d2$ and $d3$, and $2\binom{k-1}{2}$ such cycles for type $d1$. If the $(g+1)$-cycles do not include any of the excess vertices, they must contain two incident horizontal edges. Each pair of incident horizontal edges determines the rest of the $(g+1)$-cycle uniquely, and hence the number of the $(g+1)$-cycles containing two horizontal edges is equal to the number of pairs of incident horizontal edges. Each such pair is determined by its shared 'central' vertex. This observation allows us to count the number of cycles of this second type. If u is of type $d3$, the edges attaching the two excess vertices in X_u to the leaves of the Moore tree rooted at u are never attached to the same leaf (as that would make the distance between the excess vertices equal to 2). Consequently, there are exactly $2k$ leaves incident with $(k-2)$ horizontal edges, each contributing $\binom{k-2}{2}$ pairs of mutually incident pairs of horizontal edges. The remaining $k(k-1)^{(g-3)/2} - 2k$ leaves are incident with $(k-1)$ horizontal edges and therefore contribute $\binom{k-1}{2}$ pairs of mutually incident pairs of horizontal edges. In case when w_1 and w_2 are attached to the same leaf (i.e., u is of type $d2$), there are $k(k-1)^{(g-3)/2} - 2k + 1$ leaves adjacent to $(k-1)$ horizontal edges, there are $2k-2$ leaves incident with $(k-2)$ horizontal edges, and there is a single leaf (the one both w_1 and w_2 are adjacent to) with $(k-3)$ horizontal edges. Adding the number of cycles involving excess vertices and cycles including two horizontal edges yields the stated equalities for $\mathbf{c}_G(u, g+1)$ for vertices of types $d2$ and $d3$.

The situation with u of type $d1$ is simpler in that w_1, w_2 cannot be adjacent to the same leaf as that would create a 3-cycle (and we assume $g > 3$). Thus, in this case, the set of leaves consists of $k(k-1)^{(g-3)/2} - 2(k-1)$ leaves incident with $(k-1)$ horizontal edges and $2(k-1)$ leaves incident with $(k-2)$ horizontal edges. The last claim also follows. *q.e.d.*

The following theorem is now fairly obvious.

Theorem 4.2. *Let G be a (k,g)-graph of odd girth $g \geq 5$ and excess 2, let x_1 be the number of vertices of type $d1$, x_2 be the number of vertices of type $d2$, and x_3 be the number of vertices of G of type $d3$. Then the following must be satisfied:*

1. $x_1 + x_2 + x_3 = M(k,g) + 2$;
2. $g \mid x_1 \cdot \left[\frac{k}{2}(k-1)^{(g-1)/2} - k + 1\right] + (x_2 + x_3) \cdot \left[\frac{k}{2}(k-1)^{(g-1)/2} - k\right]$;
3. $(g+1) \mid$

$$x_1 \cdot \left[2\binom{k-1}{2} + (k(k-1)^{(g-3)/2} - 2(k-1))\binom{k-1}{2} + 2(k-1)\binom{k-2}{2}\right] + x_2 \cdot \left[2\binom{k}{2} + (k(k-1)^{(g-3)/2} - 2k + 1)\binom{k-1}{2} + 2(k-1)\binom{k-2}{2} + \binom{k-3}{2}\right] + x_3 \cdot \left[2\binom{k}{2} + (k(k-1)^{(g-3)/2} - 2k)\binom{k-1}{2} + 2k\binom{k-2}{2}\right].$$

In particular, if no non-negative integers x_1, x_2, x_3 satisfying the above arithmetic conditions exist, no (k,g)-graphs of excess 2 exist.

In order to take an advantage of the above theorem, we have tested a number of parameter pairs (k,g) for those that do not allow for any solutions to the above divisibility requirements. Unfortunately, we have found (often many) solutions x_1, x_2, x_3 for each of the parameter pairs considered. This might be due to having three parameters and three divisibility criteria which somehow may always allow for a solution. While an obvious solution to this problem would be to add a criterion for divisibility by $(g+2)$, this approach runs into the problem that it forces further divisions of the edges into different types. This leads to additional restrictions together with additional variables. Instead of taking this path, we have decided to focus on the case of girth 5 where we have already argued that k-regular graphs of girth 5 and order $M(k,g)+2$ must be of diameter 3. After adding the restriction $g=5$, we obtain the following constraints.

Lemma 4.3. *Let G be a $(k,5)$-graph with excess 2, let x_1 be the number of vertices of G of type $d1$, x_2 be the number of vertices of type $d2$, and x_3 be the number of vertices of type $d3$. Then*

1. *if $x_1 > 0$, then $x_1 \geq 4$;*
2. *if $x_2 > 0$, then $x_2 \geq 4$;*
3. *x_3 must be divisible by 3.*

In particular, if no non-negative integers x_1, x_2, x_3 satisfying the conditions from Theorem 4.2 and the above conditions exist, no $(k,5)$-graphs of excess 2 exist.

Proof. Let u be a vertex of type $d1$ and w_1, w_2 be the adjacent excess vertices associated with u. Then w_1 is only adjacent to $(k-1)$ of the k branches of the Moore tree rooted at u, and there exists a neighbor of u of distance greater than 2 from w_1. This vertex thus must be the second vertex from X_{w_1}, which makes w_1 into a type $d1$ vertex. Similarly, w_2 is also a type $d1$ vertex, and hence for any vertex of type $d1$, both of its excess vertices are of type $d1$, which (applied to w_1) yields that at least one of the neighbors of u is also of type $d1$; granting 4 vertices of type $d1$.

If u is of type d_2, the two vertices in $X_u = \{w_1, w_2\}$ share a neighbor, and hence $u \in X_{w_1}$, but $w_2 \notin X_{w_1}$. We claim that the second vertex in X_{w_1} different from u must be of distance 2 from u; which makes w_1 into a type $d2$ vertex. Since all edges incident to w_1 connect w_1 to leaves of the Moore tree rooted at u, w_1 has to be connected to each branch of this Moore tree (it cannot be connected twice to the same branch as that would create a cycle of length 4), and hence all neighbors of u are of distance 2 from w_1, and therefore the vertex in X_{w_1} different from u is not a neighbor of u, neither it is of distance 3 from u, and hence it is of distance 2 from u; as claimed. By a symmetric argument, w_2 is a type $d2$ vertex as well, and consequently, if u is of type $d2$, so are both of its excess vertices. Applying the same argument to w_1 yields that both excess vertices associated with w_1 are also of type $d2$, and hence we obtain at least one more vertex of type $d2$ different from the vertices u, w_1 and w_2.

Consider finally a vertex u of type $d3$. The two excess vertices w_1, w_2 in X_u which ar of distance 3 one from the other are also the only two vertices of distance 3 from u, and hence the triple u, w_1, w_2 consists of vertices any two of which are of distance 3. Thus, $X_{w_1} = \{u, w_2\}$ and $X_{w_2} = \{u, w_1\}$, and all three vertices are of type $d3$. Any vertex u' of type $d3$ distinct from the vertices u, w_1, w_2 must then come with its own pair of type $d3$ vertices, and hence $X_u \cap X_{u'} = \emptyset$ and the claim $3 \mid x_3$ follows by induction. *q.e.d.*

We have added the above conditions to our program looking for parameters $(k, 5)$ not allowing the existence of a $(k, 5)$-graph of order $M(k, 5) + 2$. In case $k = 3$, our program determined that the only possible triples (x_1, x_2, x_3) are $(9, 0, 3)$ and $(8, 4, 0)$; both of which are realized by the two graphs obtained from the Petersen graph in Figure 3. For $k = 4$, there are a number of triples satisfying the criteria, with the triple $(4, 12, 3)$ exhibited by Robertson's graph included. Thus, the restrictions obtained so far do not suffice to exclude impossible triples (x_1, x_2, x_3). This can be also seen from the fact that none of the degrees k, $5 \leq k \leq 11$, excluded by Eroh and Schwenk [11] are excluded by our program. Clearly, further, possibly more complicated, criteria would be needed in order to exclude more degrees k for which there exists no $(k, 5)$-graph of excess 2.

In the second part of this section, we consider graphs of excess 2 and even girth. One such example is the Möbius-Kantor graph of degree 3, girth 6, and order 16. This is once more an example of the situation where a Moore graph of degree 3 and girth 6 (and order smaller by 2) also exists; namely the Heawood graph. Note in addition, that the Möbius-Kantor graph is arc-transitive, and therefore must look the same with respect to every edge. Extending the definition of the excess sets to edges in the natural way, we define X_f to consist of all vertices of X whose distance from *both* end-points of the edge f is greater than $(g-1)/2$. The arc-transitivity of the Möbius-Kantor graph yields that all subgraphs induced by the excess vertices of any of its edges are isomorphic to K_2. As we will see in the forthcoming paragraphs, this observation holds for all $(k, 6)$-graphs of order $M(k, 6) + 2$ (whether vertex-transitive or not).

Graphs of excess 2 and even girth are covered by Theorem 1.2. Thus, no excess 2 graphs exist for even girths greater than 6, for $k \equiv 5, 7 \pmod 8$, or for parameters which do not allow for a double cover of an incidence graph of a symmetric $(v, k, 2)$-design. As complete classification of $(v, k, 2)$-designs is not known, we focus in the remaining part of this section on counting cycles in $(k, 6)$-graphs of order $M(k, 6)+2$. We will only consider those graphs that are bipartite due to Theorem 1.1.

Lemma 4.4. *Let $k \geq 4$. If G is a $(k, 6)$-graph of order $M(k, 6) + 2$, then G is bipartite, and*

1. $\overline{\mathbf{c}}_G(e, 6) = (k-1)^3 - (k-1)$*, for all* $e \in E(G)$*, and*
2. $\overline{\mathbf{c}}_G(e, 8) = (k-1)^3(k-2)^2 - k^3 + 6k^2 - 10k + 5$*, for all* $e \in E(G)$*.*

Proof. If G is a (k, g)-graph of order $M(k, g)+2$, then G is bipartite by Theorem 1.1. It follows that G cannot contain odd-length cycles, and consequently, none of the two extra vertices can be attached to both branches of the Moore tree rooted at e. The only way to achieve this is for each of the two vertices to be attached to just one branch of the tree. It is not possible, however, for either of the two vertices to be attached just to the leaves of one of the branches: since the branches consist of $(k-1)$ sub-branches, a vertex of degree k would have to be attached twice to the same sub-branch. But that would cause a cycle of length $g - 2 = 4$ and violate the girth of G. It follows that each of the two extra vertices is attached to only $(k-1)$ sub-branches of a different branch of the Moore tree, and therefore the two extra vertices have to be adjacent – connected through an edge. This information is sufficient to guarantee that $\overline{\mathbf{c}}_G(e, 6)$ is the same for each edge $e \in E(G)$. In fact, $\overline{\mathbf{c}}_G(e, 6) = (k-1)^3 - (k-1)$ as exactly $(k-1)$ horizontal edges are lost due to the connections to the two extra vertices. As for the 8-cycles, they come in two kinds: those that consist of two 2-paths in the different branches of the Moore tree completed via a 3-path of horizontal edges (the same kind as in the Moore graph case) and those that pass through the two extra edges. The number of 8-cycles through e and the two extra edges can be easily seen to be equal to $(k-1)^2$. The number of 8-cycles 'lost' in comparison to the Moore graph is $(k-1)(k-2)^2$, and therefore

$$\overline{\mathbf{c}}_G(e,8) = (k-1)^3(k-2)^2 - (k-1)(k-2)^2 + (k-1)^2 \ .$$

q.e.d.

We see that the additional information about the potential graphs being necessarily bipartite (due to Theorem 1.1) yields a strong restriction on the structure of the graphs considered, and, in particular, implies the existence of a single type of excess set. This situation differs quite a bit from the case of odd girth considered in the first part of this section. Thus, one might expect the restrictions obtained from Lemma 4.4 to exclude at least some pairs (k,g). This is unfortunately not the case, as we have found no pairs (k,g) that could be excluded using the divisibility criteria of Lemma 4.4. On the other hand, the fact that all excess sets must be of the same structure suggests that vertex-transitive graphs may play an important role in this case.

We conclude the section with an analogue of Lemma 2.7.

Lemma 4.5. *If $k \geq 3$ and $g \geq 3$ is odd, such that a (k,g)-graph G of excess 2 and having at least one vertex of type $d1$ exists, then there is a $(k-1,g)$-graph of order $k(k-1)^{(g-3)/2}+2$.*

If $k \geq 3$ and $g \geq 6$ is even, such that a (k,g)-graph G of excess 2 exists, then there is a $(k-1,g)$-graph of order $2(k-1)^{(g-2)/2}+2$.

Proof. If u is a $d1$-type vertex, the desired graph is obtained by taking the subgraph of G induced by the leaves of the Moore tree rooted at u and the two excess vertices in X_u, and removing the edge between the excess vertices.

In the even girth case, the graph is bipartite and all excess pairs are joined by an edge, and so one can take the graph induced in G by the leaves of the Moore tree and the excess vertices of any vertex u of G. The edge connecting the two excess vertices must again be removed. *q.e.d.*

5. Graphs of excess 3

Since both Moore bounds are even for odd degree k, no odd degree graphs of excess 3 exist. The only interesting cases are those where k is even.

Let k be even and g be odd. This case is not covered by any of the previously mentioned results, and the existence of (k,g)-graphs with even k, odd g, and order $M(k,g)+3$, is wide open. The smallest cage with excess 3 is the $(6,5)$-cage of order 40 obtained by removing the vertices of a Petersen graph from the Hoffman-Singleton graph (which is the unique $(7,5)$-cage). The graph is sometimes known as the Anstee-Robertson graph (it appeared for the first time in Robertson's thesis [22], and was independently discovered by Anstee [1]), but was first considered as a cage by O'Keefe and Wong [21].

If we assume that g is at least 5, only three of the four non-isomorphic graphs of order 3 can appear as subgraphs induced by the three excess vertices associated with any vertex of the graph (the 3-cycle is too short). Thus, the subgraphs induced by the excess vertices are either isomorphic to the graph $3K_1$ containing no edges, the union $K_2 \cup K_1$ containing exactly one edge, or the 2-path $\mathcal{P}_2$ of two edges. This makes for a relatively complicated situation, and we only list a result concerning cycles of length g.

Lemma 5.1. *Let $k \geq 3$ be even, $g \geq 5$ be odd, and G be a (k, g)-graph of order $M(k, g) + 3$. Let x_1 denote the number of vertices u of G for which the subgraph induced by X_u is isomorphic to $3K_1$, x_2 denote the number of vertices u of G for which the subgraph induced by X_u is isomorphic to $K_2 \cup K_1$, and x_3 denote the number of vertices u of G for which the subgraph induced by X_u is isomorphic to $\mathcal{P}_2$. Then the following hold:*

1. $x_1 + x_2 + x_3 = M(k, g) + 3$, *and*
2. *g divides the value*

$$(M(k,g)+3)\ \frac{k(k-1)^{(g-1)/2}}{2} - x_1\ \frac{3k}{2} - x_2(\frac{3k}{2} - 1) - x_3(\frac{3k}{2} - 2)\ .$$

Proof. We have argued the first claim of the lemma prior to its statement. The second claim follows from counting horizontal edges in G with respect to u, as each horizontal edge corresponds to a unique g-cycle through u. If the subgraph induced by X_u is isomorphic to $3K_1$, the number of horizontal edges decreases by $3 \cdot (k/2)$, and is therefore equal to $(k/2)(k-1)^{(g-1)/2} - 3k/2$ (where $(k/2)(k-1)^{(g-1)/2}$ would be the number of horizontal edges in a (k, g)-Moore graph). If the subgraph induced by X_u is isomorphic to $K_2 \cup K_1$, this number only decreases by $3 \cdot (k/2) - 1$, and if the subgraph induced by X_u is isomorphic to $\mathcal{P}_2$, it decreases by $3 \cdot (k/2) - 2$. The rest of the proof then follows by the usual argument. *q.e.d.*

As has unfortunately repeatedly been the case before, the above lemma excludes no small pairs of parameters (k, g).

For the case of even k and even g, Theorem 1.1 contains a great deal of information concerning the structure of (k, g)-graphs. Interestingly, the part of the theorem that states that the excess for these graphs must be even is only stated informally in [6]. For the sake of completeness, we include a quick proof of the claim.

Because the first degree considered by Theorem 1.1 for the case of excess 3 is $k = 5$, and the first girth it applies to is 6, we do not know whether 4-regular graphs of even girth and excess 3 or k-regular graphs with even k and girth 4 and excess 3 necessarily have to be bipartite. All the other cases of even girth and even degree are covered by the next theorem.

Theorem 5.2. *Let $k, g \geq 6$ be both even. Then there exist no (k, g)-graphs of* odd *excess $e \leq k - 2$.*

Proof. Assume throughout that $k, g \geq 6$ are even, $g = 2m$. Applying Theorem 1.1 yields that all (k, g)-graphs of excess $e \leq k - 2$ are bipartite of diameter $m + 1$. Thus, all such graphs consist of a bipartite Moore tree rooted at an edge $\{u, v\}$ and extra vertices $w_1, w_2, \cdots, w_e$ of distance $m + 1$ from either u or v. Due to the bipartiteness, u and v belong to different partition sets, and consequently the leaves of the Moore tree divide into two distinct partition sets based on whether they are of distance $m - 1$ from u or v. Also, each of the extra vertices $w_1, w_2, \cdots, w_e$ must belong to one of the partition sets. Since we have an odd number of them, the two partition sets contain different numbers of extra vertices. This means that the two leaf sets of the Moore tree are attached to distinct numbers of vertices from the set $w_1, w_2, \cdots, w_e$. This causes an imbalance contradicting the fact that all the edges emanating from one of the sets of leaves that are not adjacent to the vertices $w_1, w_2, \cdots, w_e$ must be paired with the edges emanating from the other set that are not adjacent to the vertices $w_1, w_2, \cdots, w_e$. The number of excess vertices must be even, and they have to split evenly between the two sets of leafs. *q.e.d.*

Corollary 5.3. *If $k = 3$ or $k \geq 5$, and $g \geq 6$ is even, then no (k, g)-graphs of excess 3 exist.*

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