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Geometry of harmonic maps and biharmonic maps

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Abstract. A biharmonic map is a critical point of the bienergy in the space of C^∞ maps between two Riemannian manifolds, and it is a natural extension of a harmonic map which is a critical point of the energy. In this paper, we give a brief survey on our recent works for biharmonic maps

1. Introduction

A biharmonic map is a critical point of the bienergy in the space of C^∞ maps between two Riemannian manifolds, and it is a natural extension of a harmonic map which is a critical point of the energy. In this paper, we give a brief survey on our recent works for biharmonic maps. The topics with which we treat in this paper will be as follows.

Table of Topics:

(1) from harmonic maps to biharmonic maps,
(2) classification and construction of biharmonic maps,
(3) B-Y. Chen's conjecture,
(4) bubbling phenomena of biharmonic maps,
(5) biharmonic Lagrangian submanifolds of a symplectic manifold,
(6) k-harmonic maps and k-harmonic B-Y. Chen's conjecture.

2. From harmonic maps to biharmonic maps

2.1. **From the submanifold theory.** One source of biharmonic map theory is submanifold theory due to the work of B-Y. Chen. B-Y. Chen proposed the following problem in his paper ([7]): some open problems and conjectures on submanifolds of finite type, *Soochow J. Math.*, 17(1991), 169–188.

Let us consider an isometric immersion $\varphi : (M^m, g) \hookrightarrow (\mathbb{R}^k, h_0)$ and $\varphi(x) = (\varphi_1(x), \cdots, \varphi_k(x))$ $(x \in M)$. Then it holds that

$$\Delta\varphi := (\Delta\varphi_1, \cdots, \Delta\varphi_k) = m\,\mathbf{H}\ ,$$

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where $\mathbf{H} := (1/m)\sum_{i=1}^{m} B(e_i, e_i)$, the mean curvature vector field, where B is the second fundamental form defined by

$$B(X,Y) := D^0_X(\varphi_* Y) - \varphi_*(\nabla_X Y) ,$$

for vector fields X, Y on M. Here, ∇, ∇^N, and D^0, are the Levi-Civita connections of Riemannian manifolds (M,g), (N,h), and the standard Euclidean space $(\mathbb{R}^k, h_0)$, respectively. The Laplacian Δ of (M,g) is defined by

$$\Delta f = -\sum_{i=1}^{m}\{e_i(e_i f) - \nabla_{e_i} e_i f\} \qquad (f \in C^\infty(M)) ,$$

where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M,g). Then

Definition 2.1. An isometric immersion $\varphi : (M^m, g) \hookrightarrow (\mathbb{R}^k, g_0)$ is *minimal* if $\mathbf{H} \equiv 0$. Furthermore Chen defined ([7]) that φ is to be *biharmonic* if

$$\Delta\mathbf{H} = \Delta(\Delta\varphi) \equiv 0 .$$

He showed ([7]) that

Theorem 2.2. *If* $\dim M = 2$*, any biharmonic submanifold is minimal.*

Then he raised the following conjecture ([7]).

• **B-Y. Chen's Conjecture**:

Any biharmonic isometric immersion into $(\mathbb{R}^k, g_0)$ *must be minimal.*

There is an alternative approach from theory of harmonic map. For a smooth map $\varphi : (M,g) \to (N,h)$, the *energy functional* is given by

$$E(\varphi) := \frac{1}{2}\int_M \|d\varphi\|^2\, v_g .$$

Then the *first variation formula* is:

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) = -\int_M \langle \tau(\varphi), V\rangle\, v_g .$$

Here, V is a variation vector field given by $V_x = (d/dt)|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$, and $\tau(\varphi)$ is the *tension field* of φ given by

$$\tau(\varphi) := \sum_{i=1}^{m} B(\varphi)(e_i, e_i) ,$$

where

$$B(\varphi)(X,Y) := \nabla^N_{d\varphi(X)} d\varphi(Y) - d\varphi(\nabla_X Y) \qquad (X, Y \in \mathfrak{X}(M)) .$$

Then $\varphi : (M,g) \to (N,h)$ is *harmonic* if $\tau(\varphi) = 0$.

The second variation formula for the energy functional $E(\cdot)$ for a harmonic map $\varphi : (M,g) \to (N,h)$ is given as follows.

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\varphi_t) = \int_M \langle J(V), V\rangle v_g .$$

Here, J is an elliptic second order partial differential operator acting on $\Gamma(\varphi^{-1}TN)$, called the Jacobi operator given by

$$J(V) := \overline{\Delta} V - \mathcal{R}(V) \qquad (V \in \Gamma(\varphi^{-1}TN)) ,$$

where

$$\overline{\Delta}V := \overline{\nabla}^*\,\overline{\nabla}V = -\sum_{i=1}^{m}\{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}V - \overline{\nabla}_{\nabla_{e_i}e_i}V)\}\ ,$$

$$\mathcal{R}(V) := \sum_{i=1}^{m} R^N(V, d\varphi(e_i))d\varphi(e_i)\ ,$$

where $\overline{\nabla}$ is the induced connection on the induced bundle $\varphi^{-1}TN$ from ∇^N by φ, and R^N is the Riemannian curvature tensor on (N,h) defined by $R^N(U,V)W = \nabla^N_U(\nabla^N_V W) - \nabla^N_V(\nabla^N_U W) - \nabla^N_{[U,V]}W$, for vector fields U, V, W on N.

In 1983, Eells and Lemaire [11] introduced the notion of k-harmonic map. Let us consider the *k-energy functional* due to Eells and Lemaire ([11]) is

$$E_k(\varphi) := \frac{1}{2}\int_M \left\|(d+\delta)^k\varphi\right\|^2 v_g \qquad (k = 1, 2, \cdots)\ .$$

Note that

$$E_1(\varphi) = \frac{1}{2}\int_M \|d\varphi\|^2\, v_g \quad \text{and} \quad E_2(\varphi) = \frac{1}{2}\int_M \|\tau(\varphi)\|^2\, v_g\ .$$

For $k = 2$, the *first variation formula* for $E_2(\varphi)$:

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M \langle \tau_2(\varphi), V\rangle\, v_g\ , \tag{2.1}$$

where $\tau_2(\varphi)$ is called the *bitension field* given by

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi))\ , \tag{2.2}$$

and $\varphi : (M,g) \to (N,h)$ is called *biharmonic* if $\tau_2(\varphi) = 0$.

The second variation formula for the bienergy was obtained by G.Y. Jiang [20] and C. Oniciuc [38], independently. The second variation formula for $E_2(\varphi)$ is given by

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E_2(\varphi_t) = \int_M \langle J_2(V), V\rangle v_g\ . \tag{2.3}$$

Here,

$$J_2(V) = J(J(V)) - \mathcal{R}_2(V)\ , \tag{2.4}$$

where

$$\begin{aligned}\mathcal{R}_2(V) &= R^N(\tau(\varphi), V)\tau(\varphi) + \\ &+ 2\,\mathrm{tr}\, R^N(d\varphi(\cdot), \tau(\varphi))\overline{\nabla}_{\cdot} V + 2\,\mathrm{tr}\, R^N(d\varphi(\cdot), V)\overline{\nabla}_{\cdot}\tau(\varphi) + \\ &+ \mathrm{tr}(\nabla^N_{d\varphi(\cdot)} R^N)(d\varphi(\cdot), \tau(\varphi))V + \\ &+ \mathrm{tr}(\nabla_{\tau(\varphi)} R^N)(d\varphi(\cdot), V)d\varphi(\cdot)\ .\end{aligned} \tag{2.5}$$

Then the notions of indices and nullities for E_2 are obtained as follows.

The *index* and *nullity* for a harmonic map are defined by

$$\mathrm{Index}(\varphi) := \dim(\oplus_{\lambda<0}E_\lambda) \quad , \quad \mathrm{Nullity}(\varphi) := \dim E_0\ , \tag{2.6}$$

and also, the *index* and *nullity* for a biharmonic map are defined by

$$\mathrm{Index}_2(\varphi) := \dim(\oplus_{\lambda<0}E^2_\lambda) \quad , \quad \mathrm{Nullity}_2(\varphi) := \dim E^2_0\ , \tag{2.7}$$

where E_λ, and E^2_λ are the eigenspaces of J, and J_2 with the eigenvalues λ, respectively.

Then it holds ([16]) that

Theorem 2.3. *If φ is a harmonic map, it is biharmonic and*

$$\mathrm{Index}_2(\varphi) = 0 \quad \textit{and} \quad \mathrm{Nullity}_2(\varphi) = \mathrm{Nullity}(\varphi)\ . \tag{2.8}$$

For the index and nullity for the Hopf map and further calculations, see E. Loubeau and C. Oniciuc's work ([25]).

3. Biharmonic maps into S^n

We turn to hypersurface theory in the symmetric spaces of rank one and of compact type. We first consider biharmonic hypersurfeces of the unit sphere.

Theorem 3.1 (Jiang, [20])**.** *Let $\varphi : (M^m, g) \to S^{m+1}(\frac{1}{\sqrt{c}})$ be an isometric immersion. Assume that the mean curvature of φ is nonzero constant. Then φ is biharmonic if and only if $\|B(\varphi)\|^2 = mc$.*

By using this theorem, we can give a classification of biharmonic isoparametric hypersurfaces in $S^n(1)$. To do it, let us recall the theory of isoparametric hypersurfaces in the unit sphere $S^n(1)$.

Let $\varphi : (M, g) \to S^n(1)$ be an isometric immersion, and assume that $\dim M = n-1$. Let us recall the notion of the shape operator $A_\xi : T_xM \to T_xM$ $(x \in M)$ which is defined by

$$g(A_\xi X, Y) = \langle \varphi_*(\nabla_X Y), \xi\rangle \qquad (X,\, Y \in \mathfrak{X}(M))\ ,$$

where ξ is the unit normal vector field along M. The eigenvalues of A_ξ are called the *principal curvatures*, and M is called *isoparametric* if all the principal curvatures are constant in $x \in M$. Thus we can apply Theorem 3.1 to all the isoparametric hypersurfaces, because the mean curvature of φ is constant because it is $1/(m-1)$ times the sum of all principal curvature with their multiplisities.

Now let us recall the works of E.Cartan, H.F. Münzner ([30], [31]), H. Ozeki, and M. Takeuchi ([40], [41]) (See also [28]).

Theorem 3.2. *Assume that $\varphi : (M, g) \to S^n(1)$ is an isoparametric hypersurface. Then there exists a homogeneous polynomial F on $\mathbb{R}^{n+1}$ of degree d such that M is given by*

$$M = \varphi^{-1}(t), \quad \textit{for} \quad \textit{some} \quad -1 < t < 1\ ,$$

where $\varphi := F|_{S^n(1)}$. Say $M = M(t)$.

All the principal curvatures are given as

$$k_1(t) > k_2(t) > \cdots > k_{d(t)}(t)\ ,$$

with their multiplicities $m_j(t)$ $(j = 1, \cdots, d(t))$. Here, $d = d(t)$ is constant in t, and $d = 1, 2, 3, 4,$ or 6.

Then our main result is the following classification of biharmonic isoparametric hypersurfaces in the unit sphere S^n ([16], [17]).

Theorem 3.3. *Assume that $\varphi : (M, g) \to S^n(1)$ is a biharmonic isoparametric hypersurface in the unit sphere. Then (M, g) is one of the following three cases:*

- $M = S^{n-1}(1/\sqrt{2}) \subset S^n(1)$ *(a small sphere, Oniciuc),*

- $M = S^{n-p}(1/\sqrt{2}) \times S^{p-1}(1/\sqrt{2}) \subset S^n(1)$ $(n-p \neq p-1)$, (*the Clifford torus, Jiang*),
- $\varphi : (M, g) \to S^n(1)$ *is minimal.*

4. Biharmonic maps into the projective spaces

The first case is the complex projective space $\mathbb{C}P^n$. Let us begin the following theorem ([16], [17]) which is an analogue of Theorem 3.1 in the case of the complex projective space.

Theorem 4.1. *Let (M, g) be a real $(2n-1)$-dimensional compact Riemannian manifold, $\varphi : (M, g) \to \mathbb{C}P^n(c)$, an isometric immersion into the projective space with constant holomorphic sectional curvature c. Assume that $\varphi : (M, g) \to \mathbb{C}P^n(c)$ has nonzero constant mean curvature. Then φ is biharmonic if and only if $\|B(\varphi)\|^2 = ((n+1)/2)\, c$.*

In order to apply this theorem to biharmonic isometric immersions, let us recall R. Takagi's work on classification of all the homogeneous real hypersurfaces in $\mathbb{C}P^n$ ([47]).

Theorem 4.2 ([47])**.** *Let U/K be a compact Hermitian symmetric space of rank two, and let $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan decomposition. Then*

- $\hat{M} := \mathrm{Ad}(K)A \subset \mathfrak{p}$ *is a hypersurface in S^{2n+1} for some regular element $A \in \mathfrak{p}$ with $\|A\| = 1$. Here, we put $\dim_{\mathbb{C}} \mathfrak{p} = n+1$.*
- $M = \pi(\hat{M}) \subset \mathbb{C}P^n$ *give all real homogeneous hypersurfaces in $\mathbb{C}P^n$, where*

$$\pi : \mathbb{C}^{n+1} - \{\mathbf{0}\} = \mathfrak{p} - \{\mathbf{0}\} \to \mathbb{C}P^n$$

is the natural projection.

Theorem 4.3 ([47])**.** *All the homogeneous real hypersurfaces in $\mathbb{C}P^n$ are classified into the following five types:*

- (*A type*) $U/K = \dfrac{SU(s+1) \times SU(t+1)}{S(U(s) \times U(1)) \times S(U(t) \times U(1))}$,
- (*B type*) $U/K = SO(m+2)/(SO(m) \times SO(2))$,
- (*C type*) $U/K = SU(m+2)/S(U(m) \times U(2))$,
- (*D type*) $U/K = O(10)/U(5)$,
- (*E type*) $U/K = E_6/(\mathrm{Spin}(10) \times U(1))$.

Then we can state our main result ([16], [17]):

Theorem 4.4. *All the biharmonic homogeneous real hypersurfaces in $\mathbb{C}P^n(4)$ are classified as follows. Let M be a homogeneous real hypersurface in $\mathbb{C}P^n(4)$. Then M is one of the types $A \sim E$.*

(I) *For all the types, there exists a unique orbit M which is a minimal hypersurface in $\mathbb{C}P^n(4)$ as in Theorem 4.3.*
(II) *There exists a unique orbit $M \subset \mathbb{C}P^n(4)$ which is biharmonic but not harmonic in each the types A, D and E. There are no such orbits in the types B and C.*

The next is real hypersurfeces in the quaternionic projective space $\mathbb{H}P^n(c)$. We first show the following:

Theorem 4.5. *Let $\varphi : (M, g) \to \mathbb{H}P^n(c)$ be an isometric immersion with nonzero constant mean curvature,* $\dim M = 4n - 1$. *Then φ: biharmonic if and only if* $\|B(\varphi)\|^2 = (n + 2)c$.

For cases of the non-compact duals of the $S^n(1)$, $\mathbb{C}P^n(4)$, or $\mathbb{H}P^n(c)$, it holds that

$$\|B(\varphi)\|^2 = (n-1)c,\ \|B(\varphi)\|^2 = \frac{n+1}{2}\,c \quad \text{or} \quad \|B(\varphi)\|^2 = (n+2)c \quad \text{(respectively).}$$

Therefore, every biharmonic hypersurfaces in $(\mathbb{R}^n, g_0)$, or one of the classical rank one symmetric spaces of non-compact type with *constant mean curvature* must be minimal.

Then we give a classification of all biharmonic homogeneous hypersurfaces in $\mathbb{H}P^n(4)$ ([16], [17]).

Theorem 4.6.

(I) (*J. Berndt*) *All the homogeneous real hypersurfaces in $\mathbb{H}P^n(4)$ are classified into of the three types.*
(II) *In each types, there exist minimal homogeneous real hypersurfaces in $\mathbb{H}P^n(4)$.*
(III) *In each types, there exist biharmonic nonminimal homogeneous real hypersurfaces in $\mathbb{H}P^n(4)$.*

5. Conformal change and biharmonic maps

In this section, we want to treat with the O.D.E. method to construct biharmonic non-harmonic maps, i.e., to give the reduction theorem of biharmonic map equation into the ordinary differential equation, and to give existence and non-existence theorem due to the joint work with H. Naito (cf. [32]).

Let us recall a work due to P. Baird and D. Kamissoko ([4]) on constructing biharmonic maps by using conformal change of metrics. Our setting is a little bit different from them. Consider a C^∞ mapping $\varphi : (M, \widetilde{g}) \to (N, h)$ with $\widetilde{g} = f^{2/(m-2)}g$, $f \in C^\infty(M)$, $f > 0$ $(m := \dim M > 2)$.

We give a theorem of conformal change of bienergy tension field.

Theorem 5.1. *For $\varphi \in C^\infty(M, N)$, the bienergy tension field, denoted by $\tau_2(\varphi; \widetilde{g}, h)$ of $\varphi : (M, \widetilde{g}) \to (N, h)$ is given by*

$$\begin{aligned} f^{2m/(m-2)}\tau_2(\varphi;\widetilde{g},h) = & -\frac{m-6}{m-2}\, f\, \overline{\nabla}_X \tau_g(\varphi) + f^2\, J_g(\tau_g(\varphi)) - \\ & - \left\{ \frac{4}{(m-2)^2}\, |X|_g{}^2 + \frac{2}{m-2}\, f(\Delta_g f) \right\} \tau_g(\varphi) - \\ & - f^{-1} \left\{ \frac{m^2}{(m-2)^2}\, |X|_g{}^2 + \frac{m}{m-2}\, f\,(\Delta_g f) \right\} d\varphi(X) + \\ & + \frac{m+2}{m-2}\, \overline{\nabla}_X d\varphi(X) + f\, J_g(d\varphi(X))\,, \end{aligned}$$

where $X = \nabla^g f \in \mathfrak{X}(M)$.

In the case that φ is the identity map of the Euclidean space, we give the reduction in the following.

Let $(M,g)=(\mathbb{R}^m,g_0)$, $(m\geq 3)$, the standard Euclidean space, and $f\in C^\infty(\mathbb{R}^m)$ is given by

$$f(x_1,x_2,\cdots,x_m)=f(x_1)=f(x)\ ,$$

where we denote by $x=x_1$, and its differentiation of f by f'. Then the identity map of $\mathbb{R}^m$, $\mathrm{id}:(\mathbb{R}^m,f^{2/(m-2)}g_0)\to(\mathbb{R}^m,g_0)$ is biharmonic if and only if

$$f^2\,f'''-2\,\frac{m+1}{m-2}\,f\,f'\,f''+\frac{m^2}{(m-2)^2}\,{f'}^3=0\ . \tag{5.1}$$

Then we have the following results (joint work with H. Naito, [32]).

Theorem 5.2. *Assume that $m\geq 3$. Then*

(i) ($m\geq 5$) *there exists no positive global C^∞ solution f on $\mathbb{R}$ of the ODE.*
(ii) ($m=4$) $f(x_1)=a/\cosh(b\,x_1+c)$ *is a global positive C^∞ solution of the ODE for every $a>0$, b and c.*
(iii) ($m=3$) *there exists a positive C^∞ solution f, and no positive periodic solution f on $\mathbb{R}$.*

Corollary 5.3. *The identity map of the 4-dimensional Euclidean space* $\mathrm{id}:(\mathbb{R}^4,(a/\cosh(b\,x_1+c))\,g_0)\to(\mathbb{R}^4,g_0)$, *is a proper biharmonic map, where, $(x_1,\cdots,x_4)$ is the standard coordinate of $\mathbb{R}^4$.*

One of main ingredients of our method is the following:

Theorem 5.4. *Let $\varphi:(M^2,g)\to(N^{n-1},h)$ be any harmonic map ($n\geq 2$). For a positive solution f of*

$$f^2\,f'''-8\,f\,f'\,f''+9\,{f'}^3=0\ , \tag{5.2}$$

let $f(x,t):=f(t)$, $(x,t)\in M\times\mathbb{R}$, and $\widetilde{\varphi}:\ M\times\mathbb{R}\ni(x,t)\mapsto(\varphi(x),t)\in N\times\mathbb{R}$. Then the C^∞ map $\widetilde{\varphi}:(M\times\mathbb{R},f^2\,(g+dt^2))\to(N\times\mathbb{R},h+dt^2)$ is a proper biharmonic map.

In the case of $m=4$, for every $a>0$, b, and $c\in\mathbb{R}$, the C^∞ map $\widetilde{\varphi}:\big(M\times\mathbb{R},(a/\cosh(bt+c))\,(g+dt^2)\big)\to(N\times\mathbb{R},h+dt^2)$ *is a proper biharmonic map.*

Corollary 5.5. *Let (M^2,g) be any Riemannian surface, and for a positive C^∞ solution of*

$$f^2\,f'''-8\,f\,f'\,f''+9\,{f'}^3=0\ , \tag{5.3}$$

let $f(x,t):=f(t)$, $(x,t)\in M\times\mathbb{R}$. Then

(1) *the identity map* $\mathrm{id}:(M\times\mathbb{R},f^2\,(g+dt^2))\to(M\times\mathbb{R},g+dt^2)$ *is a proper biharmonic map.*
(2) *Let $m=4$. For $a>0$, b, and $c\in\mathbb{R}$, the identity map* $\mathrm{id}:\big(M\times\mathbb{R},(a/\cosh(bt+c))\,(g+dt^2)\big)\to(M\times\mathbb{R},g+dt^2)$ *is a proper biharmonic map.*

6. Biharmonic maps into compact Lie groups

In this section, we want to describe all the harmonic maps and biharmonic maps into compact Lie groups. Let us recall the theories of harmonic maps into Lie groups (cf. K. Uhlenbeck [48], and J.C. Wood [54]). Then we have to extend them to biharmonic maps into compact Lie groups.

Let G be a compact Lie group, and h a bi-invariant Riemannian metric on G corresponding to $\mathrm{Ad}(G)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$.

Let θ be the Maurer-Cartan form on G which is defined by $\theta_y(Z_y) = Z$ ($Z \in \mathfrak{g}$, $y \in G$).

For a C^∞ map $\psi : M \to G$, let $\alpha := \psi^*\theta$. Then the tension field $\tau(\psi) \in \Gamma(\psi^{-1}TG)$ is given by

$$\langle \theta, \tau(\psi)\rangle = \theta \circ \tau(\psi) = -\delta\alpha\ , \tag{6.1}$$

i.e.,

$$\theta_{\psi(x)}(\tau(\psi)(x)) = -(\delta\alpha)_x \qquad (x \in G)\ . \tag{6.2}$$

We can calculate the bitension field as follows:

$$\theta(\tau_2(\psi)) = \theta(J_\psi(\tau(\psi)))\ . \tag{6.3}$$

Theorem 6.1. *For a C^∞ map $\psi : (M,g) \to (G,h)$,*

$$\theta(J_\psi(\tau(\psi))) = -\delta_g\, d\,(\delta\alpha) - \mathrm{Trace}_g([\alpha,\, d\,\delta\alpha])\ . \tag{6.4}$$

As a corollary, we have

Corollary 6.2. *We have*

(1) *$\psi : (M,g) \to (G,h)$ is harmonic if and only if $\delta\alpha = 0$.*
(2) *$\psi : (M,g) \to (G,h)$ is biharmonic if and only if*

$$\delta_g\, d\,\delta\alpha + \mathrm{Trace}_g([\alpha, d\delta\alpha]) = 0\ . \tag{6.5}$$

In the Case $(M,g) = (\mathbb{R}, g_0)$, let $\psi : \mathbb{R} \ni t \mapsto \psi(t) \in (G,h)$, a C^∞ curve, and consider the $\mathfrak{g}$-valued 1-form $\alpha := \psi^*\theta$ on $\mathbb{R}$. Then the 1-form $\alpha = \psi^*\theta = F(t)\,dt$ satisfies

$$F(t) = \theta\left(\psi_*\left(\frac{\partial}{\partial t}\right)\right) = L^{-1}_{\psi(t)*}\psi'(t) \quad , \quad \delta\alpha = -F'(t)\ .$$

Then ψ is harmonic if and only if

$$\delta\alpha = 0 \quad \Longleftrightarrow \quad F'(t) = 0 \Longleftrightarrow \tag{6.6}$$

$$\Longleftrightarrow \quad \psi : (\mathbb{R}, g_0) \to (G,h),\ \text{a geodesic}\ . \tag{6.7}$$

Furthermore, in the Case $(M,g) = (\mathbb{R}, g_0)$, we have the following.

To see the biharmonic map equation, we have

$$\delta_{g_0} d\delta\alpha = -\frac{\partial^2}{\partial t^2}\,(-F'(t)) = F^{(3)}\ , \tag{6.8}$$

$$\mathrm{Trace}_{g_0}[\alpha, d\delta\alpha] = -\,[F(t), F''(t)]\ . \tag{6.9}$$

Then ψ is biharmonic if and only if

$$F^{(3)} - [F(t), F''(t)] = 0\ . \tag{6.10}$$

Furthermore let us consider $\psi(t) = \exp(X(t))$, where $X : \mathfrak{g} \to \mathfrak{g}$ is a C^∞-mapping. Then $\alpha = F(t)\,dt$ is given by

$$\alpha(t)(\frac{\partial}{\partial t}) = F(t) = \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}X(t))^n}{(n+1)!}(X'(t)) \ . \tag{6.11}$$

Now let us consider the Case: $(M, g) = (\mathbb{R}, g_0)$, $G = SU(2)$. In the case $G = SU(2)$, and $\mathfrak{g} = \mathfrak{su}(2)$,

$$\langle X, Y\rangle := -2\,\mathrm{Trace}(XY) \quad , \quad X, Y \in \mathfrak{su}(2) \ .$$

We take as an orthonormal basis $\{X_i\}_{i=1}^3$ of $\mathfrak{su}(2)$,

$$X_1 = \begin{pmatrix} \sqrt{-1}/2 & 0 \\ 0 & -\sqrt{-1}/2 \end{pmatrix} \quad , \quad X_2 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix} \ ,$$

$$X_3 = \begin{pmatrix} 0 & \sqrt{-1}/2 \\ \sqrt{-1}/2 & 0 \end{pmatrix} \ .$$

It holds that

$$[X_1, X_2] = X_3 \quad , \quad [X_2, X_3] = X_1 \quad , \quad [X_3, X_1] = X_2 \ .$$

Then we have

$$\begin{aligned} F(t) = &\left(-\frac{a}{\sqrt{a^2+1}}\ \sin t\right) X_1 + \\ &+ \left(\frac{a}{\sqrt{a^2+1}}\ \cot t\right) X_2 + \frac{1}{\sqrt{a^2+1}}\, X_3 \ . \end{aligned} \tag{6.12}$$

But it is difficult for us to find $X(t)$ satisfying that

$$F(t) = \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}X(t))^n}{(n+1)!}(X'(t)) \ ,$$

for a given $F(t)$ such as the above.

7. Biharmonic maps into Lie groups and integrable systems

In this section, we consider a C^∞ map

$$\psi : (\mathbb{R}^2, g) \supset \Omega \to (G, h) \ ,$$

where $g := \mu^2\, g_0$ with $\mu > 0$, a C^∞ function on Ω, G, a compact linear Lie group, and h, a bi-invariant Riemannian metric corresponding to the $\mathrm{Ad}(G)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Then we have

$$\alpha := \psi^*\theta = \psi^{-1}d\psi \ .$$

We first want to describe the harmonic map equations.
If we put $A_x := \psi^{-1}(\partial\psi/\partial x)$, $A_y := \psi^{-1}(\partial\psi/\partial y)$, we have

$$\delta\alpha = -\mu^{-2}\left\{\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y\right\} \ .$$

Then ψ is harmonic if and only if

$$\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y = 0 \ , \tag{7.1}$$

where A_x and A_y are $\mathfrak{g}$-valued 1-forms on Ω, and satisfy the integrability condition:

$$\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x + [A_x, A_y] = 0\ . \tag{7.2}$$

Conversely, if two $\mathfrak{g}$-valued 1-forms A_x and A_y on Ω satisfy the above two equations (7.1) and (7.2), then there exists a harmonic map $\psi : \Omega \to (G, h)$ with $\psi^{-1}(\partial\psi/\partial x) = A_x$ and $\psi^{-1}(\partial\psi/\partial y) = A_y$.

Now we want to describe the biharmonic map equations. We have

Theorem 7.1.

(1) *ψ is biharmonic is and only if*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\delta\alpha) - \frac{\partial}{\partial x}[A_x, \delta\alpha] - \frac{\partial}{\partial y}[A_y, \delta\alpha] = 0\ . \tag{7.3}$$

(2) *If we define the $\mathfrak{g}$-valued 1-form β by*

$$\beta := [A_x, \delta\alpha]\, dx + [A_y, \delta\alpha]\, dy\ , \tag{7.4}$$

then,

$$\delta\beta = -\mu^{-2}\left(\frac{\partial}{\partial x}[A_x, \delta\alpha] + \frac{\partial}{\partial y}[A_y, \delta\alpha]\right)\ . \tag{7.5}$$

(3) *Thus ψ is biharmonic if and only if*

$$\delta(d\delta\alpha - \beta) = 0\ . \tag{7.6}$$

Now, we want to consider the complexifications.

Take the complex coordinate $z = x + iy$ ($i = \sqrt{-1}$). Then $dz = dx + idy$, $d\overline{z} = dx - idy$,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad , \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\ .$$

Extend α to a $\mathfrak{g}^{\mathbb{C}}$-valued 1-form on Ω as

$$\alpha = A_x\, dx + A_y\, dy = A_z\, dz + A_{\overline{z}}\, d\overline{z}\ .$$

Then

$$-\delta\alpha = \mu^{-2}\left(\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y\right) = 2\mu^{-2}\left(\frac{\partial}{\partial \overline{z}}A_z + \frac{\partial}{\partial z}A_{\overline{z}}\right)\ , \tag{7.7}$$

$$\text{the integrability:}\quad \frac{\partial}{\partial z}A_{\overline{z}} - \frac{\partial}{\partial \overline{z}}A_z + [A_z, A_{\overline{z}}] = 0\ . \tag{7.8}$$

Then the harmonic and biharmonic conditions can be described as follows. Let $\psi : (\mathbb{R}^2, g) \supset \Omega \to (G, h)$ with $g = \mu^2 g_0$.

ψ is harmonic if and only if

$$\frac{\partial}{\partial \overline{z}}A_z + \frac{\partial}{\partial z}A_{\overline{z}} = 0\ . \tag{7.9}$$

For ψ, to be biharmonic if and only if

$$\frac{\partial}{\partial \overline{z}}B_z + \frac{\partial}{\partial z}B_{\overline{z}} = 0\ . \tag{7.10}$$

Here, $B = B_z\, dz + B_{\overline{z}}\, d\overline{z}$ is a $\mathfrak{g}^{\mathbb{C}}$-valued 1-form on Ω defined by

$$\begin{cases} B_z & := \dfrac{\partial}{\partial z}(\delta\alpha) - [A_z, \delta\alpha]\ , \\ B_{\overline{z}} & := \dfrac{\partial}{\partial \overline{z}}(\delta\alpha) - [A_{\overline{z}}, \delta\alpha]\ , \end{cases} \tag{7.11}$$

where

$$\delta\alpha = -2\mu^{-2}\left(\frac{\partial}{\partial \overline{z}}A_z + \frac{\partial}{\partial z}A_{\overline{z}}\right)\ .$$

Then our problem is how to solve the biharmonic map equation.

Step 1: Solve the harmonic map equation (1):

$$\frac{\partial}{\partial \overline{z}}B_z + \frac{\partial}{\partial z}B_{\overline{z}} = 0 \quad , \quad \frac{\partial}{\partial z}B_{\overline{z}} - \frac{\partial}{\partial \overline{z}}B_z + [B_z, B_{\overline{z}}] = 0\ . \tag{7.12}$$

Step 2: For such B, solve A of the partial differential equations (2):

$$\begin{cases} \dfrac{\partial}{\partial z}(\delta\alpha) - [A_z, \delta\alpha] = B_z \quad , \quad \dfrac{\partial}{\partial \overline{z}}(\delta\alpha) - [A_{\overline{z}}, \delta\alpha] = B_{\overline{z}}\ , \\ \dfrac{\partial}{\partial z}A_{\overline{z}} - \dfrac{\partial}{\partial \overline{z}}A_z + [A_z, A_{\overline{z}}] = 0\ , \end{cases} \tag{7.13}$$

where

$$\delta\alpha := -2\mu^{-2}\left(\frac{\partial}{\partial \overline{z}}A_z + \frac{\partial}{\partial z}A_{\overline{z}}\right)\ .$$

Step 3: For such $A = A_z dz + A_{\overline{z}} d\overline{z}$, solve a C^∞ mapping $\psi : \Omega \to G$ satisfying that

$$\begin{cases} \psi(x_0, y_0) = a \in G\ , \\ \psi^{-1}\dfrac{\partial\psi}{\partial z} = A_z \quad , \quad \psi^{-1}\dfrac{\partial\psi}{\partial \overline{z}} = A_{\overline{z}}\ . \end{cases} \tag{7.14}$$

Then we have:

Theorem 7.2. *This map $\psi : (\Omega, g) \to (G, h)$ is biharmonic. Every biharmonic map can be obtained in this way. ($g := \mu^{-2} g_0$ and μ is a positive C^∞ function on Ω).*

To see biharmonic map: $\psi : (S^2, g_0) \to (G, h)$, let us recall

Theorem 7.3 (Sacks and Uhlenbeck)**.** *Every harmonic map $\psi : (\mathbb{R}^2, g) \to (G, h)$ with finite energy can be uniquely extended to a harmonic map $\tilde{\psi} : (S^2, g_0) \to (G, h)$.*

Conversely, every harmonic map $\tilde{\psi} : (S^2, g_0) \to (G, h)$ can be obtained in this way.

We wish to obtain the following:

"Theorem 7.4" *Every biharmonic map $\psi : (\mathbb{R}^2, g) \to (G, h)$ with finite bienergy can be uniquely extended to a biharmonic map $\tilde{\psi} : (S^2, g_0) \to (G, h)$.*
Conversely, every biharmonic map $\tilde{\psi} : (S^2, g_0) \to (G, h)$ can be obtained in this way.

The above works is based on our recent works (cf. [49]).

8. Biharmonic maps into symmetric spaces

Now let us recall the famous work of Dorfmeister, Pedit and Wu ([9]) on harmonic maps into symmetric spaces. This work gave a systematic scheme for constructing all harmonic maps from a Riemann surface Σ into G/K.

We want to extend it to biharmonic maps. We first begin to set framework of biharmonic maps into symmetric spaces. Let (M, g) be a compact Riemannian manifold, $(N, h) = (G/K, h)$, a Riemannian symmetric space with G-invariant Riemannian metric h on G/K, and $\pi : G \to G/K$, the natural projection. Let $\varphi : M \to G/K$, a C^∞ map with a local lift $\psi : M \to G$, i.e., $\varphi = \pi \circ \psi$. Let θ be the Maurer-Cartan form on G, i.e., $\theta_y(Z_y) = Z$, $Z \in \mathfrak{g}, y \in G$.

Let us consider a $\mathfrak{g}$-valued 1-form α on M given by $\alpha := \psi^*\theta$, and, corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, decompose it as

$$\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}} .$$

Then the tension field $\tau(\varphi)$ is given by

$$t_{\psi(x)^{-1}*}\tau(\varphi) = -\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] , \tag{8.1}$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M, g) ($\dim M = m$), and δ is the co-differentiation. Then we have

Theorem 8.1. *The bitension field $\tau_2(\varphi)$ of $\varphi : (M, g) \to (G/K, h)$ is given by*

$$\tau_2(\varphi) = \Delta_g \left(-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] \right) +$$
$$+ \sum_{s=1}^{m} \left[\left[-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \alpha_{\mathfrak{m}}(e_s) \right], \alpha_{\mathfrak{m}}(e_s) \right] .$$

As a corollary, we have

Corollary 8.2. *Let $(G/K, h)$ be a Riemannian symmetric space, $\varphi : (M, g) \to (G/K, h)$, a C^∞ map. Then*

(1) *φ is harmonic if and only if*

$$-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] = 0 . \tag{8.2}$$

(2) *φ is biharmonic if and only if the following equation holds:*

$$\Delta_g \left(-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)] \right) +$$
$$+ \sum_{s=1}^{m} \left[\left[-\delta(\alpha_{\mathfrak{m}}) + \sum_{i=1}^{m}[\alpha_{\mathfrak{k}}(e_i), \alpha_{\mathfrak{m}}(e_i)], \alpha_{\mathfrak{m}}(e_s) \right], \alpha_{\mathfrak{m}}(e_s) \right] = 0 . \tag{8.3}$$

Conversely, any $\mathfrak{g}$ valued 1-form $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$ satisfies the Maurer-Cartan equation: $d\alpha + (1/2)\,[\alpha \wedge \alpha] = 0$, i.e.,

$$\begin{cases} d\alpha_{\mathfrak{k}} + \dfrac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + \dfrac{1}{2}[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}] = 0 , \\ d\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{m}}] = 0 . \end{cases} \tag{8.4}$$

Then there exists a C^∞ map $\psi : M \to G$ such that $\alpha = \psi^*\theta$.

Therefore, every $\mathfrak{g}$-valued 1-form $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$ which satisfies (8.3) and (8.4), there exists a C^∞ map $\psi : M \to G$ such that $\alpha = \psi^*\theta$, and $\varphi := \pi \circ \psi : (M, g) \to (G/K, h)$ is a biharmonic map.

The lift ψ is *horizontal* if

$$\psi_*(T_xM) \subset H_{\psi(x)} := \{Y_{\psi(x)} \,|\, Y \in \mathfrak{m}\} \qquad (\forall\ x \in M) \tag{8.5}$$

which is equivalent to $\alpha_{\mathfrak{k}} \equiv 0$.

Now, let $\varphi : \mathbb{R} \to G/K$ be a C^∞ curve in G/K, and $\psi : \mathbb{R} \to G$, the lift, i.e., $\varphi = \pi \circ \psi$. The 1-form $\alpha := \psi^*\theta = \psi^{-1}d\psi$ can be written as $\alpha = F(t)dt$, where $F(t)$ is a $\mathfrak{g}$-valued function in t. Write as

$$F(t) = F_{\mathfrak{k}}(t) + F_{\mathfrak{m}}(t)\ .$$

Then φ is harmonic if and only if

$$F'_{\mathfrak{m}}(t) + [F_{\mathfrak{k}}(t), F_{\mathfrak{m}}(t)] = 0\ . \tag{8.6}$$

And φ is biharmonic if and only if

$$-(F'_{\mathfrak{m}}(t) + [F_{\mathfrak{k}}(t), F_{\mathfrak{m}}(t)])'' + [[F'_{\mathfrak{m}}(t) + [F_{\mathfrak{k}}(t), F_{\mathfrak{m}}(t)], F_{\mathfrak{m}}], F_{\mathfrak{m}}] = 0\ . \tag{8.7}$$

Now we treat with the case of harmonic curves: Consider the horizontal lift: $F_{\mathfrak{k}} \equiv 0$. Then $F'_{\mathfrak{m}} = 0$, i.e., $F_{\mathfrak{m}} = X \in \mathfrak{m}$ (constant). Thus $\psi(t) = x\ \exp(tX)$ and $\varphi(t) = x\ \exp(tX) \cdot o$, where $X \in \mathfrak{m}$ and $o = \{K\} \in G/K$.

Next, we treat with the case of biharmonic curves: Consider the horizontal lift: $F_{\mathfrak{k}} \equiv 0$. Then

$$-F'''_{\mathfrak{m}}(t) + [[F'_{\mathfrak{m}}(t), F_{\mathfrak{m}}(t)], F_{\mathfrak{m}}(t)] = 0\ . \tag{8.8}$$

In particular, in the case of the Euclidean space $\mathbb{R}^n$, (8.7) is $-{F_{\mathfrak{m}}}''' = 0$, so we have

$$F_{\mathfrak{m}}(t) = at^2 + bt + c \qquad (a, b, c \in \mathfrak{m} = \mathbb{R}^n,\ \text{constants})\ . \tag{8.9}$$

We have

$$\psi(t) = x\ \exp(d_t) = (I_d, d_t) \in EM(n)\ , \tag{8.10}$$

$$\varphi(t) = \psi(t)\ \cdot\ o = x\, d_t\ , \tag{8.11}$$

where

$$d_t := \frac{1}{3}\, at^3 + \frac{1}{2}\, bt^2 + ct \in \mathbb{R}^n \quad , \quad x \in EM(n)\ .$$

Now we consider biharmonic curves in the unit sphere. Let us recall that the unit sphere S^n can be realized as follows. Let $G = SO(n+1)$, $K = SO(n)$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$,

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -{}^t u \\ u & O \end{pmatrix}\ |\ u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n \right\}\ .$$

The equations (8.7) is

$$-u''' + \langle u', u\rangle\, u - \langle u, u\rangle\, u' = 0\ . \tag{8.12}$$

One of solutions of (8.12) is given by

$$(u_1, \cdots, u_n) = (0, \cdots, 0, at^2 + bt + c, 0, \cdots, 0)\ . \tag{8.13}$$

Furthermore

$$\varphi_i(t) = x \cdot \begin{pmatrix} \cos d_t \\ 0 \\ \vdots \\ 0 \\ -\sin d_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (x \in SO(n+1), \;\; i = 1, \cdots, n)$$

are biharmonic curves in S^n. In particular, φ_i are harmonic if and only if $a = b = 0$.

To solve the equation (8.12) in the case S^2, let take a curve $\mathbf{p}(s)$ in $\mathbb{R}^2$ with arc length parameter s. Then

$$\mathbf{p}' = \mathbf{e}_1 \quad , \quad \mathbf{e}_1' = \kappa\, \mathbf{e}_2 \quad , \quad \mathbf{e}_2' = -\kappa\, \mathbf{e}_1 \; .$$

Substituting this into (8.12), we have $\kappa^3 = \kappa$, i.e., $\kappa = 0, \; \pm 1$.

Then we have

Case 1: $\kappa = 0$. In this case, we have a great circle:

$$\varphi(t) = x \cdot {}^{\mathrm{t}}(\cos(t\, c), \frac{a}{c}\sin(t\, c), \frac{b}{c}\sin(t\, c)) \; , \tag{8.14}$$

where $c := \sqrt{a^2 + b^2}$, $a, b \in \mathbb{R}$ and $x \in SO(3)$.

Case 2: $\kappa = 1$. In this case, we have

$$F_{\mathfrak{m}}(t) = \left(\begin{array}{c|cc} 0 & \sin t & -\cos t \\ \hline -\sin t & 0 & 0 \\ \cos t & 0 & 0 \end{array} \right) . \tag{8.15}$$

Case 3: $\kappa = -1$. In this case, we have

$$F_{\mathfrak{m}}(t) = \left(\begin{array}{c|cc} 0 & \sin t & \cos t \\ \hline -\sin t & 0 & 0 \\ -\cos t & 0 & 0 \end{array} \right) . \tag{8.16}$$

Now we consider biharmonic curves in $\mathbb{C}P^n$.

Recall that the complex projective space $\mathbb{C}P^n$ can be realized as follows: let $G = SU(n+1)$, $K = S(U(1) \times U(n))$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$,

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -{}^{\mathrm{t}}\overline{z} \\ z & 0 \end{pmatrix} \mid z = {}^{\mathrm{t}}(z_1, \cdots, z_n) \in \mathbb{C}^n \right\} .$$

The equations (8.12) is that:

$$-z''' + 2\langle z, z' \rangle z - \langle z', z \rangle z - \langle z, z \rangle z' = 0 \; . \tag{8.17}$$

Solutions of (8.17) are

$$(z_1, \cdots, z_n) = (0, \cdots, 0, at^2 + bt + c, 0, \cdots, 0), \quad \text{or} \tag{8.18}$$

$$(z_1, \cdots, z_n) = (0, \cdots, 0, \sqrt{-1}(at^2 + bt + c), 0, \cdots, 0) \; . \tag{8.19}$$

Then biharmonic curves in $\mathbb{C}^n$ are given by

$$\varphi_i^1(t) = x \cdot \begin{pmatrix} \cos d_t \\ 0 \\ \vdots \\ 0 \\ -\sin d_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad \varphi_i^2(t) = x \cdot \begin{pmatrix} \cos d_t \\ 0 \\ \vdots \\ 0 \\ \sqrt{-1}\,\sin d_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{8.20}$$

where $x \in SU(n+1)$, $i = 1, \cdots, n$, are biharmonic curves in $\mathbb{C}P^n$, and φ_i^1 (resp. φ_i^2) are harmonic if and only if $a = b = 0$.

Next, we treat with biharmonic curves in $\mathbb{H}P^n$. In $\mathbb{H}P^n$, one can also obtain biharmonic curves:

$$x \cdot \begin{pmatrix} \cos d_t \\ 0 \\ \vdots \\ 0 \\ -\sin d_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad x \cdot \begin{pmatrix} \cos d_t \\ 0 \\ \vdots \\ 0 \\ i\,\sin d_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad x \cdot \begin{pmatrix} \cos d_t \\ 0 \\ \vdots \\ 0 \\ j\,\sin d_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \quad x \cdot \begin{pmatrix} \cos d_t \\ 0 \\ \vdots \\ 0 \\ k\,\sin d_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

($x \in Sp(n+1)$), are all biharmonic curves in $\mathbb{H}P^n$, and they are harmonic if and only if $a = b = 0$.

Now, let us consider the case $\dim M = 2$. From now on, we consider biharmonic maps of $(\mathbb{R}^2, g_0)$ into a Riemannian symmetric space $(G/K, h)$ with $g_0 = dx^2 + dy^2$.

Let $\varphi : \mathbb{R}^2 \to (G/K, h)$ be a C^∞ map, $\psi : \mathbb{R}^2 \to G$, a local lift, and $\alpha = \psi^*\theta$. Let $\alpha_{\mathfrak{k}} = R\,dx + S\,dy$, $\alpha_{\mathfrak{m}} = P\,dx + Q\,dy$, where R, S, (resp. P, and Q) are $\mathfrak{k}$-valued (resp. $\mathfrak{m}$-valued) functions on $\mathbb{R}^2$.

Then φ is *harmonic* if and only if

$$P_x + Q_y + [R, P] + [S, Q] = 0 \, , \tag{8.21}$$

where $P_x = \partial P/\partial x$, $Q_y = \partial Q/\partial y$. Furthermore φ is *biharmonic* if and only if

$$\begin{aligned} &-\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}(P_x + Q_y + [R, P] + [S, Q]) + \\ &+[[P_x + Q_y + [R, P] + [S, Q], P], P] + \\ &+[[P_x + Q_y + [R, P] + [S, Q], Q], Q] = 0 \, . \end{aligned} \tag{8.22}$$

In the case ψ is horizontal, i.e., $\alpha_{\mathfrak{k}} \equiv 0$, $R = S = 0$, φ is biharmonic if and only if

$$\begin{aligned} &-P_{xxx} - P_{xyy} - Q_{xxy} - Q_{yyy} + \\ &+[[P_x + Q_y, P], P] + [[P_x + Q_y, Q], Q] = 0 \, . \end{aligned} \tag{8.23}$$

Furthermore, the integrability condition, i.e., $d\alpha + (1/2)[\alpha\wedge\alpha] = 0$ becomes the following:

$$-R_y + S_x + [R,S] + [P,Q] = 0\ , \tag{8.24}$$

$$-P_y + Q_x + [R,Q] + [P,S] = 0\ . \tag{8.25}$$

In the case ψ is horizontal,

$$[P,Q] = 0 \quad , \quad \text{and} \quad P_y = Q_x\ . \tag{8.26}$$

Thus, when ψ is horizontal, we only have to solve the following three equations:

$$-P_{xxx} - P_{xyy} - Q_{xxy} - Q_{yyy} + \tag{1}$$

$$+[[P_x + Q_y, P], P] + [[P_x + Q_y, Q], Q] = 0\ ,$$

$$[P,Q] = 0\ , \tag{2}$$

$$P_y = Q_x\ . \tag{3}$$

Now assume that $[P,Q] = 0$ and $P_y \equiv 0$ and $Q_x \equiv 0$, i.e., $P(x,y) = P(x)$, $Q(x,y) = Q(y)$. Then (1) becomes

$$\{-P_{xxx} + [[P_x,P],P]\} + \{-Q_{yyy} + [[Q_y,Q],Q]\} +$$

$$+[[Q_y,P],P] + [[P_x,Q],Q] = 0\ . \tag{8.27}$$

Here,

$$0 = \frac{\partial}{\partial x}[[P,Q],Q] = [[P_x,Q],Q]\ , \tag{8.28}$$

and

$$0 = \frac{\partial}{\partial y}[[Q,P],P] = [[Q_y,P],P]\ . \tag{8.29}$$

Thus (1) becomes

$$\{-P_{xxx} + [[P_x,P],P]\} + \{-Q_{yyy} + [[Q_y,Q],Q]\} = 0\ . \tag{8.30}$$

Then we have

Theorem 8.3. *Let $(G/K,h)$ be a Riemannian symmetric space of rank ≥ 2, $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$, the Cartan decomposition, $\mathfrak{m}\supset\mathfrak{a}$, a maximal abelian subalgebra of $\mathfrak{g}$, and $\{X,Y\}\subset\mathfrak{a}$. Let $P(x) := (a_1x^2 + b_1x + c_1)X$, $Q(y) := (a_2y^2 + b_2y + c_2)$, where a_i, b_i, and c_i $(i = 1,2)$ are constants. Then*

1. *$\psi(x,y) := x_0\,\exp(d_x^1\,X + d_y^2\,Y)$ satisfies $\psi^{-1}d\psi = \alpha = Pdx + Qdy$.*
2. *Let $\varphi(x,y) := x_0\,\exp(d_x^1\,X + d_y^2\,Y)\cdot o$. Then $\varphi : (\mathbb{R}^2, g_0) \to (G/K,h)$ is biharmonic (where $x_0\in G$, $d_t^i = (a_i/3)t^3 + (b_i/2)t^2 + c_it$).*
3. *φ is harmonic if and only if $a_i = b_i = 0$ $(i = 1,2)$.*

Finally, we give an example.

Example (*The case* (S^n, h)). Let

$$\varphi_i(x,y) = x \cdot \begin{pmatrix} \cos(d_x^1 + d_y^2) \\ 0 \\ \vdots \\ 0 \\ -\sin(d_x^1 + d_y^2) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (x \in SO(n+1)) . \tag{8.31}$$

Then φ_i are biharmonic maps, and φ_i are harmonic if and only if $a_i = b_i = 0$. For the projective spaces, one can get the similar results which are omitted. The above works are due to the recent one (cf. [50]).

9. Chen, Caddeo, Montaldo, Piu and Oniciuc's conjecture

Let us recall again the following famous conjecture which has been open until now ([7], [8]):

- (The B-Y. Chen's conjecture) *Any biharmonic submanifold of the Euclidean space is harmonic.*
- (The generalized B-Y. Chen's conjecture) *Every immersion into a Riemannian manifold with nonpositive curvature which is biharmonic is harmonic.*

Partial answers were given by several authors (see [8]), and a negative answer to the generalized B-Y. Chen's conjecture was given by Y. Ou and L. Tang (cf. [39]):

There exist biharmonic, but not minimal hypersurfaces (which are incomplete Riemannian manifolds) into the 5*-dimensional space with strictly negative sectional curvature.*

On the other hand, B-Y. Chen's conjecture and the generalized B-Y. Chen's conjecture are true under certain additional conditions. Indeed, our first answer to the conjecture is the following ([17]):

Theorem 9.1 ([17])**.** *Assume that* (M,g) *is complete, and* $\left|\mathrm{Riem}^M\right| \leq C$, *and* $\mathrm{Riem}^N \leq 0$. *Let* $\varphi : (M,g) \to (N,h)$ *be a biharmonic map whose tension field* $\tau(\varphi)$ *satisfies*

$$\|\tau(\varphi)\| \in L^2(M) \quad \textit{and} \quad \|\overline{\nabla}\tau(\varphi)\| \in L^2(M) .$$

Then $\varphi : (M,g) \to (N,h)$ *is harmonic.*

Recently, Akutagawa and Maeta obtained ([1]) a very striking result:

Theorem 9.2 (Akutagawa and Maeta, [1])**.** *B-Y. Chen's conjecture is true for a proper isometric immersion into the Euclidean space. Namely, let* $\varphi : (M,g) \to (\mathbb{R}^n, g_0)$ *be an isometric immersion which is proper, i.e.,* $\varphi^{-1}(K)$ *is compact for every compact subset* K *in* $\mathbb{R}^n$. *Then if* $\varphi : (M,g) \to (\mathbb{R}^n, g_0)$ *is biharmonic, then it is minimal.*

Our recent results are as follows ([33], [34], [35]).

Theorem 9.3 (N. Nakauchi and H. Urakawa, [33]). *Assume that* $\varphi : (M^m, g) \to (N, h)$ *is a biharmonic isometric immersion and* $\dim M = m = \dim N - 1$. *Let* $H := (1/m)\mathrm{Tr}_g A$, *the mean curvature, where* A *is the shape operator.*

If $\mathrm{Ric}^N \leq 0$, (M, g) *is complete, and* $\int_M H^2 \, v_g < \infty$, *then,* $H = 0$, *i.e.,* $\varphi : (M, g) \to (N, h)$ *is minimal.*

Furthermore, for biharmonic isometric immersions, we have

Theorem 9.4 (N. Nakauchi and H. Urakawa, [34]). *Let* $\varphi : (M^m, g) \to (N, h)$ *be a biharmonic isometric immersion, and* $\eta := (1/m)\mathrm{Tr}_g B$, *the mean curvature vector field along* φ, $\nabla^N_{\varphi_* X} \varphi_* Y = \varphi_*(\nabla_X Y) + B(X, Y)$, $X, Y \in \mathfrak{X}(M)$. *Assume that* (N, h) *is non-positive curvature,* (M, g) *is complete and* $\int_M |\eta|^2 \, v_g < \infty$. *Then* $\eta = 0$, *i.e.,* φ *is minimal.*

For a biharmonic maps, we have

Theorem 9.5 (N. Nakauchi and H. Urakawa, [35]). *Assume that* (M, g) *is complete and* (N, h) *has non-positive curvature. Then*

(1) *any biharmonic map with finite energy* $E(\varphi) < \infty$ *and finite bienergy* $E_2(\varphi) < \infty$ *must be harmonic.*
(2) *Assume that* $\mathrm{Vol}(M, g) = \infty$. *Then any biharmonic map with finite bienergy* $E_2(\varphi) < \infty$ *must be harmonic.*

Notice that Theorem 9.5 implies Theorem 9.4 Indeed, assume that $\varphi : (M, g) \to (N, h)$ is a biharmonic isometric immersion, (M, g) complete, $\int_M |\eta|^2 \, v_g < \infty$, and $R^N \leq 0$. Then since $\tau(\varphi) = (1/m)\, \eta$ $(m = \dim M)$,

$$E_2(\varphi) = \frac{1}{2m^2} \int_M |\eta|^2 \, v_g < \infty \quad \text{and} \quad E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 \, v_g = \frac{m}{2} \mathrm{Vol}(M, g) \, .$$

(1) If $\mathrm{Vol}(M, g) < \infty$, (1) of Theorem 9.5 implies that φ is minimal.
(2) If $\mathrm{Vol}(M, g) = \infty$, (2) of Theorem 9.5 implies also that φ is minimal.

□

We have their applications to horizontally conformal submersions.

Here, recall that a submersion $\varphi : (M^m, g) \to (N^n, h)$ $(m > n \geq 2)$ is a *horizontal conformal submersion* if there exist the orthogonal direct sum: $T_x M = \mathcal{V}_x \oplus \mathcal{H}_x$, $\mathcal{V}_x = \mathrm{Ker}(d\varphi_x)$ (*vertical space*), $\mathcal{H}_x$ horizontal space $(x \in M)$, and $\lambda \in C^\infty(M)$ (*dilation*) such that

$$h(d\varphi_x(X), d\varphi_x(Y)) = \lambda^2(x) \, g(X, Y) \quad , \quad X, Y \in \mathcal{H}_x \, .$$

Then the tension field $\tau(\varphi)$ is given by

$$\tau(\varphi) = \frac{n-2}{2} \, \lambda^2 \, d\varphi(\mathrm{grad}_{\mathcal{H}}(\frac{1}{\lambda^2})) - (m-n) d\varphi(\hat{\mathbf{H}}) \, , \tag{9.1}$$

where $\hat{\mathbf{H}} := (1/(m-n))\mathcal{H}(\sum_{k=n+1}^m \nabla_{e_k} e_k)$, and e_i is a local orthonormal frame field on M.

Then let us recall

Theorem 9.6 (cf. Wang and Ou, [53]). *For a horizontally conformal submersion from a space form* (M^m, g) *of constant curvature into* (N^2, h), *it is biharmonic if and only if it is harmonic.*

On the other hand, one of our theorems which is a corollary of Theorem 9.5, is that (cf. [35]):

Theorem 9.7. *Assume that (M^m, g) $(m > 2)$ is non-compact complete, (N^2, h) has non-positive curvature, $\lambda \in L^2(M)$ and $\lambda\,|\hat{\mathbf{H}}|_g \in L^2(M)$. For a horizontally conformal submersion $\varphi : (M^m, g) \to (N^2, h)$, it is biharmonic if and only if it is harmonic.*

Proof of Theorem 9.5. The proof of Theorem 9.5 is divided into four steps.

(First step). Take a cut-off function λ on M as

$$\begin{cases} 0 \le \lambda(x) \le 1 \ , \\ \lambda(x) = 1 \ \text{ on } \ B_r(x_0) \ , \\ \lambda(x) = 0 \ \text{ outside } \ B_{2r}(x_0), \ \text{and} \\ |\nabla \lambda| \le \dfrac{2}{r} \quad \text{on } \ M \ . \end{cases}$$

The bitension field of a map $\varphi : (M, g) \to (N, h)$ is given by

$$\tau_2(\varphi) = \overline{\Delta}(\tau(\varphi)) - \sum_{i=1}^m R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) \ .$$

For a biharmonic map $\varphi : (M, g) \to (N, h)$, we have

$$\int_M \langle \overline{\Delta}(\tau(\varphi)), \eta^2\,\tau(\varphi)\rangle\, v_g =$$

$$= \int_M \eta^2 \sum_{i=1}^m \langle R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i), \tau(\varphi)\rangle\, v_g \le 0$$

since (N, h) has non-positive curvature.

Notice here that $\overline{\Delta} = \overline{\nabla}^*\,\overline{\nabla}$, where $\overline{\nabla}$ is the induced connection on $\Gamma(\varphi^{-1}TN)$.

(Second step). Thus we have

$$0 \ge \int_M \langle \overline{\Delta}(\tau(\varphi)), \eta^2\,\tau(\varphi)\rangle\, v_g = \int_M \langle \overline{\nabla}\tau(\varphi), \overline{\nabla}(\eta^2\,\tau(\varphi))\rangle =$$

$$= \int_M \sum_{i=1}^m \langle \overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i}(\eta^2\,\tau(\varphi))\rangle\, v_g =$$

$$= \int_M \left\{\eta^2\,\langle \overline{\nabla}_{e_i}\tau(\varphi), \overline{\nabla}_{e_i}\tau(\varphi)\rangle + e_i(\eta^2)\langle \overline{\nabla}_{e_i}\tau(\varphi), \tau(\varphi)\rangle\right\} =$$

$$= \int_M \eta^2\,\left|\overline{\nabla}_{e_i}\tau(\varphi)\right|^2\, v_g + 2\int_M \langle \eta\,\overline{\nabla}_{e_i}\tau(\varphi), e_i(\eta)\,\tau(\varphi)\rangle\, v_g \ .$$

Thus, letting $V_i := \eta\,\overline{\nabla}_{e_i}\tau(\varphi)$, $W_i := e_i(\eta)\,\tau(\varphi)$,

$$\int_M \eta^2\,\left|\overline{\nabla}_{e_i}\tau(\varphi)\right|^2 v_g \le -2\int_M \langle \eta\,\overline{\nabla}_{e_i}\tau(\varphi), e_i(\eta)\,\tau(\varphi)\rangle\, v_g =$$

$$= -2\int_M \sum_{i=1}^m \langle V_i, W_i\rangle\, v_g \ . \tag{9.2}$$

Use Cauchy-Schwarz inequality in (9.2),

$$\pm 2\langle V_i, W_i\rangle \le \epsilon\,|V_i|^2 + \frac{1}{\epsilon}\,|W_i|^2 \qquad (\forall\ \epsilon > 0) \ .$$

Then we have

$$-2\int_M \sum_{i=1}^m \langle V_i, W_i\rangle \, v_g \le \epsilon \int_M \sum_{i=1}^m |V_i|^2 \, v_g + \frac{1}{\epsilon}\int_M \sum_{i=1}^m |W_i|^2 \, v_g \; .$$

Therefore, we have, by putting $\epsilon = 1/2$,

$$\int_M \eta^2 \sum_{i=1}^m \left|\overline{\nabla}_{e_i}\tau(\varphi)\right|^2 \, v_g \le \frac{1}{2}\int_M \sum_i \eta^2 \left|\overline{\nabla}_{e_i}\tau(\varphi)\right|^2 \, v_g +$$

$$+2\int_M \sum_i e_i(\eta)^2 \, |\tau(\varphi)|^2 \, v_g \; .$$

Thus we have

$$\int_M \eta^2 \sum_i \left|\overline{\nabla}_{e_i}\tau(\varphi)\right|^2 \, v_g \le 4\int_M |\nabla\eta|^2 \, |\tau(\varphi)|^2 \, v_g \le$$

$$\le \frac{16}{r^2}\int_M |\tau(\varphi)|^2 \, v_g \; . \tag{9.3}$$

(Third step). Since (M, g) is complete and non-compact, we can tend r to infinity. But

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g < \infty \; ,$$

so that the right hand side of $(*)$ goes to zero if $r \to \infty$. We obtain

$$\int_M \sum_{i=1}^m \left|\overline{\nabla}\tau(\varphi)\right|^2 \, v_g = 0 \; . \tag{9.4}$$

Thus we obtain, for every vector field X on M,

$$\overline{\nabla}_X \tau(\varphi) = 0 \; . \tag{9.5}$$

Therefore $|\tau(\varphi)|$ is constant, say c. Because, for all vector field X on M,

$$X\,|\tau(\nabla)|^2 = 2\,\langle \overline{\nabla}_X\tau(\nabla), \tau(\varphi)\rangle = 0 \; .$$

Thus, in the case that $\mathrm{Vol}(M, g) = \infty$, if we assume $c \neq 0$, we have

$$\tau_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g = \frac{c^2}{2}\,\mathrm{Vol}(M, g) = \infty \; ,$$

which is a contradiction. We have (2) of Theorem 9.5.

For (1), assume $E(\varphi) < \infty$ and $E_2(\varphi) < \infty$. Define

$$\alpha(X) := \langle d\varphi(X), \tau(\varphi)\rangle \qquad (X \in \mathfrak{X}(M)) \; ,$$

Then we have

$$\begin{aligned} -\delta\alpha &= e_i(\alpha(e_i)) - \alpha(\nabla_{e_i}e_i) = \\ &= e_i\,\langle d\varphi(e_i), \tau(\varphi)\rangle - \langle d\varphi(\nabla_{e_i}e_i), \tau(\varphi)\rangle = \\ &= \langle \overline{\nabla}_{e_i}(d\varphi(e_i)) - d\varphi(\nabla_{e_i}e_i), \tau(\varphi)\rangle + \langle d\varphi(e_i), \overline{\nabla}_{e_i}\tau(\varphi)\rangle = \\ &= \langle \tau(\varphi), \tau(\varphi)\rangle = |\tau(\varphi)|^2 \; . \end{aligned}$$

Thus we have

$$\int_M |\delta\alpha| \, v_g = \int_M |\tau(\varphi)|^2 \, v_g = 2\,E_2(\varphi) < \infty \; .$$

Furthermore we have

$$\int_M |\alpha| = \int_M \left(\sum \langle d\varphi(e_i), \tau(\varphi)\rangle^2\right)^{1/2} \leq$$

$$\leq \int_M \left(\sum |d\varphi(e_i)|^2 \, |\tau(\varphi)|^2\right)^{1/2} =$$

$$= \int_M |d\varphi| \, |\tau(\varphi)| \leq \sqrt{\int_M |d\varphi|^2} \sqrt{\int_M |\tau(\varphi)|^2} \, .$$

The finiteness assumptions $E(\varphi) = (1/2) \int_M |d\varphi| \, v_g < \infty$ and also $E_2(\varphi) = (1/2) \int_M |\tau(\varphi)|^2 \, v_g < \infty$ imply that

$$\int_M |\delta\alpha| \, v_g < \infty \quad \text{and} \quad \int_M |\alpha| \, v_g < \infty \, .$$

By Gaffney' s theorem, since (M, g) is complete,

$$0 = \int_M (-\delta\alpha) \, v_g = \int_M |\tau(\varphi)|^2 \, v_g \, .$$

Namely, we obtain $\tau(\varphi) = 0$, i.e., φ is harmonic.

□

10. Bubbling phenomena of harmonic maps and biharmonic maps

We treat with the totality of harmonic maps and/or biharmonic maps, that is,

"*What is bubbling phenomena on harmonic maps and biharmonic maps?*"

Let us recall the following results (cf. [37]): For any $C > 0$, let

$$\mathcal{F} := \left\{ \varphi : (M^m, g) \to (N^n, h) \;\; \text{smooth harmonic} \;\; | \;\; \int_M |d\varphi|^m \, v_g \leq C \right\} \, .$$

Then $\mathcal{F}$ is causes a *bubbling*, a kind of compactness.

For any $C > 0$, let

$$\mathcal{F} := \left\{ \varphi : (M^m, g) \to (N^n, h) \;\; \text{smooth biharmonic} \;\; | \right.$$

$$\left. \int_M |d\varphi|^m \, v_g \leq C \;\; \text{and} \;\; \int_M |\tau(\varphi)|^2 \, v_g \leq C \right\} \, .$$

Then $\mathcal{F}$ is causes a *bubbling*, a kind of compactness.

More precisely, for a bubbling for harmonic maps, we have

Theorem 10.1. *Let (M, g), (N, h) be compact Riemannian manifolds* $\dim M \geq 3$. *For any $C > 0$, let*

$$\mathcal{F} := \left\{ \varphi : (M^m, g) \to (N^n, h) \;\; \textit{smooth harmonic} \;\; | \;\; \int_M |d\varphi|^m \, v_g \leq C \right\} \, .$$

Then for all $\{\varphi_i\} \in \mathcal{F}$, there exist $\mathcal{S} = \{x_1, \cdots, x_\ell\} \subset M$, and a harmonic map $\varphi_\infty : (M \setminus \mathcal{S}, g) \to (N, h)$ such that

(1) $\varphi_{i_j} \to \varphi_\infty$ *in the C^∞-topology on $M \setminus \mathcal{S}$ $(j \to \infty)$, and*

(2) *the Radon measures* $|d\varphi_{i_j}|^m \, v_g$ *converges to a measure given by*

$$|d\varphi_\infty|^m \, v_g + \sum_{k=1}^{\ell} a_k \, \delta_{x_k} \qquad (j \to \infty) .$$

Our bubbling of biharmonic maps is the following.

Theorem 10.2 (Bubbling)**.** *Let* (M, g)*,* (N, h) *be compact Riemannian manifolds,* $\dim M \geq 3$*. For any* $C > 0$*, let*

$$\mathcal{F} := \Big\{\varphi : (M^m, g) \to (N^n, h) \ \ \text{smooth biharmonic} \ \ \Big| \int_M |d\varphi|^m \, v_g \leq C \ \ \text{and} \ \ \int_M |\tau(\varphi)|^2 \, v_g \leq C \Big\} .$$

Then for all $\{\varphi_i\} \in \mathcal{F}$*, there exist* $\mathcal{S} = \{x_1, \cdots, x_\ell\} \subset M$*, and a biharmonic map* $\varphi_\infty : (M \setminus \mathcal{S}, g) \to (N, h)$ *such that*

(1) $\varphi_{i_j} \to \varphi_\infty$ *in the* C^∞*-topology on* $M \setminus \mathcal{S}$ $(j \to \infty)$*,*
(2) *Radon measure* $|d\varphi_{i_j}|^m \, v_g$ *converges to a measure*

$$|d\varphi_\infty|^m \, v_g + \sum_{1 \leq k \leq \ell} a_k \, \delta_{x_k} \qquad (j \to \infty) .$$

Outline of Proof of Bubbling Theorem. We need the following two propositions:

Proposition 10.3. *Assume that the sect. curvature of* (N, h) *is bounded from above by a positive constant* $C > 0$*:* $R^N \leq C < \infty$*. Then there exist* $\epsilon_0 > 0$ *and* $C' > 0$ *depending on* C *&* (M, g)*, such that, for all biharmonic map* $\varphi \in C^\infty(M, N)$ *with* $\int_{B_r(x_0)} |d\varphi|^m \, v_g \leq \epsilon_0$*, it holds that*

$$\sup_{B_{r/2}(x_0)} |\tau(\varphi)|^2 \leq \frac{C'}{r^{m/2}} \int_{B_r(x_0)} |\tau(\varphi)|^2 \, v_g , \tag{10.1}$$

where $m := \dim M$*.*

Proposition 10.4 (The C^1 estimate for biharmonic maps)**.** *Assume that* (M, g) *is compact and* $R^N \leq C < \infty$*,* $m := \dim M$*. Then there exist* $\epsilon_1 > 0$*,* $\epsilon_2 > 0$ *and* $C^* > 0$ *such that for all biharmonic map* $\varphi : (M, g) \to (N, h)$*, with* $E_2(\varphi) = (1/2) \int_M |\tau(\varphi)|^2 \, v_g < \infty$*, and*

$$\int_{B_r(x_0)} |d\varphi|^m \, v_g < \epsilon_1 \quad \text{and} \tag{10.2}$$

$$\int_{B_r(x_0)} |\tau(\varphi)|^m \, v_g < \epsilon_2 , \tag{10.3}$$

then it holds that

$$\sup_{B_{r/2}(x_0)} |d\varphi| + \sup_{B_{r/2}(x_0)} |\tau(\varphi)| \leq \frac{C^*}{r} \left[\epsilon_1^{1/m} + \epsilon_2^{1/m} + 1\right] . \tag{10.4}$$

Remark 10.5. By Propositions 10.3 and 10.4, we have the C^1 estimate for a biharmonic map with finite bienergy. The proofs of Propositions 10.3 and 10.4 need to use the Moser's iteration technique.

By using Propositions 10.3 and 10.4, we give a proof of Theorem 10.2.

Take for all $\{\varphi_i\} \in \mathcal{F}$, and $\epsilon_0 > 0$ as in Proposition 10.3. Let us define a subset $\mathcal{S}$ of M, as

$$\mathcal{S} := \{x \in M \mid \liminf_{i\to\infty} \int_{B_r(x)} |d\varphi_i|^m \, v_g \geq \epsilon_0\} \,. \tag{10.5}$$

Then it holds that $\mathcal{S}$ is a finite set. Because For for all subsets $\{x_\ell; 1 \leq \ell \leq k\}$ in $\mathcal{S}$, choose $r_0 > 0$ in such a way that $B_{r_0}(x_s) \cap B_{r_0}(x_t) = \emptyset$ $(s \neq t)$. Then it holds that

$$k\epsilon_0 \leq \sum_{1\leq \ell \leq k} \int_{B_{r_0}(x_\ell)} |d\varphi_i|^m \, v_g = \int_{\bigcup_{\ell=1}^k B_{r_0}(x_\ell)} |d\varphi_i|^m \, v_g \leq$$

$$\leq \int_M |d\varphi_i|^m \, v_g \leq C < \infty \tag{10.6}$$

for large i. Thus (10.6) implies that $k \leq C/\epsilon_0$. Therefore, we obtain that $\#\mathcal{S} \leq C/\epsilon_0 < \infty$.

□

Proof of Bubbling Theorem continued. We may assume by taking a subsequence of $\{\varphi_i\}$,

$$\mathcal{S} = \{x \in M \mid \limsup_{i\to\infty} \int_{B_r(x)} |d\varphi_i|^m \, v_g \geq \epsilon_0\} \,.$$

Now, let take $x \in M \setminus \mathcal{S}$. Then

$$\limsup_{i\to\infty} \int_{B_r(x)} |d\varphi_i|^m \, v_g < \epsilon_0 \,.$$

We have, by Proposition 10.3,

$$\sup_{B_{r/2}(x)} |\tau(\varphi_i)|^2 \leq \frac{C}{r^{m/2}} \int_{B_r(x)} |\tau(\varphi_i)|^2 \, v_g \leq \frac{C^2}{r^{m/2}} \,. \tag{10.7}$$

Thus we obtain

(C^0): the C^0-estimate on $B_r(x)$ of $\tau(\varphi_i)$.

Then for a sufficiently small $r > 0$, we obtain

$$\int_{B_r(x)} |\tau(\varphi_i)|^m \, v_g < \epsilon_2 \,, \tag{10.8}$$

where $\epsilon_2 > 0$ is a positive constant in Proposition 10.4 after a long estimation.

We may take $\epsilon_0 < \epsilon_2$ which is a positive constant in Proposition 10.4. Then it holds that

$$\sup_{B_{r/2}(x)} |d\varphi_i| + \sup_{B_{r/2}(x)} |\tau(\varphi_i)| \leq \frac{C^*}{r} \left[\epsilon_1^{1/m} + \epsilon_2^{1/m} + 1\right] . \tag{10.9}$$

Thus we obtain

(C^1): the C^1-estimate on $B_r(x)$ of φ_i.

All the φ_i are biharmonic, i.e.,

$$\overline{\Delta}(\tau(\varphi_i)) - \mathcal{R}(\tau(\varphi_i)) = 0 \iff \overline{\Delta}\sigma_i = \mathcal{R}(\sigma_i), \tau(\varphi_i) = \sigma_i \,. \tag{10.10}$$

Here $\mathcal{R}(\sigma_i) := \sum_j {}^N R(\sigma_i, d\varphi_i(e_j)) d\varphi_i(e_j)$ and $|d\varphi_i(e_j)|$ $(1 \leq j \leq m)$ are bounded uniformly on i. So that, each σ_i are solutions of the linear equations with bounded coefficients uniformly on i.

By Ladyzhenskaya and Ural'tseva [21], p. 397, Theorem 3.1, we obtain

(C^α): the C^α-estimate on $B_r(x)$ of σ_i.

By the equation $\tau(\varphi_i) = \sigma_i$, φ_i are solutions of the equations

$$\tau(\varphi_i)^\gamma = \Delta({\varphi_i}^\gamma) + \sum g^{jk}\, {}^N\Gamma^\gamma_{\alpha\beta}(\varphi_i)\, \frac{\partial {\varphi_i}^\alpha}{\partial x_j}\, \frac{\partial {\varphi_i}^\beta}{\partial x_k} = {\sigma_i}^\gamma\,, \tag{10.11}$$

Namely,

$$\tau(\varphi_i) = \Delta\varphi_i + {}^N\Gamma(\varphi_i)(d\varphi_i, d\varphi_i) = \sigma_i\ . \tag{10.12}$$

Since $|d\varphi_i|$ are bounded and one of two $d\varphi_i$ can be regarded as coefficients, φ_i are solutions of the linear equations with bounded coefficients. By Ladyzhenskaya and Ural'tseva [21], p. 399, Theorem 4.1, we obtain

$(C^{1,\alpha})$: the $C^{1,\alpha}$-estimate on $B_r(x)$ of φ_i.

Thus we obtain that φ_i are solutions of the linear equations with the C^α-coefficients $d\varphi_i$.

Due to the Schauder estimate, we have

$$|\varphi_i|_{C^{2,\alpha}(B_r(x))} \leq C(|\varphi_i|_{C^0(B_r(x))} + |\sigma_i|_{C^\alpha(B_r(x))}) \leq C(C_1 + C_2)\ , \tag{10.13}$$

due to $|\varphi_i|_{C^0(B_r(x))} \leq C_1$ and $|\sigma_i|_{C^\alpha(B_r(x))} \leq C_2$. Namely, we have

$(C^{2,\alpha})$: the $C^{2,\alpha}$-estimate on $B_r(x)$ of φ_i.

Thus we have the C^∞-estimate on $B_r(x)$ of φ_i uniformly on i. Therefore, we obtain (1) of Theorem 10.2, i.e., there exist a subsequence $\{\varphi_{i_j}\}$ and a C^∞-map $\varphi_\infty : M \setminus \mathcal{S} \to N$ such that $\varphi_{i_j} \to \varphi_\infty$ on $B_r(x)$ in C^∞-topology $(j \to \infty)$. Therefore, $\varphi_\infty : (M \setminus \mathcal{S}, g) \to (N, h)$ is biharmonic.

For (2), there exists a Radon measure μ such that the Radon measure $|d\varphi_{i_j}|^m\, v_g$ $\to \mu$ (weakly) $(j \to \infty)$. μ satisfies, for any Borel set $A \subset M$,

$$\mu(A) = \sup\{\mu(K)\ \mid\ \forall\, K \subset A \text{ (compact subset)}\} =$$

$$= \inf\{\mu(O)\ \mid\ \forall\, O \supset A \text{ (open subset)}\}\ .$$

Since $\varphi_{i_j} \to \varphi_\infty$ on $M \setminus \mathcal{S}$ (C^∞-topology) $(j \to \infty)$, $\mu = |d\varphi_\infty|^m\, v_g$ on $M \setminus \mathcal{S}$. Since $\mathcal{S} = \{x_1, \cdots, x_k\}$ (a finite set), it holds that

$$\mu - |d\varphi_\infty|^m\, v_g = \sum_{\ell=1}^k a_\ell\, \delta_{x_\ell}$$

for some $a_\ell \geq 0$ (constants).

□

The above is based on a joint work with N. Nakauchi ([37]).

11. Biharmonic maps and symplectic geometry

In this section, our problem is:

What is a relation between biharmonic maps and symplectic geometry?

One of main issues in symplectic geometry is the geometry of Lagrangian submanifolds in a symplectic manifold. Then one can ask:

When are Lagrangian submanifolds biharmonic immersions into a symplectic manifold? Take as a symplectic manifold, a Kähler manifold: When is its Lagrangian submanifold biharmonic immersion?

We begin symplectic setting for biharmonic maps.

Let (N, J, h) be a complex m-dimensional Kähler manifold, and consider a symplectic form on N by $\omega(X, Y) := h(X, JY)$, $X, Y \in \mathfrak{X}(N)$.

A real submanifold M in N of dimension m is called to be *Lagrangian* if the immersion $\varphi : M \to N$ satisfies that $\varphi^*\omega \equiv 0$, i.e.,

$$h_x(T_xM, J(T_xM)) = 0 \qquad (\forall\, x \in M)\ .$$

Problem *When is $\varphi : (M, g) \to (N, J, h)$ biharmonic?*
Here, $g := \varphi^* h$.

As examples of Lagrangian submanifolds, there is a real form of a Hermitian manifold. Let (N, J, h) be a Hermitian manifold of $\dim_{\mathbb{C}} N = m$, σ, an anti-holomorphic involutive isometry of (N, J, h), and $M := \mathrm{Fix}(\sigma) = \{x \in N \mid \sigma(x) = x\}$, (called a *real form*). Let ω be a 2-form on N defined by $\omega(X, Y) := h(X, JY)$ $(X, Y \in \mathfrak{X}(N))$, and, an m-dimensional submanifold M is said to be *Lagrangian* in (N, J, h) if $\omega|_M \equiv 0$. ($d\omega \not\equiv 0$, in general.) Then any real form is Lagrangian, and for all $x \in M$,

$$T_xN = T_xM \oplus J(T_xM) \quad \text{and} \quad h(T_xM, J(T_xM)) = 0\ .$$

Proposition 11.1. *Let (N, J, h) be a Hermitian symmetric space of compact type. Then any real form M of (N, J, h) is totally geodesic.*

Then we have

Theorem 11.2. *Let $G^{\mathbb{C}}$ be a complex Lie group with a left invariant Riemannian metric h. Then any real form G of $G^{\mathbb{C}}$ is minimal in $(G^{\mathbb{C}}, h)$.*

Proof. The proof is clear since the second fundamental form of the inclusion $G \hookrightarrow G^{\mathbb{C}}$ satisfies that

$$B(X, Y) = -\frac{1}{2} \sum_{i=1}^{m} Je_i(h(X, Y))\, Je_i\ ,$$

for all X, $Y \in \mathfrak{X}(G)$, where $\{e_i\}$ is a locally defined orthonormal frame on G $(\dim G = m)$.

□

Now we consider biharmonic submanifolds. We first need the following theorem:

Theorem 11.3 (Maeta and Urakawa, [27])**.** *Let $\varphi : (M, g) \to (N, h)$, an isometric immersion. Then it is biharmonic if and only if*

$$\mathrm{Tr}_g(\nabla A_{\mathbf{H}}) + \mathrm{Tr}_g(A_{\nabla^{\perp}_{\bullet}\mathbf{H}}(\bullet)) - \Big(\sum R^N(\mathbf{H}, e_i)e_i\Big)^T = 0\ , \tag{11.1}$$

$$\Delta^{\perp}\mathbf{H} + \mathrm{Tr}_g B(A_{\mathbf{H}}(\bullet), \bullet) - \Big(\sum R^N(\mathbf{H}, e_i)e_i\Big)^{\perp} = 0 \; . \tag{11.2}$$

Here, B, A_ξ are the 2nd fundamental form, the shape operator for φ, i.e., recall that, for $X, Y \in \mathfrak{X}(M)$, $\xi \in \Gamma(TM^{\perp})$,

$$\nabla^N_X Y = \nabla_X Y + B(X, Y) \; ,$$

$$\langle B(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle \; ,$$

$$\nabla^N_X \xi = -A_\xi X + \nabla^{\perp}_X \xi \; .$$

Then we have

Theorem 11.4 (Maeta and Urakawa, [27]). *Let (N, J, h), a Kähler manifold, and (M, g), a Lagrangian submanifold. Then it is biharmonic if and only if*

$$\mathrm{Tr}_g(\nabla A_{\mathbf{H}}) + \mathrm{Tr}_g(A_{\nabla^{\perp}_{\bullet}\mathbf{H}}(\bullet)) -$$

$$- \sum \langle \mathrm{Tr}_g(\nabla^{\perp}_{e_i} B) - \mathrm{Tr}_g(\nabla^{\perp}_{\bullet} B)(e_i, \bullet), \mathbf{H}\rangle \, e_i = 0 \; , \tag{11.3}$$

$$\Delta^{\perp}\mathbf{H} + \mathrm{Tr}_g B(A_{\mathbf{H}}(\bullet), \bullet) +$$

$$+ \sum \mathrm{Ric}^N(J\mathbf{H}, e_i)\, Je_i - \sum \mathrm{Ric}(J\,\mathbf{H}, e_i)\, Je_i -$$

$$- J\, \mathrm{Tr}_g A_{B(J\mathbf{H}, \bullet)}(\bullet) + m\, JA_{\mathbf{H}}(J\mathbf{H}) = 0 \; . \tag{11.4}$$

where $m = \dim M$ and Ric, Ric^N *are the Ricci tensors of (M, g), (N, h), respectively.*

In particular, we have

Theorem 11.5 (Maeta and Urakawa, [27]). *Let $(N, J, h) = N^m(4c)$ be the complex space form of complex dimension m with constant holomorphic curvature $4c$ (< 0, $= 0$, > 0), and, (M, g), a Lagrangian submanifold. Then it is biharmonic if and only if*

$$\mathrm{Tr}_g(\nabla A_{\mathbf{H}}) + \mathrm{Tr}_g(A_{\nabla^{\perp}_{\bullet}\mathbf{H}}(\bullet)) = 0 \; , \tag{11.5}$$

$$\Delta^{\perp}\mathbf{H} + \mathrm{Tr}_g B(A_{\mathbf{H}}(\bullet), \bullet) - (m+3)c\mathbf{H} = 0 \; . \tag{11.6}$$

Now recall that B.Y. Chen introduced the following two notions on a Lagrangian submanifold M in a Kähler manifold N:

H-umbilic: M is called *H-umbilic* if M has a local orthonormal frame field $\{e_i\}$ satisfying

$$B(e_1, e_1) = \lambda\, Je_1 \quad , \quad B(e_1, e_i) = \mu\, Je_i \; ,$$

$$B(e_i, e_i) = \mu\, Je_1 \quad , \quad B(e_i, e_j) = 0 \quad (i \neq j) \; ,$$

where $2 \leq i, j \leq m = \dim M$, B is the second fundamental form of $M \hookrightarrow N$, and λ, μ are local functions on M.

PNMC: M has a *parallel normalized mean curvature* vector field if $\nabla^{\perp}(\mathbf{H}/|\mathbf{H}|) = 0$.

Then based with the Sasahara's works [42] $\sim$ [45], and Inoguchi [18], we obtain the following theorem ([27]):

Theorem 11.6. *Let $\varphi : M \to (N^m(4c), J, h)$ be a Lagrangian H-umbilic* PNMC *submanifold. Then it is biharmonic if and only if $c = 1$ and $\varphi(M)$ is congruent to a submanifold of $P^m(4)$ given by*

$$\pi\left(\sqrt{\frac{\mu^2}{1+\mu^2}}\, e^{-(i/\mu)x}\,,\ \sqrt{\frac{1}{1+\mu^2}}\, e^{i\mu x}\, y_1, \cdots,\sqrt{\frac{1}{1+\mu^2}}\, e^{i\mu x}\, y_m\right)$$

where $x, y_i \in \mathbb{R}$ with $\sum_{i=1}^m {y_i}^2 = 1$. Here, $\pi : S^{2m+1} \to P^m(4)$ is the Hopf fibering, and $\mu = \pm\sqrt{(m + 5 \pm \sqrt{m^2 + 6m + 25})/2m}$, $(\lambda = (\mu^2 - 1)/\mu)$.

More recently, we study the Kähler cone manifolds. Then we have ([51]):

Theorem 11.7. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a Legendrian submanifold of a Sasaki manifold ($n = 2m + 1$), and $\overline{\varphi} : (C(M), \overline{g}) \to (C(N), \overline{h})$, a Lagrangian cone submanifold of a Kähler cone manifold with $\overline{g} = dr^2 + r^2 g$, $\overline{h} = dr^2 + r^2 h$. Then it holds that*

(1) $\tau(\overline{\varphi}) = r^2\, \tau(\varphi)$, *and*
(2) $\tau_2(\overline{\varphi}) = r^4\, \tau_2(\varphi)$.

Recall that

Theorem 11.8 (T. Sasahara's recent works)**.** *Immersions φ into $S^{2m+1}(1)$,*

$$\varphi(x, y_1, \cdots, y_m) = \sqrt{2}^{\,-1}(e^{-i(x/\mu)},\, e^{i\mu x}\, y_1,\, \cdots,\, e^{i\mu x}\, y_m)\,,$$

where $x, y_i \in \mathbb{R}$ with $\sum_{i=1}^m {y_i}^2 = 1$, are proper biharmonic Legendrian immersions. Here, $\mu = \pm 1$.

Then we obtain

Corollary 11.9. *The corresponding embeddings $\overline{\varphi} : C(M) - \{0\} \to \mathbb{C}^{m+1}$ are proper biharmonic embeddings into the standard complex space $\mathbb{C}^{m+1}$ ($m \geq 2$).*

This work is due to the recent paper on Sasaki manifolds and Kähler cone manifolds, and biharmonic submanifolds ([51]).

12. The k-harmonic maps and the k-harmonic B-Y. Chen's conjecture

Now, in this section, we study more general k-harmonic maps. The contents of this section will be as follows:

⟨Table of contents of this section⟩

12.1. **Introduction to k-harmonic maps and the k-harmonic B-Y. Chen's conjecture.** J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, *Regional Conference Series in Math.*, **50** (1983), AMS.

They introduced the *k-energy*: for a C^∞ map of (M,g) into (N,h),

$$E_k(\varphi) = \frac{1}{2}\int_M |(d+\delta)^k\,\varphi|^2\,v_g\ .$$

Here, $(d+\delta)^k$ is k-times iteration, $d\varphi \in A^1(\varphi^{-1}TN)$, $d\colon A^p(\varphi^{-1}TN)\to A^{p+1}(\varphi^{-1}TN)$ is the exterior differentiation with respect to the induced connection $\overline{\nabla}$ from the Levi-Civita connection ∇^N of (N,h), and δ is the co-differentiation.

Definition 12.1. $\varphi : (M^m,g)\to(N,h)$ is *k-harmonic* if

$$\left.\frac{d}{dt}\right|_{t=0} E_k(\varphi_t) = 0$$

for all C^∞ variations $\varphi_t \in C^\infty(M,N)$ $(-\epsilon < t < \epsilon)$ with $\varphi_0 = \varphi$.

We would like to expect the *first variation formula* for all $k \geq 3$,

Theorem 12.2. *There exists the* k-tension field $\tau_k(\varphi)\in\Gamma(\varphi^{-1}TN)$ *such that*

$$\left.\frac{d}{dt}\right|_{t=0} E_k(\varphi_t) = -\int_M \langle V, \tau_k(\varphi)\rangle\,v_g\ ,$$

where

$$V_x := \left.\frac{d}{dt}\right|_{t=0}\varphi_t(x)\in T_{\varphi(x)}N \qquad (x\in M)\ .$$

Let us recall the following conjecture (cf. [8]):

The k-harmonic B-Y. Chen's conjecture: *Let $\varphi : (M,g)\hookrightarrow(\mathbb{R}^n,h_0)$ be an isometric immersion. Assume that φ is* k-harmonic *($k\geq 2$). Then φ is* minimal, *i.e.,* harmonic.

Here let us recall again $\varphi : (M,g)\to(N,h)$ is *harmonic* if $\tau(\varphi)=0$, where the *tension field* $\tau(\varphi)$ is defined by

$$\tau(\varphi) := \sum_{i=1}^m(\widetilde{\nabla}_{e_i}d\varphi)(e_i) = \sum_{i=1}^m\left\{\,\overline{\nabla}_{e_i}d\varphi(e_i) - d\varphi(\nabla_{e_i}e_i)\right\} =$$

$$= \sum_{i=1}^m\left\{\nabla^N_{d\varphi(e_i)}d\varphi(e_i) - d\varphi(\nabla_{e_i}e_i)\right\}\ .$$

Then in this section, we will show our following results and also some related topics:

(1) Due to a recent joint work of S. Maeta, N. Nakauchi and H. Urakawa (cf. [26]), we obtain *the first variation formula for the trienergy* E_3 $(k=3)$. We show that *the* $(k=3)$*-harmonic B-Y. Chen's conjecture is true* for an isometric immersion into $N^n(c)$ $(c<0)$ under some L^2-, L^4-conditions.
(2) By a recent joint work of N. Nakauchi and H. Urakawa (cf. [36]), we establish *the first variation formula* for the k-energy E_k for a C^∞ map of (M,g) into the Euclidean space $(\mathbb{R}^n,h_0)$. And then, we show that *the* $(k\geq 2)$*-harmonic B-Y. Chen's conjecture is true* under some L^2-conditions.

12.2. **The first variation formula of 3-harmonic maps.** For the first variation formula for $k = 3$, we have

Theorem 12.3. *Recall the definition of* trienergy *which is defined by*

$$E_3(\varphi) := \frac{1}{2}\int_M \|(d+\delta)^3\varphi\|^2\, v_g \qquad (\varphi \in C^\infty(M,N)\ .$$

Then the first variation formula for E_3 *holds:*

$$\left.\frac{d}{dt}\right|_{t=0} E_3(\varphi_t) = -\int_M \langle \tau_3(\varphi), V\rangle v_g\ ,$$

where

$$\tau_3(\varphi) := J(\overline{\Delta}\,\tau(\varphi)) - \sum_{j=1}^m R^N(\overline{\nabla}_{e_j}\tau(\varphi), \tau(\varphi))\, d\varphi(e_j)\ , \tag{12.1}$$

$$J(W) := \overline{\Delta} W - \sum_{i=1}^m R^N(W, d\varphi(e_i))d\varphi(e_i)\ , \tag{12.2}$$

for $W \in \Gamma(\varphi^{-1}TN)$.

Definition 12.4. $\varphi : (M,g) \to (N,h)$ is *triharmonic* if $\tau_3(\varphi) = 0$.

For every C^∞ variation $\varphi_t : (M,g) \to (N,h)$ $(-\epsilon < t < \epsilon)$ with $\varphi_0 = \varphi$, it holds that

$$E_3(\varphi) = \frac{1}{2}\int_M |(d+\delta)^3\varphi|\, v_g = \frac{1}{2}\int_M |\overline{\nabla}\,\tau(\varphi)|^2\, v_g\ .$$

Because $(d+\delta)^2\varphi = d(d\varphi)+\delta(d\varphi) = 0-\tau(\varphi)$, and $(d+\delta)^3\varphi = d(-\tau(\varphi)) = -\overline{\nabla}\,\tau(\varphi)$, since $d : \Gamma(\varphi^{-1}TN) \to A^1(\varphi^{-1}TN)$ is the exterior differentiation associated with $\overline{\nabla}$.

Then we have

$$\frac{d}{dt}E_3(\varphi_t) = \int_M \langle \overline{\nabla}_{\frac{\partial}{\partial t}}\, \overline{\nabla}_{e_i}\,\tau(\varphi_t), \overline{\nabla}_{e_i}\tau(\varphi_t)\rangle\, v_g\ . \tag{12.3}$$

Let $F : M \times (-\epsilon,\epsilon) \to N$ be a C^∞ map defined by $F(t,x) := \varphi_t(x)$, $(x \in M, -\epsilon < t < \epsilon)$, and $\bigtriangledown$, the Levi-Civita connection of $(-\epsilon,\epsilon)\times M$, $\overline{\bigtriangledown}$, the induced connection on $F^{-1}TN$, the corresponding rough Laplacian

$$\overline{\triangle} V = -\sum_{i=1}^m \left\{\overline{\bigtriangledown}_{e_i}\overline{\bigtriangledown}_{e_i}V - \overline{\bigtriangledown}_{\nabla_{e_i}e_i}V\right\}\ ,$$

$V \in \Gamma(F^{-1}TN)$, $\widetilde{\bigtriangledown}$, the induced connection on $T^*((-\epsilon,\epsilon)\times M)\otimes F^{-1}TN$, and $\widetilde{R}$, the corresponding curvature tensor. Then, by (12.3),

$$\frac{d}{dt}E_3(\varphi_t) == \int_M \langle \overline{\bigtriangledown}_{\frac{\partial}{\partial t}}\,\overline{\bigtriangledown}_{e_i}\,\tau(F), \overline{\bigtriangledown}_{e_i}\tau(F)\rangle\, v_g\ .$$

Lemma 12.5. *For* $X \in \mathfrak{X}(M)$,

$$\left.\overline{\bigtriangledown}_{\frac{\partial}{\partial t}}\overline{\bigtriangledown}_X\tau(F)\right|_{t=0} = -\overline{\nabla}_X\,\overline{\Delta} V + \overline{\nabla}_X\left(R^N(V, d\varphi(e_j))d\varphi(e_j)\right) + R^N(V, d\varphi(X))\tau(\varphi)\ .$$

Proof continued. By Lemma 12.5, we have

$$\frac{d}{dt}\bigg|_{t=0} E_3(\varphi_t) = \int_M \sum_{i=1}^m \langle -\overline{\nabla}_{e_i}(\overline{\Delta} V) + $$
$$+\overline{\nabla}_{e_i}\left(R^N(V, d\varphi(e_j))d\varphi(e_j)\right) + $$
$$+R^N(V, d\varphi(e_i))\tau(\varphi), \overline{\nabla}_{e_i}(\tau(\varphi))\rangle\, v_g \ . \tag{12.4}$$

Integrating by parts,

$$\frac{d}{dt}\bigg|_{t=0} E_3(\varphi_t) = \int_M \left\{ \langle V, -\overline{\Delta}(\overline{\Delta}\tau(\varphi))\rangle + \sum_{j=1}^m \langle R^N(V, d\varphi(e_j))d\varphi(e_j), \overline{\Delta}\tau(\varphi)\rangle + \right.$$
$$\left. + \sum_{j=1}^m \langle R^N(V, d\varphi(e_j))\tau(\varphi), \overline{\nabla}_{e_j}\tau(\varphi)\rangle \right\} v_g \ . \tag{12.5}$$

Next, by $\langle R^N(v_3, v_4)v_2, v_1\rangle = \langle R^N(v_1, v_2)v_4, v_3\rangle$,

$$\frac{d}{dt}\bigg|_{t=0} E_3(\varphi_t) = \int_M \langle V, -\overline{\Delta}(\overline{\Delta}\tau(\varphi)) + $$
$$+\sum_{j=1}^m R^N(\overline{\Delta}\tau(\varphi), d\varphi(e_j))d\varphi(e_j) + $$
$$+\sum_{j=1}^m R^N(\overline{\nabla}_{e_j}\tau(\varphi), \tau(\varphi))d\varphi(e_j)\rangle\, v_g \ . \tag{12.6}$$

Finally, we obtain

$$\frac{d}{dt}\bigg|_{t=0} E_3(\varphi_t) = -\int_M \langle V, \tau_3(\varphi)\rangle\, v_g \ .$$

□

The proof of Lemma 12.5 goes as follows: Recall that

$$\overline{\triangledown}_{\frac{\partial}{\partial t}} \overline{\triangledown}_X \tau(F) = \overline{\triangledown}_X(\overline{\triangledown}_{\frac{\partial}{\partial t}} \tau(F)) + R^N(dF(\frac{\partial}{\partial t}), dF(X))\tau(F) =$$
$$= \overline{\triangledown}_X \left\{ -\overline{\triangle} dF(\frac{\partial}{\partial t}) + R^N(dF(\frac{\partial}{\partial t}), dF(e_i))dF(e_i) \right\} +$$
$$+R^N(dF(\frac{\partial}{\partial t}), dF(X))\tau(F) \ . \tag{12.7}$$

Putting $t = 0$ in the equality (12.6), we have Lemma 12.5.

We have to give a proof of the following equality in (12.7):

$$\overline{\triangledown}_{\frac{\partial}{\partial t}} \tau(F) = -\overline{\triangle} dF(\frac{\partial}{\partial t}) + R^N(dF(\frac{\partial}{\partial t}), dF(e_i))dF(e_i) \ . \tag{12.8}$$

Indeed, we have

$$\overline{\triangledown}_{\frac{\partial}{\partial t}} \tau(F) = \overline{\triangledown}_{\frac{\partial}{\partial t}} \left((\widetilde{\triangledown}_{e_i} dF)(e_i) \right) =$$
$$= \left(\widetilde{\triangledown}_{\frac{\partial}{\partial t}} \widetilde{\triangledown}_{e_i} dF \right)(e_i) + \left(\widetilde{\triangledown}_{e_i} dF \right)(\nabla_{\frac{\partial}{\partial t}} e_i) =$$

$$= \left(\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{\frac{\partial}{\partial t}} dF + \widetilde{\nabla}_{[\frac{\partial}{\partial t}, e_i]} dF + \widetilde{R}(\frac{\partial}{\partial t}, e_i) dF \right) (e_i) \ . \tag{12.9}$$

By (12.9), $\overline{\nabla}_{\partial/\partial t}\tau(F)$ can be given as follows:

$$\overline{\nabla}_{\frac{\partial}{\partial t}}\tau(F) = \widetilde{\nabla}_{e_i}\left((\widetilde{\nabla}_{\frac{\partial}{\partial t}} dF)(e_i) \right) - (\widetilde{\nabla}_{\frac{\partial}{\partial t}} dF)(\nabla_{e_i} e_i) +$$
$$+ R^N(dF(\frac{\partial}{\partial t}), dF(e_i))\, dF(e_i) =$$
$$= \widetilde{\nabla}_{e_i}\left((\widetilde{\nabla}_{e_i} dF)(\frac{\partial}{\partial t}) \right) - (\widetilde{\nabla}_{\nabla_{e_i} e_i} dF)(\frac{\partial}{\partial t}) +$$
$$+ R^N(dF(\frac{\partial}{\partial t}), dF(e_i))\, dF(e_i) \ . \tag{12.10}$$

Then (12.10) gives

$$\overline{\nabla}_{\frac{\partial}{\partial t}}\tau(F) = \widetilde{\nabla}_{e_i}\left((\widetilde{\nabla}_{e_i} dF)(\frac{\partial}{\partial t}) \right) - (\widetilde{\nabla}_{\nabla_{e_i} e_i} dF)(\frac{\partial}{\partial t}) +$$
$$+ R^N(dF(\frac{\partial}{\partial t}), dF(e_i))\, dF(e_i) =$$
$$= (\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} dF)(\frac{\partial}{\partial t}) + (\widetilde{\nabla}_{e_i} dF)(\widetilde{\nabla}_{e_i}\frac{\partial}{\partial t}) - (\widetilde{\nabla}_{\nabla_{e_i} e_i} dF)(\frac{\partial}{\partial t}) +$$
$$+ R^N(dF(\frac{\partial}{\partial t}), dF(e_i))\, dF(e_i) =$$
$$= -\overline{\triangle}\, dF(\frac{\partial}{\partial t}) + R^N(dF(\frac{\partial}{\partial t}), dF(e_i)) dF(e_i) \ .$$

12.3. **The 3-harmonic maps into $N^n(c)$ $(c < 0)$.** Then we obtain

Theorem 12.6 (cf. S. Maeta, N. Nakauchi and H. Urakawa [26]). *Let φ be an isometric immersion of a complete Riemannian manifold (M, g) into the space form $N^n(c)$ $(c < 0)$. If φ is* 3-harmonic, *and*

$$\int_M |\overline{\Delta}\tau(\varphi)|^2\, v_g < \infty \quad \text{and} \quad \int_M |\tau(\varphi)|^4\, v_g < \infty \ ,$$

then $\varphi : (M, g) \to N^n(c)$ is minimal.

Lemma 12.7. *Let $\varphi : (M, g) \to N^n(c)$ $(c < 0)$ be an isometric immersion. Then*

$$\tau_3(\varphi) = \overline{\Delta}(\overline{\Delta}\,\tau(\varphi)) - R^N(\overline{\Delta}\tau(\varphi), d\varphi(e_j)) d\varphi(e_j) -$$
$$- c\, h(\tau(\varphi), \tau(\varphi))\, \tau(\varphi) \ .$$

To prove Lemma 12.7, we only have to show:

$$R^N(\overline{\nabla}_{e_j}\tau(\varphi), \tau(\varphi))\, d\varphi(e_j) = c\, h(\tau(\varphi), \tau(\varphi))\, \tau(\varphi) \ .$$

Since $R^N(X, Y)Z = c\{h(Y, Z)X - h(X, Z)Y\}$, the left hand side is equal to

$$c\, \left\{ h(\tau(\varphi), d\varphi(e_j))\, \overline{\nabla}_{e_j}\tau(\varphi) - h(\overline{\nabla}_{e_j}\tau(\varphi), d\varphi(e_j))\, \tau(\varphi) \right\} \ .$$

Then, since $h(\tau(\varphi), d\varphi(e_j)) = 0$ for all $j = 1, \cdots, m$, and

$$h(\overline{\nabla}_{e_j}\tau(\varphi), d\varphi(e_j)) =$$
$$= e_j(h(\tau(\varphi), d\varphi(e_j))) - h(\tau(\varphi), \overline{\overline{\nabla}}_{e_j}(d\varphi(e_j))) =$$
$$= -h(\tau(\varphi), \tau(\varphi)) + d\varphi(\nabla_{e_j} e_j) =$$

$$= h(\tau(\varphi), \tau(\varphi))\ .$$

□

12.4. **The k-harmonic B-Y. Chen's conjecture.** The *k-energy* is obtained as follows. For a C^∞ map $\varphi : (M, g) \to (\mathbb{R}^n, h_0)$,

$$E_k(\varphi) := \frac{1}{2} \int_M |(d+\delta)^k \varphi|^2 v_g =$$

$$= \begin{cases} \dfrac{1}{2} \displaystyle\int_M \big|\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1} \tau(\varphi)\big|^2 v_g = \dfrac{1}{2}\displaystyle\int_M \big|W_\varphi^\ell\big|^2 v_g \qquad , \qquad (k = 2\ell)\ , \\ \dfrac{1}{2} \displaystyle\int_M \big|\overline{\nabla}(\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1} \tau(\varphi))\big|^2 v_g = \dfrac{1}{2} \displaystyle\int_M |\overline{\nabla} W_\varphi^\ell|^2 v_g \quad , \quad (k = 2\ell+1)\ , \end{cases}$$

where we put $W_\varphi^\ell := \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1}\tau(\varphi)$ $(\ell \geq 1)$, $W_\varphi^0 := \varphi$.

Then we have

Theorem 12.8 (The first variational formula)**.** *Let φ be a C^∞ map of (M, g) into $(\mathbb{R}^n, h_0)$. Then*

$$\frac{d}{dt}\bigg|_{t=0} E_k(\varphi_t) = -\int_M \langle V, \tau_k(\varphi)\rangle\, v_g \quad , \quad (k = 1, 2, 3, \cdots)\ , \tag{12.11}$$

for every variation vector field

$$V(x) = \frac{d}{dt}\bigg|_{t=0} \varphi_t(x) \in T_{\varphi(x)}N \qquad (x \in M)\ .$$

The k-tension field $\tau_k(\varphi)$ is given by

$$\tau_k(\varphi) = J(W_\varphi^{k-1}) = \overline{\Delta}(W_\varphi^{k-1}) = \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{k-1}\tau(\varphi)\ , \tag{12.12}$$

where $W_\varphi^k := \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{k-1}\tau(\varphi)$ for $k \geq 1$. Furthermore $\tau_1(\varphi) = \tau(\varphi)$ $(k = 1)$. Therefore φ is k-harmonic *if $W_\varphi^k = 0$.*

Then we obtain

Theorem 12.9. *Let φ be a k-harmonic map of a complete manifold (M, g) into $(\mathbb{R}^n, h_0)$. Assume that*

(1) $E_j(\varphi) < \infty$ *for all* $j = 2, 4, \cdots, 2k-2$,

(2) $\mathrm{Vol}(M, g) = \infty$ *or* $E_j(\varphi) < \infty$ *for all* $j = 1, 3, \cdots, 2k-3$.

Then φ is harmonic.

Remark 12.10. The condition (1) in Theorem 12.9, is equivalent to

$$\int_M |W_\varphi^\ell|^2 v_g = \int_M \big|\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1}\tau(\varphi)\big|^2 v_g < \infty$$

for all $1 \leq \ell \leq k-1$.

The condition (2) in Theorem 12.9, is equivalent to

$$\mathrm{Vol}(M,g)=\infty \quad \text{or} \quad \int_M |\overline{\nabla} W_\varphi^\ell|^2\, v_g = \int_M \big|\overline{\nabla}\,\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1}\tau(\varphi)\big|^2\, v_g < \infty$$

for all $0 \leq \ell \leq k-2$.

We can restate Theorem 12.9 as

Theorem 12.11. *Let φ be a k-harmonic map of a complete manifold (M,g) into $(\mathbb{R}^n, h_0)$.*

(1) *In the case that $\mathrm{Vol}(M,g) < \infty$, assume that $E_j(\varphi) < \infty$ for all $j = 1, 2, \cdots, 2k-2$.*
(2) *In the case that $\mathrm{Vol}(M,g) = \infty$, assume that $E_j(\varphi) < \infty$ for all $j = 2, 4, \cdots, 2k-2$.*

Then φ is harmonic.

Let us prepare Theorem 12.11 with the previous results on bi-harmonic maps as follows:

Theorem 12.12 (cf. [35])**.** *Let φ be a 2-harmonic map of a complete manifold (M,g) into (N,h) with $R^N \leq 0$. Assume that (1) in case of $\mathrm{Vol}(M,g) < \infty$, $E_j(\varphi) < \infty$ $(j = 1, 2)$, or (2) in case of $\mathrm{Vol}(M,g) = \infty$, $E_2(\varphi) < \infty$. Then φ is harmonic.*

To prove Theorem 12.11, we need

Lemma 12.13 (Key Lemma (the iteration method))**.** *Let φ be a C^∞ map from a complete manifold (M,g) into any Riemannian manifold (N,h). Assume that there exists $k \geq 2$ such that $W_\varphi^k = 0$, i.e. $\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{k-1}\tau(\varphi) = 0$, and*

$$\begin{cases} (1) \quad \displaystyle\int_M |W_\varphi^{k-1}|^2\, v_g < \infty \quad , \qquad \textit{and} \\[2ex] (2) \quad (a) \quad \displaystyle\int_M |\overline{\nabla} W_\varphi^{k-2}|^2\, v_g < \infty \quad \textit{or} \quad (b) \quad \mathrm{Vol}(M,g) = \infty\,. \end{cases}$$

Then we have $W_\varphi^{k-1} = 0$, i.e. $\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{k-2}\tau(\varphi) = 0$.

Proof of Theorem 12.11. Let $\varphi : (M,g) \to (\mathbb{R}^n, h_0)$ be k-harmonic, i.e. $W_\varphi^k = 0$ $(k \geq 2)$, and assume that the conditions (1) and (2) in Theorem 12.11 hold. By applying Key Lemma 12.13, the iteration works, and we have

$$\begin{cases} W_\varphi^{k-1} = 0\,, \\ W_\varphi^{k-2} = 0\,, \\ \cdots\cdots \\ \tau(\varphi) = W_\varphi^1 = 0\,. \end{cases}$$

□

The proof of Key Lemma 12.13 goes as follows.

(First step). For a fixed $x_0 \in M$ and $0 < r < \infty$, take a cut-off function η on M:

$$\begin{cases} 0 \leq \eta(x) \leq 1 & (x \in M)\ , \\ \eta(x) = 1 & (x \in B_r(x_0) := \{x \in M : \, d(x, x_0) < r\})\ , \\ \eta(x) = 0 & (x \notin B_{2r}(x_0))\ , \\ |\nabla \eta| \leq \dfrac{2}{r} & (x \in M)\ . \end{cases}$$

(Second step). Assume that $W_\varphi^k = \overline{\Delta} W_\varphi^{k-1} = 0$.
Our aim is to show $W_\varphi^{k-1} = 0$.

$$0 = \int_M \langle \eta^2 \, W_\varphi^{k-1}, \overline{\Delta} W_\varphi^{k-1} \rangle \, v_g = \quad (\text{by } \ \overline{\Delta} W_\varphi^{k-1} = 0)$$

$$= \int_M \langle \overline{\nabla}_{e_i}(\eta^2 \, W_\varphi^{k-1}), \overline{\nabla}_{e_i} W_\varphi^{k-1} \rangle \, v_g =$$

$$= \int_M \eta^2 |\overline{\nabla}_{e_i} W_\varphi^{k-1}|^2 v_g + 2 \int_M \eta \, e_i(\eta) \langle W_\varphi^{k-1}, \overline{\nabla}_{e_i} W_\varphi^{k-1} \rangle v_g \ . \tag{12.13}$$

Therefore, by virtue of (12.13), we have

$$\int_M \eta^2 \sum_{i=1}^m \left| \overline{\nabla}_{e_i} W_\varphi^{k-1} \right|^2 \, v_g = -2 \int_M \sum_{i=1}^m \langle S_i, T_i \rangle \, v_g \ , \tag{12.14}$$

where $S_i := \eta \, \overline{\nabla}_{e_i} W_\varphi^{k-1}$, and $T_i := e_i(\eta) \, W_\varphi^{k-1}$.

Since $0 \leq |\sqrt{\epsilon}\, S_i \pm (1/\sqrt{\epsilon}) T_i|^2$, we have $\pm 2 \langle S_i, T_i \rangle \leq \epsilon \, |S_i|^2 + (1/\epsilon) \, |T_i|^2$ for all $\epsilon > 0$. Thus we have

$$-2 \int_M \sum_{i=1}^m \langle S_i, T_i \rangle \, v_g \leq \epsilon \int_M \sum_{i=1}^m |S_i|^2 \, v_g + \frac{1}{\epsilon} \int_M \sum_{i=1}^m |T_i|^2 \, v_g \ .$$

So by putting $\epsilon = 1/2$,

$$\begin{aligned} &\int_M \eta^2 \sum_{i=1}^m \left| \overline{\nabla}_{e_i} W_\varphi^{k-1} \right|^2 \, v_g \leq \\ &\leq \frac{1}{2} \int_M \eta^2 |\overline{\nabla}_{e_i} W_\varphi^{k-1}|^2 v_g + 2 \int_M e_i(\eta)^2 |W_\varphi^{k-1}|^2 v_g \ . \end{aligned} \tag{12.15}$$

(Third step). By (12.14) and (12.15), we obtain

$$\begin{aligned} \int_M \eta^2 \sum_{i=1}^m |\overline{\nabla}_{e_i} W_\varphi^{k-1}|^2 \, v_g &\leq 4 \int_M |\nabla \eta|^2 \, |W_\varphi^{k-1}|^2 \, v_g \leq \\ &\leq \frac{16}{r^2} \int_M |W_\varphi^{k-1}|^2 \, v_g \ . \end{aligned} \tag{12.16}$$

By the last inequality in (12.16), we have

$$\int_{B_r(x_0)} |\overline{\nabla} W_\varphi^{k-1}|^2 \, v_g \leq \frac{16}{r^2} \int_M |W_\varphi^{k-1}|^2 \, v_g \ . \tag{12.17}$$

Since (M, g) is complete, $B_r(x_0)$ tends to M as $r \to \infty$. By the assumption (1) $\int_M |W_\varphi^{k-1}|^2 \, v_g < \infty$, the right hand side tends to zero if $r \to \infty$. Thus we obtain

$$\int_M |\overline{\nabla} W_\varphi^{k-1}|^2 \, v_g = 0 \, , \tag{12.18}$$

Thus we obtain

$$\overline{\nabla} W_\varphi^{k-1} = 0 \qquad \text{(everywhere on } M) \, . \tag{12.19}$$

Since $e_i \, |W_\varphi^{k-1}|^2 = 2\langle \overline{\nabla}_{e_i} W_\varphi^{k-1}, W_\varphi^{k-1} \rangle = 0$ by (12.19),

$$|W_\varphi^{k-1}|^2 \quad \text{is a constant on } M, \text{ say } C_0 \, . \tag{12.20}$$

(Fourth step). (a) In the case of $\mathrm{Vol}(M, g) = \infty$, by the condition in (1): $\int_M |W_\varphi^{k-1}|^2 \, v_g < \infty$, we have

$$\infty > \int_M |W_\varphi^{k-1}|^2 \, v_g = C_0 \, \mathrm{Vol}(M, g) \, . \tag{12.21}$$

Thus, by $\mathrm{Vol}(M, g) = \infty$, $C_0 = 0$, i.e. $W_\varphi^{k-1} = 0$.

(b) In the case $\int_M |\overline{\nabla} W_\varphi^{k-2}|^2 \, v_g < \infty$, let us define $\alpha \in A^1(M)$ by $\alpha(X) := \langle W_\varphi^{k-1}, \overline{\nabla}_X W_\varphi^{k-2} \rangle$ $(X \in \mathfrak{X}(M))$. Then we will obtain that

$$\begin{cases} \text{(i)} \quad \mathrm{div}(\alpha) = -|W_\varphi^{k-1}|^2 \quad , \quad \text{so} \displaystyle\int_M |\mathrm{div}(\alpha)| \, v_g < \infty \, , \\[2ex] \text{(ii)} \quad \displaystyle\int_M |\alpha| \, v_g < \infty \end{cases} \tag{12.22}$$

which we will give a proof. Then we can apply Gaffney's theorem, and then we have $0 = \int_M \mathrm{div}(\alpha) v_g = -\int_M |W_\varphi^{k-1}|^2 v_g$, which implies that $W_\varphi^{k-1} = 0$.

□

For the proof of (12.22) (i):

$$\begin{aligned} \mathrm{div}(\alpha) &= \sum_{i=1}^m (\nabla_{e_i} \alpha)(e_i) = \sum_{i=1}^m \{ e_i(\alpha(e_i)) - \alpha(\nabla_{e_i} e_i) \} = \\ &= e_i \left(\langle W_\varphi^{k-1}, \overline{\nabla}_{e_i} W_\varphi^{k-2} \rangle \right) - \langle W_\varphi^{k-1}, \overline{\nabla}_{\nabla_{e_i} e_i} W_\varphi^{k-2} \rangle = \\ &= \langle \overline{\overline{\nabla}_{e_i} W_\varphi^{k-1}}, \overline{\nabla}_{e_i} W_\varphi^{k-2} \rangle + \langle W_\varphi^{k-1}, \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} W_\varphi^{k-2} \rangle - \\ &\qquad - \langle W_\varphi^{k-1}, \overline{\nabla}_{\nabla_{e_i} e_i} W_\varphi^{k-2} \rangle = \\ &= \langle W_\varphi^{k-1}, -\overline{\Delta} W_\varphi^{k-2} \rangle = -\langle W_\varphi^{k-1}, W_\varphi^{k-1} \rangle = -|W_\varphi^{k-1}|^2 \, . \end{aligned}$$

To see (ii) $\int_M |\alpha| \, v_g < \infty$, we have

$$\begin{aligned} \int_M |\alpha| \, v_g &= \int_M |\langle W_\varphi^{k-1}, \overline{\nabla} W_\varphi^{k-2} \rangle| \, v_g \leq \\ &\leq \left(\int_M |W_\varphi^{k-1}|^2 v_g \right)^{1/2} \left(\int_M |\overline{\nabla} W_\varphi^{k-2}|^2 v_g \right)^{1/2} < \infty \, , \end{aligned}$$

by our assumptions that $\int_M |W_\varphi^{k-1}|^2 v_g < \infty$ and also $\int_M |\overline{\nabla} W_\varphi^{k-2}|^2 v_g < \infty$. We obtain Key Lemma 12.13.

□

12.5. **The k-harmonic maps into $\mathbb{R}^n$.** We shall get the k-tension field for a C^∞ map into $(\mathbb{R}^n, h_0)$.

Recall, for a C^∞ map of (M, g) into (N, h),

$$E_k(\varphi) = \frac{1}{2}\int_M |(d+\delta)^k\varphi|^2\, v_g \qquad (k \geq 1)\ . \tag{12.23}$$

Calculate $(d+\delta)^k\varphi$.

$k = 1$, $(d+\delta)\varphi = d\varphi$.
$k = 2$, $(d+\delta)^2\varphi = d^2\varphi + \delta d\varphi = \delta d = -\tau(\varphi)$ since

$$d(d\varphi)(X,Y) = \overline{\nabla}_X(d\varphi(Y)) - \overline{\nabla}_Y(d\varphi(X)) - d\varphi([X,Y]) = 0\ , \tag{12.24}$$

due to Pages 5 and 6 in J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, 1983, AMS.

For $k = 3$, we have

$$(d+\delta)^3\varphi = d\delta d\varphi = -d\tau(\varphi)) = -\overline{\nabla}\tau(\varphi)\ .$$

For $k = 4$,

$$(d+\delta)^4\varphi = (d+\delta)(d\delta d\varphi) = dd\delta d\varphi + \delta d\delta d\varphi =$$

$$= -dd\tau(\varphi) - \delta d\tau(\varphi) = R^{\overline{\nabla}} \otimes \tau(\varphi) - \overline{\Delta}\tau(\varphi)\ . \tag{12.25}$$

For $k \geq 5$, We have no idea to calculate $(d+\delta)^k$, in general. But we can proceed more in the case $(N,h) = (\mathbb{R}^n, h_0)$. We need the following theorem whose proof will be given later.

Theorem 12.14. *For a C^∞ map $\varphi : (M,g) \to (\mathbb{R}^n, h_0)$, $R^{\overline{\nabla}} = 0$.*

Thus, for $k = 4$, we have

$$(d+\delta)^4\varphi = \delta d\delta d\varphi = -\overline{\Delta}\tau(\varphi)\ . \tag{12.26}$$

Furthermore, for every $k \geq 1$, we have

$$(d+\delta)^k\varphi = \begin{cases} \underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi & ,\quad k = 2\ell,\ \ell = 1,2,\cdots\ , \\ d\,\underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi & ,\quad k = 2\ell+1,\ \ell = 0,1,\cdots\ . \end{cases}$$

The proof is an induction on k.

For $k = 2\ell$ $\ (\ell = 1,2,\cdots)$,

$$(d+\delta)^{k+1}\varphi = (d+\delta)(d+\delta)^k\varphi = d\,\underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi\ . \tag{12.27}$$

Because $\delta\,\underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi = 0$, since $\underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi \in \Gamma(\varphi^{-1}TN)$.

For $k = 2\ell+1$ $(\ell = 1,2,\cdots)$,

$$(d+\delta)^{k+1}\varphi = (d+\delta)(d+\delta)^k\varphi = (d+\delta)d\,\underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi =$$

$$= d^2\,\underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi + \underbrace{(\delta d)\cdots(\delta d)}_{\ell+1}\varphi = \underbrace{(\delta d)\cdots(\delta d)}_{\ell+1}\varphi\ , \tag{12.28}$$

$d^2V = R^{\overline{\nabla}} \otimes V = 0 \otimes V = 0$, where $V = \underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi \in \Gamma(\varphi^{-1}TN)$.

□

Next notice that

$$(12.29)\qquad \begin{cases} \delta d\varphi = -\tau(\varphi) =: V \in \Gamma(\varphi^{-1}TN)\ , \\ \underbrace{(\delta d)\cdots(\delta d)}_{\ell-1} V = \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1} V \end{cases}$$

because

$$\delta\, dV = -\sum_{i=1}^{m}\left\{\overline{\nabla}_{e_i}\left((dV)(e_i)\right) - dV\left(\nabla_{e_i}e_i\right)\right\} =$$
$$= -\left\{\overline{\nabla}_{e_i}\left(\overline{\nabla}_{e_i}V\right) - \overline{\nabla}_{\nabla_{e_i}e_i}V\right\} = \overline{\Delta}V \in \Gamma(\varphi^{-1}TN\ .$$

Therefore we obtain

$$(12.30)\qquad \begin{cases} \underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi = \underbrace{(\delta d)\cdots(\delta d)}_{\ell-1}(-\tau(\varphi)) = -\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1}\tau(\varphi)\ , \\ d\underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi = -\overline{\nabla}(\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1}\tau(\varphi))\ . \end{cases}$$

Thus we have

Theorem 12.15. *For a C^∞ map of (M,g) into $(\mathbb{R}^n, h_0)$, it holds that*

$$(12.31)\qquad \begin{cases} E_{2\ell}(\varphi) = \dfrac{1}{2}\displaystyle\int_M \big|\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1}\,\tau(\varphi)\big|^2 v_g \qquad (\ell = 1, 2, \cdots)\ , \\ E_{2\ell+1}(\varphi) = \dfrac{1}{2}\displaystyle\int_M \big|\overline{\nabla}(\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{\ell-1}\,\tau(\varphi))\big|^2 v_g \quad (\ell = 1, 2, \cdots)\ . \end{cases}$$

We remark that, for $k = 1$ $(\ell = 0)$, $E_1(\varphi) = (1/2)\int_M |d\varphi|^2\, v_g$ since $d\underbrace{(\delta d)\cdots(\delta d)}_{\ell}\varphi = d\varphi$.

Then we obtain the first variation formula:

Theorem 12.16 (The first variation formula)**.** *Let $k = 2, 3, \cdots$, and φ be a C^∞ map of (M,g) into $(\mathbb{R}^n, h_0)$. Then it holds that*

$$(12.32)\qquad \left.\frac{d}{dt}\right|_{t=0} E_k(\varphi_t) = -\int_M \langle \tau_k(\varphi), V\rangle\, v_g\ ,$$

where the k-tension field $\tau_k(\varphi)$ is given as

$$(12.33)\qquad \tau_k(\varphi) = \overline{\Delta}(W_\varphi^{k-1}) = \overline{\Delta}\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{k-2}\tau(\varphi) = \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{k-1}\tau(\varphi)\ .$$

*Thus, for $k = 1, 2, \cdots$, φ is k-*harmonic *if and only if*

$$(12.34)\qquad \tau_k(\varphi) = \overline{\Delta}(W_\varphi^{k-1}) = \overline{\Delta}\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{k-2}\tau(\varphi) = \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{k-1}\tau(\varphi) = 0\ .$$

The proof of Theorem 12.16 goes as follows.

Let $\varphi : (M, g) \to (N, h) = \mathbb{R}^n, h_0)$, a C^∞ map, and φ_t $(\epsilon < t < \epsilon)$, a C^∞ variation of φ with $\varphi_0 = \varphi$, and consider a C^∞ map F,

$$F : (-\epsilon, \epsilon) \times M \ni (t, x) \mapsto F(t, x) := \varphi_t(x) \in N\ .$$

Recall first the following notation. Let us take a Riemannian metric $\overline{dt^2 + g}$ on $(-\epsilon, \epsilon) \times M$, and let $\triangledown$, its Levi-Civita connection, $\overline{\triangledown}$, the induced connection on $F^{-1}TN$, and $\widetilde{\triangledown}$, the induced connection on $T^*((-\epsilon, \epsilon) \times M) \otimes F^{-1}TN$, respectively. If $\{(\partial/\partial t),\, e_i\}$ a local orthonormal frame field, then we have

$$d\varphi_t(e_i) = dF(e_i)\ ,$$
$$\left(\widetilde{\nabla}_{e_i} d\varphi_t\right)(e_j) = \left(\widetilde{\triangledown}_{e_i} dF\right)(e_j)\ ,$$
$$\left(\widetilde{\nabla}_{e_k} \widetilde{\nabla}_{e_i} d\varphi_t\right)(e_j) = \left(\widetilde{\triangledown}_{e_k} \widetilde{\triangledown}_{e_i} d\varphi_t\right)(e_j)\ .$$

For $k = 2\ell$ $(\ell = 1, 2, \cdots)$,

$$\frac{d}{dt} E_{2\ell}(\varphi_t) = \frac{1}{2}\,\frac{d}{dt} \int_M \langle \underbrace{\overline{\Delta} \cdots \overline{\Delta}}_{\ell-1} \tau(\varphi_t), \underbrace{\overline{\Delta} \cdots \overline{\Delta}}_{\ell-1} \tau(\varphi_t) \rangle\, v_g =$$

$$= \frac{1}{2}\,\frac{d}{dt} \int_M \langle \underbrace{\overline{\Delta} \cdots \overline{\Delta}}_{\ell-1} \left(\left(\widetilde{\nabla}_{e_i} d\varphi_t\right)(e_i)\right), \underbrace{\overline{\Delta} \cdots \overline{\Delta}}_{\ell-1} \left(\left(\widetilde{\nabla}_{e_i} d\varphi_t\right)(e_i)\right) \rangle\, v_g =$$

$$= \frac{1}{2}\,\frac{d}{dt} \int_M \langle \underbrace{\overline{\triangle} \cdots \overline{\triangle}}_{\ell-1} \left(\left(\widetilde{\triangledown}_{e_i} dF\right)(e_i)\right), \underbrace{\overline{\triangledown} \cdots \overline{\triangledown}}_{\ell-1} \left(\left(\widetilde{\triangle}_{e_i} dF\right)(e_i)\right) \rangle\, v_g\ . \tag{12.35}$$

Here, by using that (N, h) is the standard Euclidean space, we have

$$\frac{d}{dt} E_{2\ell}(\varphi_t) = \int_M \langle \overline{\triangledown}_{\frac{\partial}{\partial t}} \left(\underbrace{\overline{\triangle} \cdots \overline{\triangle}}_{\ell-1} \left(\left(\widetilde{\triangledown}_{e_i} dF\right)(e_i)\right) \right), \underbrace{\overline{\triangle} \cdots \overline{\triangle}}_{\ell-1} \left(\left(\widetilde{\triangledown}_{e_i} dF\right)(e_i)\right) \rangle\, v_g =$$

$$= \int_M \langle \underbrace{\overline{\triangle} \cdots \overline{\triangle}}_{\ell-1} \left(\overline{\triangledown}_{\frac{\partial}{\partial t}} \left(\left(\widetilde{\triangledown}_{e_i} dF\right)(e_i)\right)\right), \underbrace{\overline{\triangle} \cdots \overline{\triangle}}_{\ell-1} \left(\left(\widetilde{\triangledown}_{e_i} dF\right)(e_i)\right) \rangle\, v_g\ . \tag{12.36}$$

Because by using $\overline{\triangle} = -\sum_{i=1}^m \{\overline{\triangledown}_{e_i} \overline{\triangledown}_{e_i} - \overline{\triangledown}_{\nabla_{e_i} e_i}\}$, to see the equality in (12.36), we only have to see $\overline{\triangledown}_{\partial/\partial t}(\overline{\triangledown}_X W) = \overline{\triangledown}_X(\overline{\triangledown}_{\partial/\partial t} W)$ which follows from $\overline{\triangledown}_{[\partial/\partial t, X]} = 0$ and $R^{\overline{\triangledown}} = 0$ (due to Theorem 12.17. In the equations of (12.36), we have

$$\overline{\triangledown}_{\frac{\partial}{\partial t}} \left(\left(\widetilde{\triangledown}_{e_i} dF\right)(e_i)\right) = \left(\widetilde{\triangledown}_{\frac{\partial}{\partial t}} \overline{\triangledown}_{e_i} dF\right)(e_i) = \left(\widetilde{\triangledown}_{e_i} \widetilde{\triangledown}_{\frac{\partial}{\partial t}} dF\right)(e_i) =$$

$$= \widetilde{\triangledown}_{e_i} \left\{ \left(\widetilde{\triangledown}_{\frac{\partial}{\partial t}} dF\right)(e_i) - \left(\widetilde{\triangledown}_{\frac{\partial}{\partial t}} dF\right)(\nabla_{e_i} e_i) \right\} =$$

$$= \left(\widetilde{\triangledown}_{e_i} \widetilde{\triangledown}_{e_i} dF\right)\left(\frac{\partial}{\partial t}\right) - \left(\widetilde{\triangledown}_{\nabla_{e_i} e_i} dF\right)\left(\frac{\partial}{\partial t}\right)\ . \tag{12.37}$$

Substituting (12.37) into (12.36), turns out the following.

$$\frac{d}{dt} E_{2\ell}(\varphi_t) = \int_M \langle \underbrace{\overline{\triangle} \cdots \overline{\triangle}}_{\ell-1} \left\{ \left(\widetilde{\triangledown}_{e_i} \widetilde{\triangledown}_{e_i} dF\right)\left(\frac{\partial}{\partial t}\right) - \left(\widetilde{\triangledown}_{\nabla_{e_i} e_i} dF\right)\left(\frac{\partial}{\partial t}\right) \right\},$$

$$\underbrace{\overline{\triangle}\cdots\overline{\triangle}}_{\ell-1}\Big(\Big(\widetilde{\nabla}_{e_j}dF\Big)(e_j)\Big)\,\rangle\, v_g =$$

$$= \int_M \langle \Big(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}dF\Big)\left(\frac{\partial}{\partial t}\right) - \Big(\widetilde{\nabla}_{\nabla_{e_i}e_i}dF\Big)\left(\frac{\partial}{\partial t}\right), \tag{12.38}$$

$$\underbrace{\overline{\triangle}\cdots\overline{\triangle}}_{2(\ell-1)}\Big(\Big(\widetilde{\nabla}_{e_j}dF\Big)(e_j)\Big)\,\rangle\, v_g\;,$$

which is equal to

$$\int_M \langle dF\left(\frac{\partial}{\partial t}\right), \Big\{\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{\nabla_{e_i}e_i}\Big\}\Big(\underbrace{\overline{\triangle}\cdots\overline{\triangle}}_{2(\ell-1)}\Big(\Big(\widetilde{\nabla}_{e_j}dF\Big)(e_j)\Big)\Big)\rangle\, v_g\;. \tag{12.39}$$

Putting $t=0$ in (12.35) $\sim$ (12.39), $(d/dt)|_{t=0}\,E_{2\ell}(\varphi_t)$ turns to be equal to

$$\int_M \langle V\,, -\overline{\Delta}\{\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2(\ell-1)}\,\tau(\varphi)\}\rangle\, v_g = -\int_M \langle V\,, \tau_{2\ell}(\varphi)\rangle\, v_g\;,$$

where

$$\tau_{2\ell}(\varphi) = \overline{\Delta}(W_\varphi^{2\ell-1}) = \overline{\Delta}\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2(\ell-1)}\,\tau(\varphi)\;.$$

$\square$

For $k = 2\ell+1$ ($\ell = 1, 2, \cdots$),

$$\frac{d}{dt}E_{2\ell+1}(\varphi_t) = \int_M \langle\overline{\nabla}_{\frac{\partial}{\partial t}}\big(\overline{\nabla}_{e_k}\big((\underbrace{\overline{\triangle}\cdots\overline{\triangle}}_{\ell-1}\Big(\widetilde{\nabla}_{e_i}dF\Big)(e_i)\big)\big)\big),$$

$$\overline{\nabla}_{e_k}\big((\underbrace{\overline{\triangle}\cdots\overline{\triangle}}_{\ell-1}\Big(\widetilde{\nabla}_{e_i}dF\Big)(e_i)\big)\big)\rangle\, v_g =$$

$$= \int_M \langle\overline{\nabla}_{e_k}\big((\underbrace{\overline{\triangle}\cdots\overline{\triangle}}_{\ell-1}\Big(\overline{\nabla}_{\frac{\partial}{\partial t}}\Big(\widetilde{\nabla}_{e_i}dF\Big)(e_i)\Big)\big)\big), \tag{12.40}$$

$$\overline{\nabla}_{e_k}\big((\underbrace{\overline{\triangle}\cdots\overline{\triangle}}_{\ell-1}\Big(\widetilde{\nabla}_{e_i}dF\Big)(e_i)\big)\big)\rangle\, v_g\;,$$

Then (12.40) is equal to the following:

$$\frac{d}{dt}E_{2\ell+1}(\varphi_t) = \int_M \langle\underbrace{\overline{\triangle}\cdots\overline{\triangle}}_{\ell-1}\Big(\overline{\nabla}_{\frac{\partial}{\partial t}}\Big(\widetilde{\nabla}_{e_i}dF\Big)(e_i)\Big),$$

$$\overline{\triangle}(\underbrace{\overline{\triangle}\cdots\overline{\triangle}}_{\ell-1}\Big(\widetilde{\nabla}_{e_i}dF\Big)(e_i))\rangle\, v_g =$$

$$= \int_M \langle\overline{\nabla}_{\frac{\partial}{\partial t}}\Big(\widetilde{\nabla}_{e_i}dF\Big)(e_i), \underbrace{\overline{\triangle}\cdots\overline{\triangle}}_{2\ell-1}\Big(\Big(\widetilde{\nabla}_{e_i}dF\Big)(e_i)\Big)\,\rangle\, v_g =$$

$$(12.41)\quad \begin{aligned} &= \int_M \langle \left(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} dF\right)\left(\frac{\partial}{\partial t}\right) - \left(\widetilde{\nabla}_{\nabla_{e_i}e_i} dF\right)\left(\frac{\partial}{\partial t}\right), \\ &\qquad \underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2\ell-1}\left(\left(\widetilde{\nabla}_{e_j} dF\right)(e_j)\right)\rangle\, v_g\ , \end{aligned}$$

which is equal to the following:

$$(12.42)\qquad \int_M \langle dF\left(\frac{\partial}{\partial t}\right), \left\{\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{\nabla_{e_i}e_i}\right\}\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2\ell-1}\left(\left(\widetilde{\nabla}_{e_j} dF\right)(e_j)\right)\rangle\, v_g\ .$$

Putting $t = 0$ in (12.40) $\sim$ (12.42), $d/dt|_{t=0}\, E_{2\ell+1}(\varphi_t)$ turns out to be equal to

$$\int_M \langle V, -\overline{\Delta}\{\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2\ell-1}\,\tau(\varphi)\}\rangle\, v_g = -\int_M \langle V, \tau_{2\ell+1}(\varphi)\rangle\, v_g\ ,$$

where

$$\tau_{2\ell+1}(\varphi) = \overline{\Delta}(W^{2\ell}_{\varphi}) = \overline{\Delta}\underbrace{\overline{\Delta}\cdots\overline{\Delta}}_{2\ell-1}\,\tau(\varphi)\ .$$

□

For the flatness of the induced connection from the Euclidean space, we have the following theorem.

Theorem 12.17. *Let $\varphi : (M, g) \to (N, h) = (\mathbb{R}^n, h_0)$, a C^∞ map, $\overline{\nabla}$, the induced connection on $\varphi^{-1}TN$ of the Levi-Civita one ∇^N of (N, h), and $R^{\overline{\nabla}}$, its curvature tensor. Then $R^{\overline{\nabla}}(X, Y)s = 0$ for all $X, Y \in \mathfrak{X}(M)$, $s \in \Gamma(\varphi^{-1}TN)$.*

Proof. Let $(y^1, \cdots, y^n)$ be the coordinate of $\mathbb{R}^n$, and $\varphi \in C^\infty(M, \mathbb{R}^n)$. Every $s \in \Gamma(\varphi^{-1}TN)$ can be written as

$$s(x) = \sum_{\alpha=1}^{n} s_\alpha(x)\left(\frac{\partial}{\partial y^\alpha}\right)_{\varphi(x)} \qquad (x \in M)\ ,$$

where $s_\alpha \in C^\infty(M)$, $(\alpha = 1, \cdots, n)$. Let $(x^1, \cdots, x^m)$ be a local coordinate on $U \subset M$, $\varphi(x) = (\varphi^1(x), \cdots, \varphi^n(x))$ $(x \in U)$, and $X = \sum_{i=1}^m X_i(\partial/\partial x^i), Y = \sum_{j=1}^m Y_j(\partial/\partial x^j) \in \mathfrak{X}(M)$. Then

$$\begin{aligned} \overline{\nabla}_X s &= \sum_i X_i \overline{\nabla}_{\frac{\partial}{\partial x^i}}\, s = \sum_{i,\,\beta} X_i\left\{\frac{\partial s_\beta}{\partial x^i}\,\frac{\partial}{\partial y^\beta} + s_\beta \overline{\nabla}_{\frac{\partial}{\partial x^i}}\,\frac{\partial}{\partial y^\beta}\right\} = \\ &= \sum_{\beta=1}^{n}\sum_{i=1}^{m} X_i(x)\left(\frac{\partial s_\beta}{\partial x^i}\right)(x)\left(\frac{\partial}{\partial y^\beta}\right)_{\varphi(x)} \qquad (x \in U)\ , \end{aligned}$$

since $\overline{\nabla}_{\partial/\partial x^i}\partial/\partial y^\beta = \nabla^N_{\varphi_*(\partial/\partial x^i)}\partial/\partial y^\beta = 0$.

By the bracket relation

$$[X, Y] = \sum_{i,j=1}^{m}\left\{X_i\,\frac{\partial Y_j}{\partial x^i} - Y_i\,\frac{\partial X_j}{\partial x^i}\right\}\frac{\partial}{\partial x^j} \qquad (\text{for } x \in U)\ ,$$

we have

$$\overline{\nabla}_{[X,Y]}\, s = \sum_{i,j,\beta}\left\{X_i\,\frac{\partial Y_j}{\partial x^i} - Y_i\,\frac{\partial X_j}{\partial x^i}\right\}(x)\,\frac{\partial s_\beta}{\partial x^j}(x)\left(\frac{\partial}{\partial y^\beta}\right)_{\varphi(x)}\ .$$

On the other hand, we have

$$\overline{\nabla}_X \left(\overline{\nabla}_Y s\right) = \overline{\nabla}_X \left(\sum_{\beta,j} Y_j \frac{\partial s_\beta}{\partial x^j} \frac{\partial}{\partial y^\beta} \right) =$$

$$= \sum_{i,j,\beta} \left\{ X_i \frac{\partial Y_j}{\partial x^i} \frac{\partial s_\beta}{\partial x^j} + X_i\, Y_j \frac{\partial^2 s_\beta}{\partial x^j \partial x^i} \right\} \frac{\partial}{\partial y^\beta} ,$$

and also

$$\overline{\nabla}_Y \left(\overline{\nabla}_X s\right) = \sum_{i,j,\beta} \left\{ Y_i \frac{\partial X_j}{\partial x^i} \frac{\partial s_\beta}{\partial x^j} + Y_i\, X_j \frac{\partial^2 s_\beta}{\partial x^j \partial x^i} \right\} \frac{\partial}{\partial y^\beta} .$$

Finally, by the above, and $\partial^2 s_\beta/\partial x^j \partial x^i = \partial^2 s_\beta/\partial x^i \partial x^j$, we have

$$\overline{\nabla}_X(\overline{\nabla}_Y s) - \overline{\nabla}_Y(\overline{\nabla}_X s) = \overline{\nabla}_{[X,Y]}\, s .$$

□

Some results of this section are due to the joint works with N. Nakauchi (cf. [36]), and also S. Maeta and N. Nakauchi (cf. [26]).

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