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Nonlinear time-harmonic Maxwell equations in $\mathbb{R}^3$: recent results and open questions

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Abstract. We investigate the existence of solutions $E : \mathbb{R}^3 \to \mathbb{R}^3$ of the time-harmonic semilinear Maxwell equation

$$\nabla \times (\nabla \times E) + V(x)E = \partial_E F(x, E) \qquad \text{in } \mathbb{R}^3$$

where $V : \mathbb{R}^3 \to \mathbb{R}$, $\nabla\times$ denotes the curl operator in $\mathbb{R}^3$ and $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a nonlinear function in E. We present recent results and open questions concerning the existence of ground states and bound states. In applications F is responsible for the nonlinear polarization and $V(x) = -\mu\omega^2\varepsilon(x)$ where $\mu > 0$ is the magnetic permeability, ω is the frequency of the time-harmonic electric field $\Re\{E(x)e^{i\omega t}\}$ and ε is the linear part of the permittivity in an inhomogeneous medium.

1. Introduction

We study the propagation of electromagnetic waves $(\mathcal{E}, \mathcal{B})$ in the absence of charges, currents and magnetization. The constitutive relations between the electric displacement field $\mathcal{D}$ and the electric field $\mathcal{E}$ as well as between the magnetic induction $\mathcal{H}$ and the magnetic field $\mathcal{B}$ are given by

$$\mathcal{D} = \varepsilon\mathcal{E} + \mathcal{P}_{NL} \quad \text{and} \quad \mathcal{H} = \frac{1}{\mu}\,\mathcal{B}\,, \tag{1.1}$$

where ε is the (linear) permittivity of an inhomogeneous material, and $\mathcal{P}_{NL}$ stands for the nonlinear polarization which depends nonlinearly on the electric field $\mathcal{E}$. In inhomogeneous media ε and $\mathcal{P}_{NL}$ depend on the position $x \in \mathbb{R}^3$ and we assume that the magnetic permeability is constant $\mu > 0$. As usual, the Maxwell equations

$$\begin{cases} \nabla \times \mathcal{H} = \partial_t\mathcal{D} & , \quad \mathrm{div}(\mathcal{D}) = 0\,, \\ \partial_t\mathcal{B} + \nabla \times \mathcal{E} = 0 & , \quad \mathrm{div}(\mathcal{B}) = 0\,, \end{cases} \tag{1.2}$$

together with the constitutive relations (1.1) lead to the equation (see Saleh and Teich [25])

$$\nabla \times \left(\frac{1}{\mu}\,\nabla \times \mathcal{E}\right) + \partial_t^2(\varepsilon\mathcal{E}) = -\partial_t^2\mathcal{P}_{NL}\,.$$

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In the time-harmonic case the fields $\mathcal{E}$ and $\mathcal{P}_{NL}$ are of the form $\mathcal{E}(x,t) = \Re\{E(x)e^{i\omega t}\}$, $\mathcal{P}_{NL}(x,t) = \Re\{P(x)e^{i\omega t}\}$, where $E(x), P(x) \in \mathbb{R}^3$ and we arrive at the time-harmonic Maxwell equation

$$\nabla \times (\nabla \times E) + V(x)E = f(x,E) \qquad \text{in } \mathbb{R}^3 , \tag{1.3}$$

where $V(x) = -\mu\omega^2\varepsilon(x) \le 0$ and $f(x,E) = \mu\omega^2 P(x,E)$; see [6, 20]. Here $E : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field and $V : \mathbb{R}^3 \to \mathbb{R}$. In a Kerr-like medium the strong electric field $\mathcal{E}$ of high intensity causes the refractive index to vary quadratically with the field and then the polarization has the form $\mathcal{P}_{NL} = \alpha(x)\langle|\mathcal{E}|^2\rangle\mathcal{E}$, where $\langle|\mathcal{E}|^2\rangle$ stands for the time average of the intensity of $\mathcal{E}$, hence $P(x,E) = (1/2)\alpha(x)|E|^2E$ (see Nie [21] and Stuart [26]). In applications, for low intensity $|\mathcal{E}|$ the Kerr effect is often considered to be linear, $\mathcal{P}_{NL}$ is negligible and therefore we may assume that $\mathcal{P}_{NL}$ decays rapidly as $|\mathcal{E}| \to 0$. In order to model these nonlinear phenomena we consider nonlinearities of the form

$$f(x,E) = \Gamma(x)\min\{|E|^{p-2}, |E|^{q-2}\}E \quad , \quad 2 < p \le q , \tag{1.4}$$

where $\Gamma \in L^\infty(\mathbb{R}^3)$ is positive, $\mathbb{Z}^N$-periodic and bounded away from 0. Case $p = 4$ corresponds to the Kerr effect for the strong field $\mathcal{E}$. In order to treat the problem in a variational way we assume, in general, that $f(x,E) = \partial_E F(x,E)$ is a Carathéodory map for some function $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ such that $F(x,0) = 0$ for a.e. $x \in \mathbb{R}^3$ and $|f(x,E)| \le c\min\{|E|^{p-1}, |E|^{q-1}\}$ for some constant $c > 0$, a.e. $x \in \mathbb{R}^3$ and all $E \in \mathbb{R}^3$; see also conditions (F1)-(F5) in Section 3.

The aim of this article is to present mathematical difficulties underlying problem (1.3) and recent results. There are only few works concerning semilinear equations involving the $\nabla\times\nabla\times(\cdot)$ and most of them concentrate on a cylindrically symmetric situation, where the standard variational methods can be applied together with the Palais principle of symmetric criticality [22]; see Section 2.1. In Section 3, however, we treat equation (1.3) in a general setting when the symmetry is not present and we recall the existence and the nonexistence results from [20]. Finally, the recently obtained results have given rise a number of open questions and some of them are posed in the last Section 4.

2. Variational formulation of the problem and difficulties

In what follows we always assume that $2 < p \le q$ and keep example (1.4) in mind. Note that problem (1.3) has a variational structure and (weak) solutions correspond to critical points of the energy functional

$$\mathcal{E}(E) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \times E|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|E|^2\,dx - \int_{\mathbb{R}^3} F(x,E)\,dx \tag{2.1}$$

defined on a space $\mathcal{D}(\mathrm{curl},p,q) \cap L^2_{|V|}(\mathbb{R}^3,\mathbb{R}^3)$, where $\mathcal{D}(\mathrm{curl},p,q)$ is the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3,\mathbb{R}^3)$ with respect to the norm

$$\|E\|_{\mathrm{curl},p,q} := (|\nabla \times E|_2^2 + |E|_{p,q}^2)^{1/2}$$

and $L^2_{|V|}(\mathbb{R}^3,\mathbb{R}^3)$ consists of square integrable vector fields with respect to the measure $|V|\,dx$. Here and in the sequel $|\cdot|_r$ denotes the L^r-norm and $|\cdot|_{p,q}$ is the norm in

$$L^{p,q} := L^p(\mathbb{R}^3,\mathbb{R}^3) + L^q(\mathbb{R}^3,\mathbb{R}^3)$$

given by

$$|E|_{p,q} = \sup\Big\{ \frac{\int_{\mathbb{R}^3} \langle E, F\rangle\, dx}{|F|_{p/(p-1)} + |F|_{q/(q-1)}} \ | \ F \in L^{p/(p-1)}(\mathbb{R}^3,\mathbb{R}^3) \cap L^{q/(q-1)}(\mathbb{R}^3,\mathbb{R}^3), F \neq 0 \Big\} .$$

Obviously if $p = q$, then $L^{p,q}$ coincides with the usual Lebesgue space $L^p(\mathbb{R}^3,\mathbb{R}^3)$; see [4] for more properties of $L^{p,q}$. Hence the norm of E in $\mathcal{D}(\mathrm{curl}, p, q) \cap L^2_{|V|}(\mathbb{R}^3,\mathbb{R}^3)$ is given by

$$(|\nabla \times E|_2^2 + |E|_{p,q}^2 + |V(x)E|_2^2)^{1/2} .$$

Observe that one difficulty from a mathematical point of view is that the curl-curl operator $\nabla \times \nabla \times (\cdot)$ has an infinite-dimensional kernel, namely all gradient vector fields, i.e. $\nabla \times (\nabla\varphi) = 0$ for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3,\mathbb{R}^3)$. Moreover, e.g. for the model nonlinearity (1.4), the functional $\mathcal{E}$ is unbounded from above and from below, even on subspaces of finite codimension and its critical points have infinite Morse index. Therefore the problem has the strongly indefinite nature. We intend to show that in some cases $\mathcal{E}$ has the linking geometry in the spirit of Benci and Rabinowitz [9, 24]. In order to investigate the underlying geometry, let us introduce the subspace of divergence-free vector fields

$$\begin{aligned}\mathcal{U} &= \{E \in \mathcal{D}(\mathrm{curl}, p, q) \cap L^2_{|V|}(\mathbb{R}^3,\mathbb{R}^3) \ : \ \mathrm{div}(E) = 0\} \\ &= \{E \in \mathcal{D}(\mathrm{curl}, p, q) \cap L^2_{|V|}(\mathbb{R}^3,\mathbb{R}^3) \ : \ \int_{\mathbb{R}^3} \langle E, \nabla\varphi\rangle\, dx = 0 \\ &\qquad \text{for any } \ \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3,\mathbb{R}^3)\} .\end{aligned}$$

Note that for $E \in \mathcal{U}$ we have $\nabla \times \nabla \times E = -\Delta E$,

$$\mathcal{E}(E) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla E|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|E|^2\, dx - \int_{\mathbb{R}^3} F(x,E)\, dx \tag{2.2}$$

and critical points of $\mathcal{E}\,|_{\mathcal{U}}$ can be found by means of standard variational methods. However, in general, we do not know whether they are critical points of the free functional $\mathcal{J}$ in $\mathcal{D}(\mathrm{curl}, p, q) \cap L^2_{|V|}(\mathbb{R}^3,\mathbb{R}^3)$.

It turns out that in the following situation $\mathcal{E}$ has the linking geometry [9, 5, 24]. Suppose that $2 < p \le 6 \le q$ and we assume that $V : \mathbb{R}^3 \to \mathbb{R}$ satisfies the following condition:

(V) $V \in L^{p/(p-2)}(\mathbb{R}^3) \cap L^{q/(q-2)}(\mathbb{R}^3)$, $V(x) \le 0$ for a.e. $x \in \mathbb{R}^3$ and $|V|_{3/2} < S$, where

$$S := \inf_{u \in \mathcal{D}^{1,2}\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2\, dx}{|u|_6^2}$$

is the classical best Sobolev constant.

Then in view of [20][Lemma 3.2] we have the following Helmholtz decomposition

$$\mathcal{D}(\mathrm{curl}, p, q) \cap L^2_{|V|}(\mathbb{R}^3,\mathbb{R}^3) = \mathcal{D}(\mathrm{curl}, p, q) = \mathcal{U} \oplus \nabla\mathcal{W} ,$$

where $\mathcal{W}$ is the completion of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|w\|_{\mathcal{W}} := |\nabla w|_{p,q}$ and

$$\nabla\mathcal{W} := \{\nabla w \in L^{p,q} \ : \ w \in \mathcal{W}\} .$$

Moreover the $\|\cdot\|$ in $\mathcal{U}$ is equivalent with $|\nabla(\cdot)|_2$ and $\mathcal{D}(\mathrm{curl}, p, q)$ embeds continuously into $L^{p,q}$.

Proposition 2.1. *Suppose that there is $u_0 \in \mathcal{U} \setminus \{0\}$ such that*

$$\int_{\mathbb{R}^3} \frac{F(x, t_n u_0 + \nabla w_n)}{t_n^2}\, dx \to \infty \tag{2.3}$$

for any $t_n \in \mathbb{R}$ and $\nabla w_n \in \nabla\mathcal{W}$ such that $\|t_n u_0 + \nabla w_n\| \to \infty$. Then there are $R > r > 0$ such that

$$\sup_{E \in \partial Q} \mathcal{E}(E) \le 0 = \mathcal{E}(0) < \inf_{E \in S} \mathcal{E}(E)\ , \tag{2.4}$$

where

$$Q := \{E = tu_0 + \nabla w \in \mathcal{D}(\mathrm{curl}, p, q) |\ w \in \mathcal{W},\ \|E\| \le R, t \ge 0\}$$
$$S := \{E \in \mathcal{U},\ \|E\| = r\}\ .$$

Proof. Observe that for $E \in \mathcal{U}$

$$\mathcal{E}(E) \ge \frac{1}{2}\,|\nabla E|_2^2 - \frac{1}{2}\,|V|_{3/2}|E|_6^2 - \int_{\mathbb{R}^3} F(x, E)\, dx \ge$$

$$\ge \frac{1}{2}(1 - |V|_{3/2} S^{-1})|\nabla E|_2^2 - \int_{\mathbb{R}^3} F(x, E)\, dx\ .$$

Since $F(x, 0) = 0$ and $|f(x, E)| \le c \min\{|E|^{p-1}, |E|^{q-1}\}$ for a.e. $x \in \mathbb{R}^3$ and all $E \in \mathbb{R}^3$, then $|F(x, E)| \le C \min\{|E|^p, |E|^q\} \le C|E|^6$ for some constant $C > 0$. Then

$$E(E) \ge \frac{1}{2}(1 - |V|_{3/2} S^{-1})|\nabla E|_2^2 - CS^{-3}|\nabla E|_2^6 \ge \inf_{E \in S} \mathcal{E}(E) > 0$$

for some $r > 0$. On the other hand $\mathcal{E}(E) \le 0$ for $E \in \nabla\mathcal{W}$ and in view of (2.3) we get $\sup_{E \in \partial Q} \mathcal{E}(E) \le 0$ for sufficiently large R.

□

Note that (1.4) satisfies super-quadratic condition (2.3) (see [20]) and (2.4) means that Q links with S, so that $\mathcal{E}$ has the linking geometry in this case [9, 5, 24]. Although $\mathcal{E}$ has the recognized geometry, we encounter the lack of sufficient regularity of functional $\mathcal{E}$. Namely $\mathcal{E}'$ is not (sequentially) weak-to-weak* continuous, i.e. the weak convergence $E_n \rightharpoonup E$ in $\mathcal{D}(\mathrm{curl}, p, q)$ does not imply that $\mathcal{E}'(E_n) \rightharpoonup \mathcal{E}'(E)$ in $\mathcal{D}(\mathrm{curl}, p, q)^*$. Indeed, take for instance model nonlinearity (1.4) with $\Gamma \equiv 1$ and observe that $E_n := \nabla w_n \rightharpoonup E := \nabla w$ in $L^{p,q}$ does not imply

$$\min\{|E_n|^{p-2}, |E_n|^{q-2}\}E_n \rightharpoonup \min\{|E|^{p-2}, |E|^{q-2}\}E$$

in $(L^{p,q})^* = L^{p/(p-1)}(\mathbb{R}^3, \mathbb{R}^3) \cap L^{q/(q-1)}(\mathbb{R}^3, \mathbb{R}^3)$. Hence we do not know if we are able to apply linking results to get any Palais-Smale sequence E_n, i.e. $E_n \in \mathcal{D}(\mathrm{curl}, p, q)$, $\lim_{n\to\infty} \mathcal{E}(E_n) > 0$ and $\lim_{n\to\infty} \mathcal{E}'(E_n) \to 0$.

Observe that, even if we find somehow a bounded Palais-Smale sequence $E_n \rightharpoonup E$ we do not know if E is a critical point of $\mathcal{E}$. This is caused, again, by the lack of weak-to-weak* continuity of $\mathcal{E}'$. Moreover, $\mathcal{D}(\mathrm{curl}, p, q)$ is not locally compactly embedded in $L^{p,q}$ and analysis of compactness in the spirit of Lions [19] is not possible to preform.

The underlying geometry and the lack of sufficient regularity makes this problem difficult to treat with the available variational methods.

2.1. **Recent results and the role of cylindrical symmetry.** Recall that semilinear equations involving the the curl-curl operator $\nabla \times \nabla \times (\cdot)$ in $\mathbb{R}^3$ have been recently studied by Benci and Fortunato in [10]. They introduce a model for a unified field theory for classical electrodynamics which is based on a semilinear perturbation of the Maxwell equations. In the magnetostatic case, in which the electric field vanishes and the magnetic field is independent of time, they are lead to an equation of the form

$$\nabla \times (\nabla \times A) = W'(|A|^2)A \qquad \text{in } \mathbb{R}^3 \tag{2.5}$$

for the gauge potential A related to the magnetic field $H = \nabla \times A$. Here $F(A) = (1/2)W(|A|^2)$ is nonlinear in A. In [3] Azzollini et al. use the cylindrical symmetry of the equation to find solutions of (2.5) of the form

$$A(x) = \alpha(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \quad , \quad r = \sqrt{x_1^2 + x_2^2} \,.$$

A field of this form is divergence-free and

$$\nabla \times \nabla \times A = -\Delta A \,,$$

hence standard methods of nonlinear analysis apply. In [17] D'Aprile and Siciliano find another kind of cylindrical solutions of the equation again using symmetry arguments and the scaling properties of (2.5). Symmetry also plays an important role in the paper [6] by Bartsch et al. which is concerned with (1.3), where $F(x, E) = \Gamma(x)|E|^p$, $2 < p < 6$ with $V \in L^\infty(\mathbb{R}^3)$ and $\Gamma \in L^\infty_{loc}(\mathbb{R}^3)$ being cylindrically symmetric, say functions of $r = \sqrt{x_1^2 + x_2^2}$ and x_3, and periodic in x_3-direction. According to our notation, roughly speaking the approach in [10, 6] is based on the following observation: If the action of a topological group G on $\mathcal{D}(\mathrm{curl}, p, q)$ is isometric, $\mathcal{E}$ is invariant with respect to G, the space of invariant points

$$\mathrm{Fix}(G) := \{E \in D(\mathrm{curl}, p, q) \,:\, gE = E \ \text{ for any } \ g \in G\} \subset \mathcal{U} \,, \tag{2.6}$$

then we may find a critical point of $\mathcal{E}\ |_{\mathrm{Fix}(G)}$ by means of standard variational methods since $\mathcal{E}\ |_{\mathrm{Fix}(G)}$ has the form (2.2). In view of the Palais principle of symmetric criticality [22], the critical points of $\mathcal{E}\ |_{\mathrm{Fix}(G)}$ are critical points of $\mathcal{E}$, hence they solve (1.3). Note that the cylindrical symmetry enables us to find G such that (2.6) holds; see [3].

We also mention the papers [26, 27, 29, 31, 32, 30, 28] by Stuart and Zhou, who studied transverse electric and transverse magnetic solutions to (1.2) for asymptotically linear polarizations and if again the cylindrical symmetry is present. The search for these solutions reduces to a one-dimensional variational problem or an ODE, which simplifies the problem considerably.

Observe that in general (1.3) cannot be treated neither by the Palais principle of symmetric criticality [22] nor by the rescaling arguments due to the presence of nonsymmetric $V \in L^{3/2}(\mathbb{R}^3)$. We would like to emphasize that we want to deal with functions $F(x, E)$ that depend on x and are not radial in E. Moreover we look for a gound state solution, which is a nontrivial solution with the least possible energy $\mathcal{E}$ and, in general, the Palais principle of symmetric criticality does not provide this information.

3. Ground states and lack of symmetry

In this section we present results from [20]. In order to find solutions to (1.3) we use a generalization of the Nehari manifold technique for strongly indefinite functionals obtained recently by Bartsch and the author in [7] (see also Szulkin and Weth [35, 36]). Namely we introduce a Nehari-Pankov manifold (cf. [23]) which is homeomorphic with the unit sphere in $\mathcal{U}$. This allows to find a minimizing sequence on the sphere and hence on the Nehari-Pankov manifold. However in [7] we are in a position to find a limit point of the sequence being a critical point because the space of divergence-free vector fields on a bounded domain is compactly embedded into certain L^p spaces and a variant of the Palais-Smale condition is satisfied. Since (1.3) is modelled in $\mathbb{R}^3$, the minimizing sequences are no longer compact. Therefore the critical point theory developed in [7][Section 4] is insufficient to find a solution to (1.3). In a recent paper [8] Bartsch and the author extended the theory in order to treat a wider class of nonlinearties F, but still the Palais-Smale condition played a crucial role. Recall that the lack of the weak-to-weak* continuity of $\mathcal{E}'$ makes this problem impossible to treat by a concentration-compactness argument in the spirit of Lions [19] in $\mathcal{D}(\mathrm{curl}, p, q)$. Our approach is based on a careful analysis of a bounded sequence (E_n) of the Nehari-Pankov manifold (Theorem 3.2) with a possibly infinite splitting (3.8) of the limit

$$\lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\times E_n|^2\,dx-\mathcal{E}(E_n)\right).$$

This result enables us to obtain the the weak-to-weak* continuity of $\mathcal{E}'$ on the Nehari-Pankov manifold. Moreover, in the spirit of the global compactness result of Struwe [34, 33] or Coti Zelati and Rabinowitz [16], we are able to find a finite splitting of the ground state level $\lim_{n\to\infty}\mathcal{E}(E_n)$ with respect to a minimizing sequence (E_n) of the Nehari-Pankov manifold (Theorem 3.3). Finally comparisons of energy levels will imply the existence of solutions to (1.3); see Theorem 3.1.

Now we collect assumptions on the nonlinearity $F(x,u)$.

(F1) $F:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ is differentiable with respect to the second variable $u\in\mathbb{R}^3$, and $f=\partial_u F:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ is a Carathéodory function (i.e. measurable in $x\in\mathbb{R}^3$, continuous in $u\in\mathbb{R}^3$ for a.e. $x\in\mathbb{R}^3$). Moreover f is $\mathbb{Z}^3$-periodic in x i.e. $f(x,u)=f(x+y,u)$ for $x,u\in\mathbb{R}^3$ and $y\in\mathbb{Z}^3$.

(F2) If $V<0$ a.e. on $\mathbb{R}^3$ then F is convex in $u\in\mathbb{R}^3$, otherwise F is uniformly strictly convex with respect to $u\in\mathbb{R}^3$, i.e. for any compact $A\subset(\mathbb{R}^3\times\mathbb{R}^3)\setminus\{(u,u):\ u\in\mathbb{R}^3\}$

$$\inf_{\substack{x\in\mathbb{R}^3\\(u_1,u_2)\in A}}\left(\frac{1}{2}\left(F(x,u_1)+F(x,u_2)\right)-F\left(x,\frac{u_1+u_2}{2}\right)\right)>0\,.$$

(F3) There are constants $c_1,c_2>0$ such that

$$F(x,u)\geq c_1\min(|u|^p,|u|^q)$$

and

$$|f(x,u)|\leq c_2\min(|u|^{p-1},|u|^{q-1})$$

for all $x,u\in\mathbb{R}^3$.

(F4) For any $x\in\mathbb{R}^3$ and $u\in\mathbb{R}^3$, $u\neq 0$

$$\langle f(x,u),u\rangle>2F(x,u)\,.$$

(F5) If $\langle f(x,u),v\rangle = \langle f(x,v),u\rangle \neq 0$ then

$$F(x,u) - F(x,v) \leq \frac{\langle f(x,u),u\rangle^2 - \langle f(x,u),v\rangle^2}{2\langle f(x,u),u\rangle}.$$

If in addition $F(x,u) \neq F(x,v)$ then the strict inequality holds.

It is easy to check that (F3) implies the super-quadratic behaviour (2.3). Let us consider the following examples

$$F(x,u) = \begin{cases} \Gamma(x)\left(\frac{1}{p}\,|Mu|^p + \frac{1}{q} - \frac{1}{p}\right) & \text{if } |Mu| > 1\,, \\ \Gamma(x)\,\frac{1}{q}\,|Mu|^q & \text{if } |Mu| \leq 1\,, \end{cases} \tag{3.1}$$

$$F(x,u) = \Gamma(x)\,\frac{1}{p}\Big((1+|Mu|^q)^{p/q} - 1\Big) \tag{3.2}$$

with $\Gamma \in L^\infty(\mathbb{R}^3)$ is $\mathbb{Z}^3$ periodic, positive and bounded away from 0, $M \in GL(3)$ is an invertible 3×3 matrix. Then all assumptions on F are satisfied. Observe that these functions are not radial when M is not an orthogonal matrix. Of course, if $M = \text{id}$, then for (3.1), $f(x,u)$ takes the form of (1.4). Other examples can be provided by considering radial functions of the form $F(x,u) = W(|u|^2)$, where $W \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$, $W(0) = W'(0) = 0$ and $W'(t)$ is strictly increasing on $(0,+\infty)$. Then we check that (F1), (F2), (F4) and (F5) are satisfied.

Our first main result reads as follows.

Theorem 3.1 ([20])**.** *Assume that* (F1)-(F5) *and* (V) *hold. Then there is a solution to* (1.3)*. If $V < 0$ a.e. on $\mathbb{R}^3$ or $V = 0$ then* (1.3) *has a ground state solution, i.e. there is a critical point $E \in \mathcal{M}$ of $\mathcal{E}$ such that*

$$\mathcal{E}(E) = \inf_{\mathcal{M}} \mathcal{E} > 0\,, \tag{3.3}$$

where

$$\begin{aligned} \mathcal{M} := \{E \in \mathcal{D}(\text{curl},p,q) \mid\, \mid E \neq 0,\ \mathcal{E}'(E)(E) = 0, \\ \textit{and } \ \mathcal{E}'(E)(\nabla\varphi) = 0 \ \textit{ for any } \ \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)\}\,. \end{aligned} \tag{3.4}$$

Since $\mathcal{M}$ contains all nontrivial critical points of $\mathcal{E}$, then a ground state solution is a nontrivial solution with the least possible energy $\mathcal{E}$. Moreover any $E \in \mathcal{M}$ admits the Helmholtz decomposition $E = u + \nabla w \in \mathcal{U} \oplus \nabla\mathcal{W}$, and since $\mathcal{E}(E) > 0$ then $u \neq 0$.

We provide a careful analysis of bounded sequences in $\mathcal{M}$ which plays a crucial role in proof of Theorem 3.1. Namely, setting

$$I(E) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \times E|^2\,dx - \mathcal{E}(E) = -\frac{1}{2}\int_{\mathbb{R}^3} V(x)|E|^2\,dx + \int_{\mathbb{R}^3} F(x,E)\,dx \tag{3.5}$$

we get the following result.

Theorem 3.2 ([20])**.** *Assume that* (F1)-(F5) *and* (V) *hold. If $(E_n)_{n=0}^\infty \subset \mathcal{M}$ is bounded then, up to a subsequence, there is $N \in \mathbb{N} \cup \{\infty\}$, $\bar{E}_0 \in \mathcal{D}(\text{curl},p,q)$ and* *there are sequences* $(\bar{E}_i)_{i=1}^N \subset \mathcal{M}_0$ *and* $(x_n^i)_{n\geq i} \subset \mathbb{Z}^3$ *with* $x_n^0 = 0$ *such that the following conditions hold:*

$$\begin{aligned} &E_n(\cdot + x_n^i) \rightharpoonup \bar{E}_i \;\text{ in }\; \mathcal{D}(\mathrm{curl}, p, q) \;\text{ and} \\ &fE_n(\cdot + x_n^i) \to \bar{E}_i \;\text{ a.e. in }\; \mathbb{R}^3 \;\text{ as }\; n \to \infty\,, \end{aligned} \tag{3.6}$$

for any $0 \leq i < N+1$, *and*

$$\begin{aligned} &E_n - \sum_{i=0}^{\min\{n,N\}} \bar{E}_i(\cdot - x_n^i) \to 0 \\ &\text{in }\; L^{p,q} = L^p(\mathbb{R}^3,\mathbb{R}^3) + L^q(\mathbb{R}^3,\mathbb{R}^3) \;\text{ as }\; n \to \infty\,. \end{aligned} \tag{3.7}$$

Moreover

$$\lim_{n\to\infty} I(E_n) = I(\bar{E}_0) + \sum_{i=1}^N I_0(\bar{E}_i) < \infty\,, \tag{3.8}$$

where $\mathcal{M}_0$ *and* I_0 *are given by* (3.4) *and* (3.5) *under assumption* $V = 0$.

As a consequence of Theorem 3.2 we get the sequentially weak-to-weak* continuity of $\mathcal{E}'$ in $\mathcal{M} \cup \{0\}$; see [20]Corollary 5.3. Moreover, in the spirit of the global compactness result of Struwe [34, 33] or Coti Zelati and Rabinowitz [16], we obtain a finite splitting of energy levels with respect to a Palais-Smale sequence in $\mathcal{M}$.

Theorem 3.3 ([20]). *Assume that* (F1)-(F5) *and* (V) *hold. If* $(E_n)_{n=0}^\infty \subset \mathcal{M}$ *is a* $(PS)_c$*-sequence at level* $c > 0$, *i.e.* $\mathcal{E}(E_n) \to c$ *and* $\mathcal{E}'(E_n) \to 0$, *then, up to a subsequence, there is* $\bar{E}_0 \in \mathcal{D}(\mathrm{curl}, p, q)$ *and a finite sequence* $(\bar{E}_i)_{i=1}^N \subset \mathcal{M}_0$ *of critical points of* $\mathcal{E}_0$ *such that* (3.6), (3.7) *hold and*

$$c = \mathcal{E}(\bar{E}_0) + \sum_{i=1}^N \mathcal{E}_0(\bar{E}_i)\,,$$

where $\mathcal{E}_0$ *is the energy functional given by* (2.1) *under assumption* $V = 0$.

Observe that if $0 < c < \inf_{\mathcal{M}_0} \mathcal{J}_0$ then $N = 0$, $\mathcal{J}(\bar{E}_0) = c$ and $\bar{E}_0$ is a nontrivial critical point of $\mathcal{J}$. In this way the comparison of energy levels implies the existence of nontrivial solutions. See [20] for details.

3.1. **Nonexistence results.** Finally we analyse situation when $2 < p \leq q < 6$ or $6 < p \leq q$. We recall a variant of the Pohozaev identity for the $\nabla\times\nabla\times(\cdot)$ operator.

Theorem 3.4 ([20]). *Suppose that* $V = 0$, F *is independent of* x *and satisfies* (F1). *If* $E = u + \nabla w$ *is a classical solution to* (1.3) *such that* $\mathrm{div}(u) = 0$,

$$u \in \mathcal{C}^2(\mathbb{R}^3,\mathbb{R}^3)\,,\; w \in \mathcal{C}^2(\mathbb{R}^3) \tag{3.9}$$

and

$$F(E), \langle f(E), \nabla w\rangle \quad \text{and} \quad |f(E)||w| \in L^1(\mathbb{R}^3)\,, \tag{3.10}$$

then

$$\int_{\mathbb{R}^3} |\nabla \times E|^2\,dx = 6\int_{\mathbb{R}^3} F(E)\,dx\,. \tag{3.11}$$

Observe that for any $2 < p \leq q$ the following growth condition

(F6) For any $x \in \mathbb{R}^3$ and $u \in \mathbb{R}^3$, $u \neq 0$

$$qF(x,u) \geq \langle f(x,u), u\rangle \geq pF(x,u) > 0$$

is satisfied by nonlinearities given by (3.1), (3.2) and implies the first inequality in (F3). Now we formulate nonexistence results as a consequence of Theorem 3.4.

Corollary 3.5. *Suppose that F is independent of x,* (F1) *and* (F6) *hold.*

(*a*) *If $V = 0$, and $2 < p \leq q < 6$ or $6 < p \leq q$, then there is no classical solution to* (1.3) *of the form $E = u + \nabla w$ with $u \neq 0$, $\operatorname{div}(u) = 0$ satisfying* (3.9) *and* (3.10).

(*b*) *If V is constant and negative, $2 < p \leq q \leq 6$, then there is no classical solution to* (1.3) *of the form $E = u + \nabla w$ with $u \neq 0$, $\operatorname{div}(u) = 0$ satisfying* (3.9), (3.10) *and $u \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, $w \in H^1(\mathbb{R}^3)$.*

In particular, for the Kerr nonlinearity, i.e. $p = q = 4$ and $f(x,E) = |E|^2 E$ there exist no classical solutions to (1.3) for constant $V \leq 0$. Therefore example (1.4) with $p = 4$ and $q > 6$ incorporates the Kerr effect only for strong fields $\mathcal{E}$ in order to solve (1.3).

4. Open questions

Periodic V. Even if V is periodic and cylindrically symmetric as in [6] we do not know whether there is a ground state solution that minimizes energy $\mathcal{E}$ among all nontrivial solutions. It is worth mentioning that usually naturally occurring materials have the permittivity positive (periodic or almost periodic) and bounded away from zero, i.e. $V(x) = -\mu\omega^2\varepsilon(x)$ is negative and bounded away from 0. This situation has not been investigated in [6]. It is not clear in which space one should seek weak solutions of this problem with such V and with a nonlinearity of the form (1.4), and whether any variational method can be used. In view of Corollary 3.5 the classical solutions do not exist. Therefore in [20] we restrict our considerations to optical metamaterials having permittivity ε close to zero, i.e. the so-called epsilon-near-zero (ENZ) media (see e.g. [2, 15, 18] and references therein).

Symmetry of ground states. It is not known if ground state solutions of (1.3) exhibit any symmetry. It seems that we cannot expect radial solutions, since in view of [6][Lemma 4] one may get $\nabla \times E = 0$. Hence if $E = u + \nabla w$ with $\operatorname{div}(u) = 0$ solves (1.3) and $\nabla \times E = 0$, then $E = \nabla w$ implies $\mathcal{E}(E) \leq 0$. This contradicts the fact that ground states have positive energy; see (4.1).

Localization of ground states. We look for weak solutions to (1.3) in $\mathcal{D}(\operatorname{curl}, p, q)$ space, where p and q are provided by the growth of f. Note that a solution E of (1.3) determines $\mathcal{P}_{NL}$ and $\mathcal{D}$ by the first constitutive relation in (1.1) whereas $\mathcal{B}$ and $\mathcal{H}$ are obtained from $\nabla \times \mathcal{E}$ by time-integration. In [20] we show that if $E \in \mathcal{D}(\operatorname{curl}, p, q)$ solves (1.3), then the total electromagnetic energy

$$\mathcal{L}(t) := \frac{1}{2} \int_{\mathbb{R}^3} \mathcal{E}\mathcal{D} + \mathcal{B}\mathcal{H}\, dx \tag{4.1}$$

is finite. We do not know whether the fields $\mathcal{E}$, $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{H}$ are localized, i.e. decay to zero as $|x| \to \infty$, however $\mathcal{D}(\operatorname{curl}, p, q)$ lies in the sum of Lebesgue spaces $L^{p,q} := L^p(\mathbb{R}^3, \mathbb{R}^3) + L^q(\mathbb{R}^3, \mathbb{R}^3)$ and therefore it does not contain the usual nontrivial travelling waves E propagating in a given direction $z \in \mathbb{R}^3$ such that $E(x) = E(x + z)$ for all $x \in \mathbb{R}^3$. The finiteness of the electromagnetic energy and the localization problem attract a strong attention in the study of self-guided beams of light in a nonlinear medium; see e.g. [26, 27].

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