Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 10(2011), pp. 115–136.

The stochastic solution of the Dirichlet problem and controlled convergence

Lucian Beznea

Abstract[1]. We expose the construction of the classical stochastic solution of the Dirichlet problem, using the hitting time of the boundary of the associated Markov process, the Brownian motion in the case of the Laplace operator. We present then the controlled convergence initiated by A. Cornea and its application to the Dirichlet problem. Finally, we show that the stochastic solution solves the Dirichlet problem with general boundary data, in the sense of the controlled convergence.

1. Introduction

Let D be an open set in $\mathbb{R}^d, d \geq 1$. Recall that a function $h \in C^2(D)$ is called *harmonic* on D provided that

$$\Delta h := \sum_{j=i}^{d} \frac{\partial^2 h}{\partial x_j^2} = 0 \qquad \text{on } D.$$

Let further ∂D be the boundary of D. A solution of the *classical Dirichlet problem on D with boundary data* $f : \partial D \to \mathbb{R}$ is a harmonic function $h : D \to \mathbb{R}$ such that

$$\lim_{\substack{x \to y \\ x \in D}} h(x) = f(y) \qquad \text{for all } y \in \partial D.$$

Our first aim is to solve the classical Dirichlet problem by means of the stochastic solution (Theorem 3.8). Necessary results on the Markov processes and Brownian motion are exposed in Section 2. The presentation of the strong Markov property and of the stochastic solution for the classical Dirichlet problem (Section 3) follows closely the lecture notes [9]. Several detailed proofs of the results from Section 2 are given in Appendix (A.5). Some complements on probability theory used in this paper were also collected in Appendix (A.2)-(A.4).

[1]Author's address: L. Beznea, Institutul de Matematică "Simion Stoilow" al Academiei Române, Research unit No. 2, P.O. Box 1-764, RO-014700 Bucureşti, România, and Universitatea din Bucureşti, Facultatea de Matematică şi Informatică, Str. Academiei 14, RO-010014 Bucureşti, România; e-mail: lucian.beznea@imar.ro .

Keywords. Dirichlet problem, Brownian motion, stochastic solution, boundary behavior, controlled convergence.

AMS Subject Classification. 31B25, 60J50, 60J65, 35J25, 60J45.

Lecture notes based on a series of four talks given at Basilicata University Potenza, Italy, May, 6-12, 2012.

The next remark will lead to our second aim, we present its proof in Appendix (A.1).

Remark 1.1. *If the classical Dirichlet problem with boundary data f has a solution then f is a continuous function on ∂D.*

Taking into account the statement of Remark 1.1, the following question arises: *What happens with the Dirichlet problem in the case when f is not a continuous function?*

Recall that the classical potential theory answers this question, with many axiomatical developments, mainly in the frame of harmonic spaces (see the monograph [5] an the references therein), with applications to elliptic and parabolic partial differential operators, not necessary with smooth coefficients.

The second aim for us is to indicate a short way to arrive to an answer for the above question, combining analytic and probabilistic tools. It is a rather new method which turned out to be efficient in solving the Dirichlet problem in an infinite dimensional frame too, like for the Gross-Laplace operator on an abstract Wiener space (cf. [2]). A main ingredient of this approach is the method of controlled convergence introduced by A. Cornea (cf. [6], [7], and [10]); we shortly present the method in Section 4. The final result (Theorem 4.8) shows that the stochastic solution solves the Dirichlet problem not only for continuous boundary data, but also for general onces, the solution controlled converging to the boundary data in this case.

We are deeply indebted to Elisabetta Barletta and Sorin Dragomir for their essential role and infinite patience in elaborating this work.

2. Markov processes, the Brownian motion

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T := [0, \infty)$. Let further $(\mathcal{F}_t)_{t\in T}$ be an increasing family of sub-σ-algebras of $\mathcal{F}$; $\mathcal{F}_t \subset \mathcal{F}_s$ whenever $t < s$. The family $(\mathcal{F}_t)_{t\in T}$ is called *filtration.*

A stochastic process with state space $(E, \mathcal{B})$ and parameter set T is a collection $X = (X_t)_{t\in T}$ of maps $X_t : \Omega \to E$ such that $X_t \in \mathcal{F}/\mathcal{B}$, $t \in T$. The map $[0, \infty) \ni t \mapsto X_t(\omega)$ is called the *path (trajectory) corresponding to ω*. X is called *adapted* to the filtration $(\mathcal{F}_t)_{t\in T}$ provided that $X_t \in \mathcal{F}_t/\mathcal{B}$ for each $t \in T$.

Example. A stochastic process $X = (X_t)_{t\in T}$ is always adapted to the *minimal filtration* $\mathcal{F}^0_t := \sigma(X_s, s \leq t), t \in T$; if $\mathcal{A} \subset \{f : \Omega \to (E, \mathcal{B})\}$ then $\sigma(\mathcal{A})$ denotes the smallest σ-algebra on Ω making measurable all the functions from $\mathcal{A}$.

We denote by $p\mathcal{B}$ the set of all numerical, positive $\mathcal{B}$-measurable functions on E.

The Gaussian semigroup. We define first *the density of the Gaussian kernel*, $g_t : \mathbb{R}^d \to \mathbb{R}$, $t \geq 0$, $d \geq 1$, as

$$g_t(x) = \frac{1}{(2\pi t)^{d/2}}\, e^{-|x|^2/2t} \quad , \quad x \in \mathbb{R}^d .$$

For every $x \in \mathbb{R}^d$ and $t \geq 0$, $1 \leq i \leq n$, the densities of the Gaussian kernels on $\mathbb{R}^d$ have the following properties (cf., e.g., [3], Ch. 0, Lemma 2.1):

$$\frac{\partial g_t}{\partial x_i}(x) = -\frac{x_i}{t} g_t(x) \quad ; \quad \frac{\partial^2 g_t}{\partial x_i^2}(x) = \left(\frac{x_i^2}{2t^2} - \frac{1}{t}\right) g_t(x) \, ;$$

(2.1)

$$\frac{\partial g_t}{\partial t}(x) = \frac{1}{2}\left(\frac{|x|^2}{t^2} - \frac{d}{t}\right) g_t(x) \, ;$$

In particular, $(1/2)\Delta g_t = \partial/\partial t \, g_t$.

We define the *Gaussian kernel* P_t as the convolution kernel (see (A.4) in Appendix) induced by the density g_t, $P_t f := g_t * f$ for all $f \in p\mathcal{B}$, that is

$$P_t f(x) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/2t} f(y)\, dy \quad \text{for all } \; t > 0, \; x \in \mathbb{R}^d, \; f \in p\mathcal{B} \, .$$

For each $t > 0$ let μ_t be the *Gaussian measure on* $\mathbb{R}^d$ *with* (*variance*) *parameter* t,

$$\mu_t(A) := \frac{1}{(2\pi t)^{d/2}} \int_A e^{-|x|^2/2t}\, dx \quad , \quad A \in \mathcal{B}(\mathbb{R}^d) \, .$$

The Gaussian semigroup on $\mathbb{R}^d$ is the family of kernels $(P_t)_{t\geq 0}$, where P_0 is the identity operator, that is, $P_0 f := f$ for all f.

The Gaussain semigroup and the Gaussian measures μ_t on $\mathbb{R}^d$, $t > 0$, have the following properties.

(i) $P_t(x, A) = \int_A g_t(y - x)\, dy$, $A \in \mathcal{B}(\mathbb{R}^d)$, and the map $A \mapsto P_t(x, A)$ is a probability on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

(ii) $P_t(0, \cdot) = \mu_t$ and $P_t f(x) = \int_{\mathbb{R}^d} f(y) P_t(x, dy) = \int_{\mathbb{R}^d} f(x + y)\mu_t(dy)$.

(iii) $\mu_{t\cdot s}(A) = \mu_t(1/\sqrt{s} \cdot A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$ (the time scaling property).

(iv) $P_t(x+y, A+y) = P_t(x, A)$ for all $x, y \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$ (P_t is translation invariant). In particular, $P_t(x, A) := \mu_t(A - x)$.

Transition function. A family of kernels $(T_t)_{t\geq 0}$ on a measurable space $(E, \mathcal{B})$ is called (time homogenous) *Markovian transition function* provided that for all $t \geq 0$ T_t is a Markovian kernel on $(E, \mathcal{B})$ and $T_t \circ T_s = T_{t+s}$ for all $t, s \geq 0$ i.e.,

$$T_{t+s}(x, A) = \int_E T_s(y, A) T_t(x, dy) \qquad \text{for all } \; A \in \mathcal{B} \text{ and } \; x \in E$$

(the Chapman-Kolmogorov equation). Assume that T_0 is the identity operator.

Example. The Gaussian semigroup is a transition function on $\mathbb{R}^d$. For the proof one uses the convolution property of the densities of the Gaussian kernel, i.e. $g_s * g_t = g_{s+t}$.

Let $(C_0(\mathbb{R}^d), \| \, \|_\infty)$ be the Banach space of all real-valued continuous functions on $\mathbb{R}^d$, vanishing at infinity, equipped with the supremum norm $\| \, \|_\infty$.

Proposition 2.1. *The following assertions hold.*

(*i*) *For every* $f \in b\mathcal{B}(\mathbb{R}^d)$ *and* $t > 0$ *it follows that* $P_t f \in C^2(\mathbb{R}^d)$ *and*

$$\frac{1}{2}\,\Delta P_t f = \frac{\partial}{\partial t} P_t f$$

(*ii*) $(P_t)_{t\geq 0}$ *induces a* C_0*-semigroup of contractions on* $C_0(\mathbb{R}^d)$*, i.e.* $P_t f \in C_0(\mathbb{R}^d)$ *and* $\lim_{t\searrow 0}||P_t f - f||_\infty = 0$ *for all* $f \in C_0(\mathbb{R}^d)$*;* $P_t \circ P_s = P_{t+s}$ *and* $||P_t|| \leq 1$ *for all* $t, s \geq 0$.

For the proof see [3], Lemma 2.3 from Ch. 0. Note that assertion (*i*) is a direct consequence of (2.1).

Markov processes with transition function. Let $X = (X_t)_{t\geq 0}$ be a stochastic process with state space $(E, \mathcal{B})$, adapted to the filtration $(\mathcal{F}_t)_{t\in T}$. Let $(T_t)_{t\geq 0}$ be a transition function on $(E, \mathcal{B})$.

One says that X is a *Markov process* with respect to $(\mathcal{F}_t)_{t\in T}$ having $(T_t)_{t\geq 0}$ as transition function provided

$$E[f \circ X_{t+s}|\mathcal{F}_t] = T_s f(X_t) \qquad \text{for all } f \in p\mathcal{B},\ t \geq 0,\ s > 0\,.$$

Taking conditional expectations with respect to $\sigma(X_t)$ in the above equality and using the properties of the conditional expectation (cf. (A.3.1) in Appendix) we obtain the *Markov property*:

$$E[f \circ X_{t+s}|\mathcal{F}_t] = E[f \circ X_{t+s}|X_t] \qquad \text{for all } f \in p\mathcal{B},\ t \geq 0,\ s > 0\,.$$

Define the *initial distribution* of X as the distribution of X_0, that is the measure $\mu := P \circ X_0^{-1}$ on $(E, \mathcal{B})$, i.e., $\mu(A) = P(X_0 \in A)$ for all $A \in \mathcal{B}$. The initial distribution of X is a probability on $(E, \mathcal{B})$.

Using the Markov property repeatedly one obtains the following formula for the *finite-dimensional distributions* of X: if $0 \leq t_1 < t_2 < \cdots < t_n$ and $f \in b\mathcal{B}(E^n)$, then

$$E(f(X_{t_1}, \cdots, X_{t_n})) =$$
$$= \int \mu(dx_0) \int T_{t_1}(x_0, dx_1) \cdots \int T_{t_n - t_{n-1}}(x_{n-1}, dx_n) f(x_1, \cdots, x_n)\,.$$

Consequently, the finite-dimensional distributions of X are expressible in terms of its initial distribution and transition function. Taking $n = 1$ in the above formula and $t_1 = t$ we get

$$E(f \circ X_t) = \int_E T_t f\, d\mu,\ \ f \in p\mathcal{B}\,. \tag{2.2}$$

Question. Suppose we are given a probability measure μ on $(E, \mathcal{B})$ and a transition function $(T_t)_{t\geq 0}$ on $(E, \mathcal{B})$. Does there exist a Markov process with state space $(E, \mathcal{B})$ which has $(T_t)_{t\geq 0}$ as transition function and μ as initial distribution?

The answer is affirmative due to a celebrated theorem of Kolmogorov (Theorem 2.2 below), if $(E, \mathcal{B})$ is a Radon measurable space (i.e., $(E, \mathcal{B})$ is measurably isomorphic with an universally measurable subset of a compact metric space).

Let $\Omega := E^{\mathbb{R}_+}$ be *the space of all trajectories* and consider the coordinate mappings $X_t : \Omega \to E$, $t \geq 0$,

$$X_t(\omega) := \omega(t) \quad , \quad \omega \in \Omega\,.$$

Endow Ω with the product σ-algebra $\mathcal{F} := \sigma(X_t, t \geq 0)$ and the filtration $\mathcal{F}_t := \mathcal{F}_t^0 = \sigma(X_s, s \leq t)$, $t \geq 0$ (the minimal filtration).

Theorem 2.2 (Kolmogorov's theorem on the construction of Markov processes)**.** *Let $(E,\mathcal{B})$ be a Radon measurable space, μ a probability and $(T_t)_{t\geq 0}$ a transition function on $(E,\mathcal{B})$. Then there exists a unique probability P^μ on $(\Omega,\mathcal{F})$ such that $(X_t)_{t\geq 0}$ is a Markov process with state space E, transition function $(T_t)_{t\geq 0}$, and initial distribution μ, i.e.,*

$$E^\mu[f\circ X_{s+t}|\mathcal{F}_t] = T_s f(X_t) \text{ for all } f\in p\mathcal{B},\ t\geq 0, s>0\,. \tag{2.3}$$

and

$$\mu(A) = P^\mu(X_0\in A) \text{ for all } A\in\mathcal{B}\,.$$

For the proof see, e.g., Corollary 52, Ch. III from [8] and [4], page 17.

Remark. (i) For each initial distribution μ, the space of trajectories Ω of the process is the same and it is called *the canonical realization.*

(ii) If μ is the Dirac measure at x, $\mu=\varepsilon_x$, we write P^x instead of P^{ε_x}. One says that the probability P^x describes the evolution of the process starting at x.

(iii) Taking $\mu=\varepsilon_x$ in (2.2) we obtain the following representation for the transition function:

$$T_t f(x) = E^x(f\circ X_t) \qquad \text{for all } f\in p\mathcal{B},\ x\in E \text{ and } t\geq 0\,.$$

If $f=1_A$ and $A\in\mathcal{B}$ then $T_t(1_A)(x)=T_t(x,A)$ and the above representation for T_t has a version which justifies its name of "transition function of the process X":

$$P^x(X_t\in A) = T_t(x,A) \text{ for all } A\in\mathcal{B}, x\in E, \text{ and } t\geq 0\,. \tag{2.4}$$

Equality (2.3) becomes the *Markov property:*

$$E^x[f\circ X_{s+t}|\mathcal{F}_t] = E^{X_t}(f\circ X_s) \text{ for all } f\in p\mathcal{B},\ x\in E,\ t\geq 0,\ s>0\,. \tag{2.5}$$

Taking the expectation in both sides we obtain

$$E^x(f\circ X_{t+s}) = E^x(E^{X_t}(f\circ X_s)) \text{ for all } t\geq 0,\ s>0,\ f\in p\mathcal{B},\ x\in E\,. \tag{2.6}$$

Brownian Motion. *The d-dimensional Brownian motion* is the Markov process with state space $\mathbb{R}^d$, having the Gaussian semigroup as transition function.

From now on we assume that $X=(X_t)_{t\geq 0}$ is a d-dimensional Brownian motion.

One can show that the corresponding probability P^μ is carried by the set

$$\{\omega\in\Omega \mid t\mapsto X_t(\omega) \text{ is continuous on } [0,\infty)\}$$

which means that the Brownian motion has continuous paths. In addition, since $P_0=Id$, taking $t=0$ and $A=\{x\}$ in (2.4), it follows that for all $x\in\mathbb{R}^d$

$$X_0 = x \quad P^x\text{-almost surely.}$$

One says that the process X is *normal*.

Stopping times. For $t\geq 0$ define $\mathcal{F}_{t+} := \bigcap_{s>t}\mathcal{F}_s$. Clearly $(\mathcal{F}_{t+})_{t\in T}$ is also a filtration and $\mathcal{F}_t\subset\mathcal{F}_{t+}$ for all $t\geq 0$.

A filtration $(\mathcal{F}_t)_{t\in T}$ is called *right continuous* provided that $\mathcal{F}_t=\mathcal{F}_{t+}$ for each $t\geq 0$. Note that $(\mathcal{F}_{t+})_{t\in T}$ is always right continuous.

A function $S:\Omega\to[0,\infty]$ is called *stopping time* (with respect to $(\mathcal{F}_{t+})_{t\in T}$) provided $[S\leq t]\in\mathcal{F}_{t+}$ for all $t>0$.

Remark. S is a *stopping time* (*with respect to* $(\mathcal{F}_{t+})_{t\in T}$) if and only if $[S < t] \in \mathcal{F}_t$ for all $t > 0$. Indeed, the assertion follows from the equalities $[S \le t] = \bigcap_n [S < t + 1/n]$ and $[S < t] = \bigcup_n [S \le t - 1/n]$.

Clearly, every nonnegative constant is a stopping time.

Hitting time. The *first hitting time* of a set $A \subset \mathbb{R}^d$ is defined as

$$T_A(\omega) := \inf\{t > 0 \mid X_t(\omega) \in A\} \quad , \quad \omega \in \Omega .$$

Proposition 2.3. *The following assertions hold.*

(i) If A is an open set then T_A is a stopping time.
(ii) Let A be a bounded subset of $\mathbb{R}^d$, then $T_{A^c} < \infty$ almost surely (the Brownian motion leaves every compact set).

Proof. (i) Because the paths $t \mapsto X_t(\omega)$, $\omega \in \Omega$, of the Brownian motion are continuous we have $[T_A < t] = \bigcup_{\substack{a<r<t \\ r\in\mathbb{Q}}} [X_t \in A] \in \mathcal{F}_t$.

(ii) We follow the proof of Lemma 4.2 from [9]. We may assume that A is a compact set. Indeed, if B is a closed ball such that $A \subset B$ then clearly $T_{A^c} \le T_{B^c}$.

Let $x \in \mathbb{R}^d$. We show that

$$\text{(2.7)} \qquad \text{for every } \ \varepsilon > 0, \text{ there exists } \ t > 0 \text{ such that } \ P^x(X_t \in A) < \varepsilon .$$

For, let $r > 0$ be such that $A - x \subset B(0, r)$. Then $(1/k)(A - x) \subset (1/k)\, B(0, r) = B(0, r/k) \underset{k\to\infty}{\searrow} \{0\}$ and since $\mu_t(\{0\}) = 0$ we get $\lim_k \mu_t((1/k)(A-x)) = \mu_t(\{0\}) = 0$. By the time scaling property of the Gaussian measures it follows that $P^x(X_{k^2t} \in A) = \mu_{k^2t}(A - x) = \mu_t((1/k)(A - x)) \underset{k\to\infty}{\searrow} 0$ and thus (2.7) holds.

From $[T_{A^c} = \infty] \subset [X_t \in A] \in \mathcal{F}_t$ if t is as in (2.7), one has $P^x([T_{A^c} = \infty]) \le P^x(X_t \in A) < \varepsilon$ and letting ε tend to 0 we conclude that $P^x([T_{A^c} = \infty]) = 0$, $P^x(T_{A^c} < \infty) = 1$ for all $x \in \mathbb{R}^d$.

□

Remark. Assertion (i) of Proposition 2.3 remains true if $A \subset \mathbb{R}^d$ is a union of compact sets (see, e.g., [9], Proposition 4.1).

The shift operator. Fix $s \ge 0$, we define the *shift operator* $\theta_s : \Omega \to \Omega$ by $\theta_s(\omega)(t) := \omega(s + t)$, that is,

$$X_t \circ \theta_s = X_{t+s} .$$

It is clear that θ_s is $\mathcal{F}/\mathcal{F}$-measurable and the map $\theta_s : (\Omega, \mathcal{F}_{s+t}) \to (\Omega, \mathcal{F}_t)$ is $\mathcal{F}_{s+t}/\mathcal{F}_t$-measurable for all $t \ge 0$.

Let S be a stopping time and define

$$\mathcal{F}_S^+ := \{A \in \mathcal{F} \mid A \cap [S < t] \in \mathcal{F}_t\} \qquad \text{for all } \ t > 0 .$$

Remark. (i) $\mathcal{F}_S^+$ is a σ-algebra. Because we should think of $\mathcal{F}_t$, as containing all the information in some physical process up to the time moment t, then $\mathcal{F}_S^+$ contains all the information up to the random time S.

(ii) If S is constant, i.e., $S = s \in \mathbb{R}^+$, then $\mathcal{F}_s^+ = \mathcal{F}_{s+}$. For, if $\Lambda \in \mathcal{F}$ then

$$\Lambda \cap [S < t] = \begin{cases} \emptyset & , \quad \text{if } \ t \le s \\ \Lambda & , \quad \text{if } \ t > s . \end{cases}$$

Proposition 2.4. *Let S, T, T_n be stopping times. Then the following assertions hold.*

(i) $S \wedge T$, $S \vee T$, $S + T$ *are stopping times.*
(ii) $\inf_n T_n$, $\sup_n T_n$, $\underline{\lim}_n T_n$, $\overline{\lim}_n T_n$ *are stopping times.*
(iii) *If* $t > 0$ *then* $t + S \circ \theta_t$ *is a stopping time.*
(iv) S *is* $\mathcal{F}_S^+$*-measurable and the sets* $[S < T]$, $[S = T]$, *and* $[S > T]$ *belong to* $\mathcal{F}_S^+ \cap \mathcal{F}_T^+$.
(v) *If* $S \leq T$ *then* $\mathcal{F}_S^+ \subseteq \mathcal{F}_T^+$ *and* $\mathcal{F}^+_{\inf_n T_n} = \bigcap_n \mathcal{F}^+_{T_n}$.

Proof. We only prove assertion (i), the proofs of the other assertions are similar. For every $t > 0$ we have $[S \wedge T < t] = [S < t] \cup [T < t] \in \mathcal{F}_t$, $[S \vee T < t] = [S < t] \cap \{T < t\} \in \mathcal{F}_t$, and $[S + T < t] = \bigcup_{\substack{0<r<t \\ r \in \mathbb{Q}}} [S < r] \cap [T < t - r] \in \mathcal{F}_t$. □

The Strong Markov Property. The presentation below is based on [9], the omitted proofs are presented in Appendix (A.5).

Proposition 2.5 (The Strong Markov Property)**.** *Let S be a stopping time and $t \geq 0$. Then for every $f \in p\mathcal{B}(\mathbb{R}^d)$,*

$$E^x(f \circ X_{S+t}; S < \infty) = E^x(E^{X_S}(f \circ X_t); S < \infty)\,.$$

Remark. Taking $f = 1_A$ in the relation (2.7) we obtain

$$P^x(X_{t+s} \in A) = E^x(P^{X_t}(X_s \in A))\,.$$

The same holds in the case of the strong Markov property:

$$P^x(X_{S+t} \in A; S < \infty) = E^x(P^{X_S}(X_t \in A); S < \infty) \text{ for all } A \in \mathcal{B}(\mathbb{R}^d)\,.$$

Proposition 2.6. *For every stopping time S the mapping $X_S : [S < \infty] \to \mathbb{R}^d$ defined by*

$$X_S(\omega) := X_{S(\omega)}(\omega)$$

is $\mathcal{F}_S^+$-measurable.

Corollary 2.7. *Let S be a stopping time and $t \geq 0$. Then for each $f \in p\mathcal{B}(\mathbb{R}^d)$ and every function $Z : [S < \infty] \to [0, \infty]$ which is $\mathcal{F}_S^+-$ measurable,*

$$E(Z \cdot (f \circ X_{S+t}); S < \infty) = E^x(Z \cdot E^{X_S}(f \circ X_t); S < \infty)$$

or equivalently

$$E[f \circ X_{S+t} \mid \mathcal{F}_S^+] = E^{X_S}(f \circ X_t)\,.$$

If S is a stopping time then the shift operator $\theta_S : [S < \infty] \to \Omega$ is defined as

$$\theta_{S(\omega)}(t) := \omega(S(\omega) + t) \quad , \quad \omega \in [S < \infty],\ t \geq 0\,.$$

Proposition 2.8. *For every stopping time S and each $\mathcal{F}$-measurable function $Y : [S, \infty] \to [0, \infty]$ we have*

$$E^x(Y \circ \theta_S; S < \infty) = E^x(E^{X_S}(Y); S < \infty)\,.$$

The next result gives the general version of the strong Markov propoerty.

Theorem 2.9. *Let S be a stopping time and $Y : [S < \infty] \to [0,\infty]$ be $\mathcal{F}$-measurable. Then*

$$E^x[Y \circ \theta_S \mid \mathcal{F}_S^+] = E^{X_S}(Y)\,.$$

Equivalently, for every $\mathcal{F}_S^+$−measurable function $Z : [S < \infty] \to [0,\infty]$,

$$E^x(Z \cdot Y \circ \theta_S; S < \infty) = E^x(Z \cdot E^{X_S}(Y); S < \infty)\,.$$

3. Stochastic solution for the classical Dirichlet problem

In this section we follow the approach of [9]. Fix $x \in \mathbb{R}^d$, $r > 0$, and define

$$B(x,r) = \{y \in \mathbb{R}^d \,:\, |y-x| < r\} \quad , \quad S(x,r) = \{y \in \mathbb{R}^d \,:\, |y-x| = r\}\,.$$

We denote the uniform distribution of mass 1 on $B(x,r)$ by $\lambda_{x,r}$, that is, the normed volume measure on $B(x,r)$ and the uniform distribution of mass 1 on $S(x,r)$ by $\sigma_{x,r}$, that is, the normed surface measure on $S(x,r)$.

Let T be a stopping time and $\alpha \geq 0$. We define the kernel P_T^α by

$$P_T^\alpha f(x) = E^x(e^{-\alpha T} f \circ X_T; T < \infty)\,.$$

If $\alpha = 0$, we write P_T instead of P_T^0.

For $A \in \mathcal{B}$ we write P_A^α instead of $P_{T_A}^\alpha$ and it is called the α*-order hitting kernel* and the measure $P_A^\alpha(x,\cdot)$ is called the α*-order hitting distribution* of A starting from x, or the α*-order harmonic measure* of A relative to $x \in \mathbb{R}^d$.

Remark. (i) Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a Banach space B, then $(e^{-\alpha t}T_t)_{t\geq 0}$ is also a C_0-semigroup on B. If L is the generator of $(T_t)_{t\geq 0}$ ($e^{Lt} = T_t$), then the generator of $(e^{-\alpha t}T_t)_{t\geq 0}$ is $L - \alpha$, where recall that

$$Lu := \lim_{t\searrow 0} \frac{T_t u - u}{t},\ u \in D(L)\,.$$

- (ii) P_T^α is the 0-order hitting kernel of the Markov process having the transition function $(e^{-\alpha t}P_t)_{t\geq 0}$ and the generator $L - \alpha$.
- (iii) The (right) continuity of the paths implies that all the measures $P_A^\alpha(x,\cdot)$ are concentrated on $\bar{A}$ (the closure of A). Indeed, if $f = 0$ on $\bar{A}$, from $P_A^\alpha f(x) = E^x(e^{-\alpha T_A} f \circ X_{T_A};\, T_A < \infty)$, where $X_{T_A}(\omega) \in \bar{A}$, it follows that $f(X_{T_A}(\omega)) = 0$ and thus $P_A^\alpha f(x) = 0$.
- (iv) Because the paths are continuous, if A is closed and $x \notin A$ then $P_A^\alpha(x,\cdot)$ is concentrated on the boundary of A.

Lemma 3.1. *Let $x \in \mathbb{R}^d$. Then the following assertions hold.*

- *(i) If A is a bounded subset of $\mathbb{R}^d$, a countable union of compact sets, then the distribution of $X_{T_{A^c}}$ is $P_{A^c}(x,\cdot)$, i.e.,*

$$P_{A^c}(x,F) = P^x(X_{T_{A^c}} \in F) \text{ for all } F \in \mathcal{B}(\mathbb{R}^d)\,.$$

  In particular, $P_{A^c}(x,\cdot)$ is a probability on $\mathbb{R}^d$ for all $x \in \mathbb{R}^d$.
- *(ii) If $r > 0$ and $a \in \mathbb{R}^d$ then $P_{B(a,r)^c}(a,\cdot) = \sigma_{a,r}$.*

Proof. (i) By assertion (ii) of Proposition 2.3 $T_{A^c} < \infty$ almost surely, hence $P_{A^c}(x,F) = P^x(X_{T_{A^c}} \in F, T_{A^c} < \infty) = P^x(X_{T_{A^c}} \in F)$.

(ii) We may assume that $a = 0$ and let $B = B(0,r)$. By assertion (i) and the above Remark (ii), it follows that $P_{B^c}(0,\cdot)$ is a probability on $S(0,r)$. Since μ_t is invariant under rotations one can check that $P^0(X_{T_{B^c}} \in \cdot)$ is also invariant under rotations, hence coincides with $\sigma_{0,r}$.

□

Let D be a bounded open subset of $\mathbb{R}^d$. For $f : \partial D \to \mathbb{R}$ bounded below, $f \in \mathcal{B}(\partial D)$, and $x \in D$ define

$$H_D f(x) := E^x(f \circ X_{T_{D^c}}) = P_{D^c} f(x)\,.$$

By Lemma 3.1 (i) we have $H_D 1(x) = 1$ and $H_D f(x)$ is the expected value of f at the points where the Brownian motion starting at x, exits from D.

Proposition 3.2 (Minimum principle; cf. [9], Proposition 6.1)**.** *Let $f : D \to (-\infty, +\infty]$ be lower semicontinuous such that $\liminf_{D\ni x\to z} f(x) \geq 0$ for every $z \in \partial D$. Then f is positive on D if it has one of the following two properties.*

(i) For every $x \in D$ there exists a radius $r_x > 0$, such that $\overline{B(x, r_x)} \subset D$ and

$$\int f\, d\sigma_{x,r_x} \leq f(x)\,.$$

(ii) $f \in C^2(D)$ and $\Delta f \leq 0$.

Proof. Assume the contrary, that is $f(x_0) < 0$ for some $x_0 \in D$. Choose $0 < \varepsilon < |f(x_0)|$ and define

$$g := f + \varepsilon\,.$$

Then $g(x_0) < 0$ and because $\liminf_{D\ni x\to\partial D} f(x) \geq 0$ we get that $g \geq 0$ outside a compact subset of D. Therefore

$$-\infty < \alpha := \inf g(D) < 0 \quad (g \text{ is lower semicontinuous on } D)\,,$$

and the set $K := [g = \alpha]$ is a non-empty compact subset of D.

Suppose that (i) holds and consider $x \in K$ with minimal distance to D^c. Then $S(x, r_x)$ is not a subset of K, hence $S(x, r_x) \cap [g > \alpha]$ is a non-empty set, so $\sigma_{x,r_x}([g > \alpha] \cap S(x, r_x)) > 0$ and

$$\alpha = f(x) + \varepsilon \geq \int g\, d\sigma_{x,r_x} = \int_{[g>\alpha]\cap S(x,r_x)} g\, d\sigma_{x,r_x} + \int_{[g=\alpha]\cap S(x,r_x)} g\, d\sigma_{x,r_x} =$$

$$= \int g_{[g>\alpha]\cap S(x,r_x)} + \alpha \cdot \sigma_{x,r_x}([g = \alpha] \cap S(x, r_x)) >$$

$$> \alpha \cdot \sigma([g > \alpha] \cap S(x, r_x)) + \alpha \cdot \sigma([g > \alpha] \cap S(x, r_x)) = \alpha\,,$$

a contradiction.

Suppose now that (ii) holds and choose an open set V such that $\overline{V}$ is a compact subset of D and $g \geq 0$ on $D \setminus V$. Let $a > 0$ be such that $\overline{V} \subset B(0, a)$ and choose $\delta > 0$ such that the function $\widetilde{g}$ defined as

$$\widetilde{g}(x) := g(x) + \delta(a - \sum_{j=1}^{d} x_j^2) \quad , \quad x \in D$$

satisfies $\widetilde{g}(x_0) < 0$ (we have $g(x_0) < 0$ and we choose δ such that $\delta < |g(x_0)|$). Clearly $\Delta\widetilde{g} = \Delta f - 2\delta d < 0$. Since $\widetilde{g} > g \geq 0$ on ∂V and $\widetilde{g}(x_0) < 0$, there exists $x \in V$ where $\widetilde{g}$ attains its minimum on the compact set $\overline{V}$. For every $1 \leq j \leq d$ the function $\phi_j \mapsto \widetilde{g}(x_1, \cdots, x_{j-1}, t, x_{j+1}, \cdots, x_d)$ has a minimum at x_j, hence $\phi_j''(x_j) \geq 0$. It follows that $\Delta\widetilde{g}(x) = \sum_{j=1}^{d} \phi_j''(x_j) \geq 0$ and this is a contradiction.

□

Corollary 3.3. *Let h_1, h_2 be harmonic functions on D such that*

$$\lim_{D \ni x \to z} (h_1(x) - h_2(x)) = 0 \quad \textit{for every } z \in \partial D\,.$$

Then $h_1 = h_2$. In particular, if the classical Dirichlet Problem has a solution then it is unique.

Proof. It is sufficient to apply Propostion 3.2 for the functions $f = h_1 - h_2$ and $f = h_2 - h_1$.

□

Poisson kernel and integral. Fix $a \in \mathbb{R}^d, r > 0$ and let $\sigma = \sigma_{a,r}$, $B = B(a,r)$, and $S = S(a,r)$. The *Poisson kernel* is defined as

$$K(x,z) := r^{d-2}\, \frac{r^2 - |x-a|^2}{|x-z|^2}\,,$$

where z $\in S$ and $x \in B$.

For every $f \in L^1(S,\sigma)$ define the *Poisson integral* as

$$H_f^B(x) := \int_S K(x,z) f(z)\sigma(dz)\,.$$

For an open set D we denote by $\mathcal{H}(D)$ the vector space of all harmonic functions on D.

Proposition 3.4 (cf. [9], Proposition 6.3, and [3], Proposition 1.2 from Ch. 0)**.** *The following assertions hold.*

(*i*) *If $z \in S$ then $K(\cdot, z)$ is harmonic on B. For every $f \in L^1(S,\sigma)$ the Poisson integral H_f^B is a harmonic function on B, $H_f^B(a) = \int_S f\, d\sigma$, and $H_1^B = 1$.*

(*ii*) (*The boundary behavior of the Poisson integral*)*. For every $y \in S$ and $f \in L^1(S,\sigma)$ we have $\liminf_{B \ni x \to y} H_f^B(x) \geq \liminf_{S \ni z \to y} f(z)$. In particular, if f is continuous at $y \in S$ then*

$$\lim_{B \ni x \to y} H_f^B(x) = f(y)\,.$$

(*iii*) *If $f \in L^1(S,\sigma)$ then $H_f^B \in L^1(B,\lambda)$. The operator $H^B : L^1(S,\sigma) \to L^1(B,\lambda)$ is linear, bounded, with $\|H^B\| = 1$. We have $\lambda \circ H^B = \sigma$, i.e.,*

$$\int_B H_f^B\, d\lambda = \int_S f\, d\sigma \qquad \textit{for all } f \in L^1(S,\sigma)\,.$$

Proof. We show first that

(3.1) if $z \in S$ and $\varepsilon > 0$ then $\lim_{x \to z} K(x, \widetilde{z}) = 0$ uniformly for $\widetilde{z} \in S \setminus B(z,\varepsilon)$.

We may assume that $z = 0$ (since we may translate B). If $\widetilde{z} \in S \setminus B(0,\varepsilon)$ then $|\widetilde{z}| \geq \varepsilon$ and $|x - \widetilde{z}| \geq |\widetilde{z}| - |x| \geq \varepsilon - |x|$, we have $1/|x-\widetilde{z}|^d \leq 1/(\varepsilon - |x|)^d$ for $|x| < \varepsilon$, so

$$K(x,\widetilde{z}) \leq r^{d-2}\, \frac{1}{(\varepsilon - |x|)^d}(r^2 - |x-a|^2) \underset{x \to 0}{\longrightarrow} 0\,.$$

We prove now assertion (i). Since $z = 0$ we have $|a|^2 = r^2$ and

$$K(x,0) = r^{d-2}\, \frac{r^2 - \sum_{i=1}^d (x_i - a_i)^2}{|x|^d} = r^{d-2} \left(-|x|^{2-d} + \sum_{i=1}^d 2a_i\, \frac{x_i}{|x|^d} \right)\,.$$

One can easily check that both $|x|^{2-d}$ and $x_i/|x|^d$ belong to $\mathcal{H}(\mathbb{R}^d \setminus \{0\})$ and the first assertion follows.

Fix $x \in B$. Since $K(x,\cdot) \in C(S)$ and $K(x,\cdot) > 0$ on S, there exist $\alpha, \beta > 0$ such that $\alpha < K(x,z) \leq \beta$ for all $z \in S$. Consequently we have $f \in L^1(S,\sigma)$ if and only if $fK(x,\cdot) \in L^1(S,\sigma)$. Because the function $x \mapsto K(x,z)$ is harmonic on B and the partial derivatives are continuous, we are allowed to interchange the Laplace operator with the integral. It follows that H_f^B is also harmonic on B.

From $K(a,z) = 1$ we get $H_f^B(a) = \int_S K(a,z)f(z)\sigma(dz) = \int_S f\, d\sigma$. Fix $0 < \rho < r$. By rotational invariance H_1^B is constant on $S(a,\rho)$. Indeed, if $x', x'' \in S(a,\rho)$ there exists a rotation R in $\mathbb{R}^d$ such that $R(a) = a$ and $Rx' = x''$. Then $|Rx' - a| = |x'' - a| = |x' - a| = \rho$ and $|Rx' - Ry| = |x' - y|$, if $y \in S$. Since $\sigma_{a,r}$ is rotational invariant we have

$$H_1^B(x'') = H_1^B(Rx') = r^{d-2}\int_S \frac{r^2 - |Rx' - a|^2}{|Rx' - y|^d}\sigma(dy) =$$
$$= r^{d-2}\int_S \frac{r^2 - |x' - a|^2}{|Rx' - Ry|^d}\sigma(dy) = H_1^B(x')\,.$$

Hence the harmonic function H_1^B is constant on $S(a,\rho)$, $H_1^B = \alpha_\rho$ on $S(a,\rho)$. By Proposition 3.2 we have that $H_1^B = \alpha_\rho$ on $B(a,\rho)$. In particular, $\alpha_\rho = H_1^B(a) = 1$. It follows that $H_1^B = 1$ on $B(a,\rho)$ for all $\rho < r$ and therefore $H_1^B = 1$ on B.

(ii) Step I. Let $g \geq 0$ on S and V be a neighborhood of $y \in S$ such that $g = 0$ on V and $g \in L^1(S,\sigma)$. Then $\lim_{B \ni x \to y} H_g^B(x) = 0$. Indeed, let $(x_n)_n \subset B$ with $\lim_n x_n = y$. From (3.1) we have $\lim_n K(x_n,\cdot) = 0$ on $S \setminus V$ and there exists $M > 0$ such that $|K(x_n,\cdot)g| \leq M|g| \in L^1(S,\sigma)$. By dominated convergence we get $\lim_n H_g^B(x_n) = 0$.

Step II. We may assume that $\liminf_{S \ni z \to y} f(z) > -\infty$ and let $\alpha \in \mathbb{R}$, $\alpha < \liminf_{S \ni z \to y} f(z)$. Thus, there exists a neighborhood V of y such that $f(z) > \alpha$ for all $z \in V$ and let $g := \alpha - \min(f,\alpha)$. Then $g = 0$ on V and $g \in L^1(S,\sigma)$. By Step I we have $\lim_{B \ni x \to y} H_g^B(x) = 0$. Clearly, from $f \geq \alpha - g$ we get $H_f^B \geq \alpha - H_g^B$. It follows that $\liminf_{B \ni x \to y} H_f^B(x) \geq \alpha - \lim_{B \ni x \to y} H_g^B(x) = \alpha$ for all $\alpha < \liminf_{S \ni z \to y} f(z)$. Taking a sequence $\alpha_n \nearrow \liminf_{S \ni z \to y} f(z)$ we conclude that

$$\liminf_{S \ni z \to y} f(z) \leq \liminf_{B \ni x \to y} H_f^B(x)\,.$$

One can replace f with $-f$ in the above argument and the second assertion follows.

(iii) Since $|H_f^B| \leq H_{|f|}^B$ we may assume $f \geq 0$. By assertion (i) we know that the function $h = H_f^B$ is harmonic on B. Using polar coordinates, the mean value property, and again assertion (i) we have

$$\int_B H_f^B\, d\lambda = \frac{1}{\mathrm{vol}(B)}\int_0^r \int_{S(a,\rho)} h\, d\sigma_{a,\rho}\, d\rho = H_f^B(a) = \int_S f\, d\sigma\,.$$

□

Corollary 3.5 (cf. [9], Corollary 6.4)**.** *Let $h : D \to \mathbb{R}$ be locally bounded and Borel measurable. Then the following assertions are equivalent.*

(*i*) *The function h is harmonic on D.*

(*ii*) *The function h has the mean value property, that is, for all $x \in D$ and $r > 0$ such that $\overline{B(x,r)} \subset D$ we have $h(x) = \int h\, d\sigma_{x,r}$.*

Proof. Let $B = B(x,r)$ with $\overline{B} \subset D$ and $S = S(x,r)$.

(i) $\Longrightarrow$ (ii). By Proposition 3.4 (i) $H^B_{h|_S} \in \mathcal{H}(B)$ and since $h = H^B_{h|_S}$ on S, Corollary 3.3 implies $h = H^B_h\big|_S$ on B. Using assertion (iii) of Proposition 3.4 we get $h(x) = H^B_h\big|_S(x) = \int_S h\, d\sigma$.

(ii) $\Longrightarrow$ (i). From $h(x) = \int_S h\, d\sigma_{x,r'}$ for all $r' \le r$, it follows that $h(x) = \int_B h\, d\lambda_{x,r}$ for all x. By dominated convergence we obtain that h is continuous. If we set

$$g := h|_B - H^B_h\big|_S$$

then g has the mean value property on B and $\lim_{B\ni x\to z} g(x) = 0$ for all $z \in S$. From Proposition 3.2 (Minimum principle) we get that $g = 0$ on B, that is $h = H^B_{h|_S}$ on B. It follows that $h|_B \in \mathcal{H}(B)$ for every ball B such that $\overline{B} \subset D$ and therefore $h \in \mathcal{H}(D)$.

□

Proposition 3.6 (cf. [9], Proposition 7.1)**.** *If $f \in C(\partial D)$ then $H_D f$ is harmonic on D.*

Proof. From $|H_D f| \le \|f\|_\infty < \infty$ it follows that $H_D f$ is bounded. By Corollary 3.5 it suffices to show that $H_D f$ has the mean value property. Let $x \in D$, $\overline{B(x,r)} \subset D$, $T = T_{D^c}$, $S = T_{B^c}$. Because D and B are bounded sets, by Proposition 2.3 (ii) the stopping times T and S are finite almost surely and by Lemma 3.1 (ii),

$$P^x \circ X_S^{-1} = \sigma_{x,r}\,.$$

Because $D^c \subset B^c$ and since the Brownian motion has continuous paths we have $S < T$ P^x-almost surely. By Proposition 2.8 (the strong Markov property) and the above equality we get

$$H_D f(x) = E^x(f \circ X_T) = E^x(E^{X_S}(f \circ X_T)) =$$
$$= \int E^z(f \circ X_T)\, d\sigma_{x,r}(z) = \int H_D f\, d\sigma_{x,r}\,.$$

□

A Borel measurable function $u : D \to (-\infty, +\infty]$ is called *supermedian* on D if it is locally bounded and

$$\int u\, d\sigma_{a,r} \le u(a) \qquad \text{for all } r > 0,\ \overline{B(a,r)} \subset D\,.$$

The function u is called *hyperharmonic* on D if it is supermedian and lower semicontinuous.

Remark. (i) A real-valued function h is harmonic on D if and only if h and $-h$ are supermedian on D (cf. Corollary 3.5).

(ii) The pointwise infimum of finitely many hyperharmonic functions on D is hyperharmonic.

Proposition 3.7 (cf. [9], Proposition 7.2)**.** *Let $u \ge 0$ be a lower semicontinuous function on $\overline{D}$ which is hyperharmonic on D. Then $H_D u \le u$ on D.*

Proof. Define inductively $T_n : \Omega \to [0, \infty]$, $n = 0, 1, 2, \cdots$, by $T_0 = 0$,

$$T_{n+1}(\omega) = \inf \left\{ t > T_n(\omega) \,:\, |X_t(\omega) - X_{T_n}(\omega)| \geq \frac{1}{2}\, \mathrm{dist}(X_{T_n}(\omega), \partial D) \right\},$$

if $T_n(\omega) < \infty$ and $T_{n+1}(\omega) = \infty$ if $T_n(\omega) = \infty$. Equivalently we have $T_{n+1} = T_n + S \circ T_n$, where

$$S(\omega) := \inf \left\{ t > 0 \,:\, |X_t(\omega) - X_0(\omega)| \geq \frac{1}{2}\, \mathrm{dist}(X_0(\omega), \partial D) \right\}.$$

It is easy to see that S is a stopping time, $(T_n)_n$ is an increasing sequence of stopping times, and $T_n \leq T_{D^c} < \infty$ P^x-almost surely, $x \in D$, for every n (the finiteness follows from Proposition 2.3 (ii)). We claim that P^x-almost surely

$$\lim_n T_n = T_{D^c}.$$

Indeed, fix $\omega \in \Omega$ with $X_0(\omega) \in D$ and assume that $t := \lim_n T_n(\omega) < T_{D^c}(\omega)$. Then $K := [X_s(\omega) \,:\, 0 \leq s \leq t]$ is a compact subset of D and therefore $\varepsilon := \mathrm{dist}(K, D^c) > 0$. There exists $\delta > 0$ such that

$$|X_s(\omega) - X_{s'}(\omega)| < \frac{\varepsilon}{2} \qquad \text{if } 0 \leq s \leq s' \leq t,\ s' - s < \delta.$$

By the definition of T_n and since $T_n(\omega) \in K$ for all n, it follows that $T_{n+1}(\omega) - T_n(\omega) \geq \delta$ for all $n \in \mathbb{N}$. We get $t \geq T_{N+1}(\omega) \geq N\delta$ for all $N \in \mathbb{N}$, contradiction. Therefore $t = T_{D^c}(\omega)$. Fixing $x \in D$ we obtain $\lim_n X_{T_n} = X_{T_{D^c}}$ P^x-almost surely. Since u is lower semicontinuous on $\overline{D}$ we conclude that

$$u(X_{T_{D^c}}) \leq \liminf_n u(X_{T_n}) \quad P^x\text{-almost surely}.$$

Using Fatou's lemma and since $u \geq 0$ on ∂D we have

$$H_D u(x) = E^x(u(X_{T_{D^c}})) \leq E^x(\liminf_n u(X_{T_n})) \leq \liminf_n E^x(u(X_{T_n})). \tag{3.2}$$

For $y \in D$ let $r(y) := \mathrm{dist}(y, D^c)/2$. Since the Brownian motion is normal we have P^y-almost surely

$$S(\omega) = \inf \left\{ t > 0 \,:\, |X_t(\omega) - y| \geq \frac{1}{2}\, \mathrm{dist}(y, \partial D) \right\} = T_{B(y,r(y))^c}(\omega)$$

and by Lemma 3.1 (ii)

$$E^y(u(X_S)) = E^y(u(T_{B(y,r(y))^c})) = \int u\, d\sigma_{y,r(y)} \leq u(y) \qquad \text{for every } y \in D.$$

By the strong Markov property (Proposition 2.8) and the preceding inequalities

$$E^x(u(X_{T_n})) = E^x(u(X_S) \circ \theta_{T_{n-1}}) = E^x\left(E^{X_{T_{n-1}}}(u(X_S))\right) \leq$$
$$\leq E^x\left(u(X_{T_{n-1}})\right) \qquad \text{for all } n$$

and by recurrence we obtain $E^x(u(X_{T_n})) \leq E^x(u(X_{T_0})) = E^x(u(X_0)) = u(x)$. From (3.2) we conclude that for all $x \in D$

$$H_D u(x) \leq \liminf_n E^x(u(X_{T_n})) \leq u(x).$$

□

Theorem 3.8. *Let $f \in C(\partial D)$ and assume that the classical Dirichlet problem for f admits a solution h. Then $h = H_D f$.*

Proof. Since h is bounded we may assume that $|h| \leq 1$ and therefore $|f| \leq 1$. The functions $1-h$ and $1+h$ are harmonic on D and positive on $\overline{D}$. By Proposition 3.7 (since $h|_{\partial D} \equiv f$) we have

$$H_D(1-f) = H_D(1-h) \leq 1-h \quad \text{and} \quad H_D(1+f) \leq 1+h\,.$$

Because $H_D 1 = 1$ we conclude that $H_D f = h$.

□

Remark. If $D = B(x,r) =: B$ then

$$H_f^B = H_B f \qquad \text{for all } f \in L^1(S,\sigma)\,.$$

Indeed, if f is continuous then the equality follows from Theorem 3.8 and the uniqueness of the solution of the classical Dirichlet problem.

4. Aurel Cornea's method of controlled convergence

Zaremba's example of an open set having no solution for the classical Dirichlet problem. Let D be the punctured unit ball, $D = B(0,1) \setminus \{0\}$. Then $\partial D = S \cup \{0\}$ and consider $f : \partial D \to \mathbb{R}$, $f \in C(\partial D)$, given by

$$f(y) := \begin{cases} 0 & , \quad \text{if } |y| = 1 \\ 1 & , \quad \text{if } y = 0 \end{cases}$$

Then the classical Dirichlet problem has no solution for the boundary data f. Indeed, assume that there exists a harmonic function $h : D \to \mathbb{R}$ such that

$$\lim_{D \ni x \to y} h(x) = f(y) \qquad \text{for all } y \in \partial D\,,$$

that is $\lim_{x\to y} h(x) = 0$ for all $y \in S(0,1)$ and $\lim_{x\to 0} h(x) = 1$. Let $u_0(x) := 1/|x|^{d-2}$, $d \geq 3$, and $u_n := (1/n)u_0 - h$, $n \geq 1$. Then u_n is harmonic on D for all $n \geq 0$ (because u_0 has this property, being the Newtonian kernel). Since

$$\lim_{D \ni x \to y} u_n(x) = \begin{cases} \dfrac{1}{n}\, u_0(y) & , \quad \text{if } y \in S(0,1) \\ +\infty & , \quad \text{if } y = 0 \end{cases}$$

we get $\lim_{D\ni x\to y} u_n(x) \geq 0$ for all $y \in \partial D$ and by the Minimum principle (Proposition 3.2) $u_n \geq 0$ on D for all n. Hence $(1/n)u_0 \geq h$ on D for all $n \geq 1$ and letting $n \to \infty$ we have $h \leq 0$ on D, contradicting the relation $\lim_{x\to 0} h(x) = 1$.

Remark. Although the classical Dirichlet problem has no solution, if we set $h_1 := H_f^B\Big|_D$ then h_1 is a harmonic function on D such that

$$\lim_{D \ni x \to y} h_1(x) = f(y) \qquad \text{for all } y \in \partial D \setminus \{0\}\,.$$

Controlled convergence. Let $D \subset \mathbb{R}^d$ be a bounded open set, $f : \partial D \to \mathbb{R}$ and $h, k : D \to \overline{\mathbb{R}}$, $k \geq 0$.

The function h *converges to* f *controlled by* k (we write $h \xrightarrow{k} f$) if the following conditions hold.

For all $(x_n)_n \subset D$, $x_n \to y \in \partial D$, we have:

($*$) If k is bounded on the sequence $(x_n)_n$ then $f(y) \in \mathbb{R}$ and $\lim_n h(x_n) = f(y)$.

$(**)$ If $(k(x_n))_n$ is unbounded then $\inf_n \dfrac{h(x_n)}{1+k(x_n)} = 0$.

Remark. (i) According to [6] and [7], the controlled convergence offers a method for setting and solving the Dirichlet problem for general open sets and general boundary data. The above function f should be interpreted as being the boundary data of the harmonic function h. The function k is called *control function*.

(ii) The function h converges to f controlled by k if and only if the following conditions hold.
For all $(x_n)_n \in D$, $x_n \to y \in \partial D$ with $k(x_n) \to k_o (\in \overline{\mathbb{R}}_+)$ we have:
$(\overline{*})$ If $k_o \in \mathbb{R}$ then $f(y) \in \mathbb{R}$ and $\lim_n h(x_n) = f(y)$.
$(\overline{**})$ If $k_o = +\infty$ then $\lim_n \dfrac{h(x_n)}{1+k(x_n)} = 0$.
Indeed, we clearly have that $(*)$ and $(**)$ imply $(\overline{*})$ and $(\overline{**})$. By passing to subsequences one can see that conditions $(\overline{*})$ and $(\overline{**})$ also imply $(*)$ and $(**)$.

A function $f : \partial D \to \overline{\mathbb{R}}$ is called *resolutive* provided that there exists $h \in \mathcal{H}(D)$ and a superharmonic function k on D, positive and continuous, such that $h \overset{k}{\to} f$. The function h is called the *solution of the Dirichlet problem with boundary data f*.

Remark 4.1. A harmonic function h on D is the solution for the classical Dirichlet problem with boundary data f if and only if h converges to f controlled by a bounded function k.

Theorem 4.2 ([7], Theorem 1.5)**.** *Let $f : \partial D \to \overline{\mathbb{R}}$, $h : D \to \mathbb{R}_+$, and $k : D \to \mathbb{R}_+$. Then the following assertions are equivalent.*

(*i*) *The function h converges to f controlled by k.*
(*ii*) *For any $y \in \partial D$ we have:*
(*ii**) *If $\liminf_{D\ni x\to y} k(x) < +\infty$ then $f(y) \in \mathbb{R}$ and $\lim_{D\ni x\to y} \dfrac{h(x)-f(y)}{1+k(x)} = 0$.*
(*ii***) *If $\lim_{D\ni x\to y} k(x) = +\infty$ then $\lim_{x\to y} \dfrac{h(x)}{1+k(x)} = 0$.*
(*iii*) *For every $\varepsilon > 0$ and $y \in \partial D$*

$$+\infty \neq \limsup_{D\ni x\to y} (h(x) - \varepsilon k(x)) \leq f(y) \leq \liminf_{D\ni x\to y} (h(x) + \varepsilon k(x)) \neq -\infty\,.$$

Corollary 4.3. *If the Dirichlet problem has a solution then it is unique.*

Proof. Let h, h' be two solutions, $h \overset{k}{\to} f$ and $h' \overset{k'}{\to} f$. Then by Theorem 4.2

$$\liminf_{x\to y} (h(x) + \varepsilon k(x)) \geq f(y) \geq \limsup_{x\to y} (h'(x) - \varepsilon k'(x))\,,$$

so

$$\liminf_{x\to y} (h + \varepsilon k + \varepsilon k' - h')(x) \geq \liminf_{x\to y} (h+\varepsilon k)(x) + \liminf_{x\to y} (\varepsilon k' - h')(x) \geq 0\,.$$

By Proposition 3.2 (Minimum principle) we have $h - h' + \varepsilon(k + k') \geq 0$ and letting ε tend to zero we get $h \geq h'$, $h = h'$. □

Theorem 4.4. *Every continuous function on ∂D is resolutive.*

The case of the Euclidean ball. Let $a \in \mathbb{R}^d$, $r > 0$, $B = B(a, r)$, $S = S(a, r)$, and $\sigma = \sigma_{a,r}$.

Theorem 4.5 (A. Cornea)**.** *Every $f \in L^1(S, \sigma)$ is resolutive. More precisely, there exists $k : B \to \mathbb{R}$ harmonic and positive such that for all $(x_n)_n \subset B$, $x_n \to y \in S$, we have:*

(*i*) *If k is bounded on $(x_n)_n$ then $f(y) \in \mathbb{R}$ and $H_f^B(x_n) \to f(y)$.*

(*ii*) *If $(k(x_n))_n$ is unbounded then $\inf_n |H_f^B(x_n)|/(1 + k(x_n)) = 0$.*

Proof. There exists a decreasing sequence $(f_n)_n$ of lower semicontinuous, lower bounded functions, $f_n : S \to \overline{\mathbb{R}}$, $f_n \geq f$ for all n, such that $\lim_n \int_S f_n \, d\sigma = \int_S f \, d\sigma$. We may assume that

$$\left| \int_S f_n \, d\sigma - \int_S f \, d\sigma \right| \leq \frac{1}{2^n} \qquad \text{for all } n \, .$$

Analogously, there exist an increasing sequence $(g_n)_n$ of upper semicontinuous, upper bounded functions, such that $g_n \leq f$ and

$$\left| \int_S f \, d\sigma - \int_S g_n \, d\sigma \right| \leq \frac{1}{2^n} \qquad \text{for all } n \, .$$

Let us set

$$g := \sum_n (f_n - g_n) \qquad \text{and} \quad k := H_g^B \, .$$

Since $g_n \leq f \leq f_n$ it follows that $g \geq 0$ and by the above inequalities we deduce that $g \in L^1(S, \sigma)$, $\int_S g \, d\sigma \leq 2$. Let $\varepsilon > 0$ and fix n such that $1/n \leq \varepsilon$. The function k is harmonic as the sum of a series of harmonic functions (the mean value property passes over the sum) which is finite in one point (Harnack inequality). Let $\phi := (1/n) \sum_{j=1}^n f_j$, The function ϕ is lower semicontinuous, lower bounded, and from $f_j \geq f$ for all j it follows that $\phi \geq f$. In addition we have

$$H_f^B + \varepsilon k \geq H_\phi^B \, . \tag{4.1}$$

Indeed

$$H_f^B + \varepsilon k \geq H_f^B + \frac{1}{n} \, k \geq H_f^B + \frac{1}{n} \sum_{j \leq n} H^B_{(f_j - g_j)} = H_\phi^B + H^B_{(f - (1/n) \sum_{j \leq n} g_j)} \geq H_\phi^B \, .$$

We check now conditions (i) and (ii). Let $(x_n)_n \subset B$, $x_n \to y \in S$, and $M := \sup_n k(x_n)$.

If $M < \infty$ then the sequence $(k(x_n))_n$ is bounded. We prove that $f(y) \in \mathbb{R}$. Indeed, $M < \infty$ implies that $\liminf_n \, k(x_n) < \infty$ and therefore $\liminf_{B \ni x \to y} \, k(x) < \infty$. The function g is lower semicontinuous on S, hence $g(y) \leq \liminf_{S \ni z \to y} \, g(z)$ and by the boundary behavior of the Poisson integral (Proposition 3.4 (ii)) we have

$$g(y) \leq \liminf_{x \to y} \, H_g^B(x) = \liminf_{B \ni x \to y} \, k(x) < \infty \, .$$

Hence $\sum_n (f_n - g_n)(y) < \infty$ and in particular $f_1(y), g_1(y) \in \mathbb{R}$. But $g_1(y) \leq f(y) \leq f_1(y)$, so $f(y) \in \mathbb{R}$. We show now that $\lim_m \, H_f^B(x_m) = f(y)$. As before, since the function ϕ is lower semicontinuous, using again Proposition 3.4 (ii) we obtain

$$\liminf_{B \ni x \to y} \, H_\phi^B(x) \geq \phi(y) \geq f(y) \, .$$

On the other hand by (4.1)

$$H_f^B(x_m) + \varepsilon M \geq H_f^B(x_m) + \varepsilon k(x_m) \geq H_\phi^B(x_m) \qquad \text{for all } m \, .$$

It follows that $\liminf_m H_f^B(x_m) \geq f(y)$. Analogously, applying the same procedure to $-f$ we have $-f_n \leq -f \leq -g_n$ for all n and observe that $\sum_n(-g_n - (-f_n)) = g$. So, $\liminf_m H_{-f}^B(x_m) \geq -f(y)$, or equivalently, $f(y) \geq \limsup_m H_f^B(x_m)$ and we conclude that $\lim_m H_f^B(x_m) = f(y)$.

If $M = \infty$ then the sequence $(k(x_n))_n$ is unbounded. Passing to a subsequence we may assume that there exists $\lim_n k(x_n) = +\infty$ (actually we check condition ($\overline{**}$)). Using (4.1) we have

$$\frac{H_f^B(x_n)}{1+k(x_n)} = \frac{H_f^B(x_n) + \varepsilon k(x_n)}{1+k(x_n)} - \varepsilon\,\frac{k(x_n)}{1+k(x_n)} \geq \frac{H_\phi^B(x_n)}{1+k(x_n)} - \varepsilon \geq \frac{\alpha}{1+k(x_n)} - \varepsilon$$

for all n, where $\alpha \in \mathbb{R}$ is such that $\phi \geq \alpha$.
It follows that $\liminf_n H_f^B(x_n)/(1+k(x_n)) \geq -\varepsilon$ for all $\varepsilon > 0$, so $\liminf_n H_f^B(x_n)/(1+k(x_n)) \geq 0$. Applying the above procedure for $-f$ we get $0 \leq \liminf_n H_{-f}^B(x_n)/(1+k(x_n)) = -\limsup_n H_f^B(x_n)/(1+k(x_n))$. Therefore $\limsup_n H_f^B(x_n)/(1+k(x_n)) \leq 0$ and we conclude that $\lim_n H_f^B(x_n)/(1+k(x_n)) = 0$, hence ($\overline{**}$) holds.

□

Let $f \in L^1(S,\sigma)$. A point $y \in S$ is called *exceptional* if it does not hold $\lim_{B\ni x\to y} H_f^B(x) = f(y)$.

Remark. If $y \in S$ is exceptional, then by Theorem 4.5 $\liminf_{x\to y} k(x) = +\infty$.

Let $E := \{y \in S : \liminf_{x\to y} k(x) = +\infty\}$.

Proposition 4.6. *The set E is σ-null (i.e. $\sigma(E) = 0$).*

Proof. Let $E_n := \{y \in S : \liminf_{x\to y} k(x) > n\}$. Then E_n is open and $E \subset E_n$ for all n. Let $\varphi \in \mathcal{C}(S)$ be such that $\operatorname{supp}\varphi \subset E_n$, $0 \leq \varphi \leq 1$ and consider the harmonic function $h := k - n\,H_\varphi^B$. We show that

$$\liminf_{B\ni x\to y} h(x) \geq 0 \qquad \text{for all } y \in S\,.$$

Indeed, $\liminf_{B\ni x\to y} h(x) \geq \liminf_{B\ni x\to y} k(x) - \lim_{B\ni x\to y} n\,H_\varphi^B(x) =$
$= \liminf_{B\ni x\to y} k(x) - n\,\varphi(y)$.
If $y \notin E_n$, then $\varphi(y) = 0$, so $\liminf_{x\to y} h(x) \geq 0$. If $y \in E_n$, since $\liminf_{B\ni x\to y} k(x) \geq n$, it follows that $\liminf_n k(x_n) - n\,\varphi(y) \geq n - n\,\varphi(y) \geq 0$.

By the Minimum principle (Proposition 3.2) we get now $h \geq 0$ on B, hence $H_\varphi^B \leq k/n$ on B for all n. In particular, using Proposition 3.4 (iii), $\int_S \varphi\, d\sigma = H_\varphi^B(0) \leq k(0)/n$ for all $\varphi \in C(S)$, $\operatorname{supp}\varphi \subset E_n$. Because there exists a sequence $(\varphi_k)_k \subset C(S)$ such that $\varphi_k \nearrow \mathbf{1}_{E_n}$, we deduce that $\sigma(E_n) \leq k(0)/n$ for all n, $\sigma(E) = 0$.

□

Controlled convergence outside an exceptional set (cf. Section 5 in [2]). Let $f \in \partial D \to \overline{\mathbb{R}}$, $D_o \subset D$, and $h,\ k : D \to \overline{\mathbb{R}}$, $k \geq 0$, such that $h|_{D_o}$, $k|_{D_o}$ are real-valued. We say that h *converges to f controlled by k on D_o* if for every $A \subset D_o$ and $y \in \partial D \cap \bar{A}$ the following conditions hold:

(C1) If $\limsup_{A\ni x\to y} k(x) < \infty$ then $f(y) \in \mathbb{R}$ and $f(y) = \lim_{A\ni x\to y} h(x)$
(C2) If $\lim_{A\ni x\to y} k(x) = +\infty$ then $\lim_{A\ni x\to y} h(x)/(1+k(x)) = 0$.

In [6] and [7] it was considered only the case $D_o = D$ for the controlled convergence. One can see that h converges to f controlled by k if and and only if h converges to f controlled by k on D.

The following result is a version of Proposition 1.7 from [7].

Proposition 4.7. *Let be $h_n \overset{k}{\to} f_n$ for all $n > 0$ on D_o. Let $(\alpha_n)_n \subset \mathbb{R}$, $\alpha_n \nearrow +\infty$ such that*

$$l := \sum_n \alpha_n |h_n| < \infty$$

and the series $\sum_n f_n$ converges on D_o. Then $\sum_n h_n$ converges to $\sum_n f_n$ controlled by $k + l$ on D_o.

Recall that a subset M of D is called *polar* provided that P^x-a.s. one has $T_M = \infty$ for all $x \in D$. Note that a Borel subset M of D is polar if and only if there exists a hyperharmonic function u on D which is finite on a dense subset of D, such that $M = [u = \infty]$.

The next result shows that the stochastic solution solves the Dirichlet problem with general boundary data; it is a special case of Theorem 5.3 and Remark 5.4 (ii) from [2].

Theorem 4.8. *Let $D \subset \mathbb{R}^d$ be a bounded domain, λ a finite measure on D, $\sigma := \lambda \circ H_D$, such that the classical Dirichlet problem has a solution. Let $f \in L^1(\partial D, \sigma)$, then there exists a polar set $M \subset D$ and $g \in p\mathcal{B}(\partial D)$, such that $k := H_D g \in L^1_+(D, \lambda)$ and $H_D f$ converges to f controlled by k on $D \setminus M$.*

Proof. (Cf. the proof of Theorem 5.3 from [2]). Let $\mathcal{M}$ be the set of all functions $f \in L^1_+(\partial D, \sigma)$ such that there exists $g \in p\mathcal{B}(\partial D)$ and $H_D f \overset{k}{\to} f$ on $[k < \infty]$, where $k := H_D g \in L^1(D, \lambda)$. Since for every $f \in C(\partial D)$ the classical Dirichlet problem has a solution and by Theorem 3.8 the solution is precisely $H_D f$, from Remark 4.1 we get $C_+(\partial D) \subset \mathcal{M}$.

We claim that it is sufficient to prove that:

$$\text{(4.2)} \qquad \text{if } (f_n)_n \subset \mathcal{M},\ f_n \nearrow f \in L^1(\partial D, \sigma), \text{ then } f \in \mathcal{M}.$$

Indeed, we apply first the monotone class theorem (cf. (A.2) in Appendix) for $\mathcal{H} = b\mathcal{B}(\partial D)$ and $\mathcal{K} = C_+(\partial D)$. It follows that $b\mathcal{B}(\partial D) \subset \mathcal{M}$. Let now $f \in L^1(\partial D, \sigma)_+$. From the above considerations the sequence $(f \wedge n)_n$ lies in $\mathcal{M}$ and applying again (4.2) we conclude that $f \in \mathcal{M}$. Hence $\mathcal{M} = L^1(\partial D, \sigma)$. Note that since the control function k is hyperharmonic on D, it belongs to $L^1(D, \lambda)$, and D is a domain, it follows that the set $[k = \infty]$ is polar.

To prove (4.2) let $(f_n)_n \subset \mathcal{M}$, $f_n \nearrow f \in L^1(\partial D, \sigma)$, and set $h_n := H_D f_n$ and $h := H_D f$. Then $h_n \nearrow h \in L^1(D, \lambda)$. By hypothesis $h_n \overset{k_n}{\to} f_n$ on $[k_n < \infty]$ for all n. We may assume that $\lambda(k_n) = 1$ for all n and define $k_o := \sum_n k_n / 2^n$. It follows that $h_n \overset{k_0}{\to} f_n$ on $[k_o < \infty]$ for all n. Let

$$l := \sum_{n \geq 1} n(h_{n+1} - h_n) = \sum_{n \geq 1} (h - h_n)\,.$$

From $h_n \nearrow h$ we have $\lambda(h_n) \nearrow \lambda(h) < \infty$. Passing to a subsequence we may assume that $\sum_n (\lambda(h) - \lambda(h_n)) < \infty$. Consequently we get $l \in L^1_+(D, \lambda)$ and $l = H_D g$ with $g \in p\mathcal{B}(\partial D)$. By Proposition 4.7 we conclude that $h \overset{k_o + l}{\longrightarrow} f$ on

$[k_o + l < \infty]$, so $f \in \mathcal{M}$ and therefore (4.2) holds, completing the proof.

$\square$

Remark. As it was mentioned in Remark 5.4 from [2], the result from Theorem 4.8 holds in a more general setting, e.g., for a path continuous Borel right process (see [1] and [11] for details on this type of Markov processes), provided that $H_D f$ solves the classical Dirichlet problem with boundary data $f \in C(\partial D)$.

Acknowledgement. Support from the Romanian Ministry of Education, Research, Youth and Sport (CNCS, PCCE-55/2008) is gratefully acknowledged.

Appendix

(A.1). **Proof of Remark 1.1.** Let $y_n,\ y \in \partial D$ such that $\lim_n y_n = y$. Since $\lim_{\substack{x \to y_n \\ x\in D}} h(x) = f(y_n)$, there exists $(x_n)_n \subset D$ such that

$$|h(x_n) - f(y_n)| < \frac{1}{n} \quad , \quad |x_n - y_n| < \frac{1}{n} \quad \text{for all } \ n\,.$$

So, $|f(y_n)| \le 1/n + |h(x_n)|$ and we obtain that the sequence $(f(y_n))_n$ is bounded. Passing to a subsequence we may assume that the limit $\lim_n f(y_n)$ exists and it is a real number α. Because $\lim_n y_n = y$ we have $\lim_n x_n = y$, $\lim_n h(x_n) = f(y)$. Therefore

$$|h(x_n) - \alpha| \le |h(x_n) - f(y_n)| + |f(y_n) - \alpha| \le \frac{1}{n} + |f(y_n) - \alpha|\,,$$
$$f(y) = \lim_n \ h(x_n) = \alpha = \lim_n \ f(y_n)\,.$$

(A.2). **The monotone class theorem.** Cf. [11], (A0.6). Let $\mathcal{H}$ be a vector space of bounded, real-valued functions such that $1 \in \mathcal{H}$ and if $(f_n)_n \subset \mathcal{H}_+,\ f_n \nearrow f$, f bounded, then $f \in \mathcal{H}$. If $\mathcal{K} \subset \mathcal{H}$ is multiplicative (i.e., $fg \in \mathcal{K}$ provided that $f,\ g \in \mathcal{K}$), then $b\sigma(\mathcal{K}) \subset \mathcal{H}$.

(A.3). **Conditional expectation.** Let $(\Omega, \mathcal{F}, P)$ be a *probability space*, i.e., $\mathcal{F}$ is a σ-algebra of subsets of Ω (that is: $\emptyset \in \mathcal{F}$, $A^c \in \mathcal{F}$ provide that $A \in \mathcal{F}$, and if $(A_n)_n \subset \mathcal{F}$ then $\bigcup_n A_n \in \mathcal{F}$) and P is a probability on the measurable space $(\Omega, \mathcal{F})$. Let $(E, \mathcal{B})$ be a second measurable space.

A map $X : \Omega \to (E, \mathcal{B})$ is called *random variable* provided that $X \in \mathcal{F}/\mathcal{B}$, i.e., $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}$.

If X is a numerical valued random variable such that $X^- \in L^1(\Omega, P)$, then *the expectation of* X is defined by $E(X) := \int_\Omega X\, dP$, we write $E(X; \Lambda) := \int_\Lambda X\, dP$ if $\Lambda \in \mathcal{F}$.

Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and $X : \Omega \to \overline{\mathbb{R}}$ such that $X^- \in L^1(\Omega, P)$. *The conditional expectation of* X *with respect to* $\mathcal{G}$ is a function $f : \Omega \to \overline{\mathbb{R}}$ which is $\mathcal{G}/\mathcal{B}(\overline{\mathbb{R}})$-measurable and $E(f; \Lambda) = E(X; \Lambda)$ for all $\Lambda \in \mathcal{G}$.

• *For every* $X : \Omega \to \overline{\mathbb{R}}$ *such that* $X^- \in L^1(\Omega, \mathcal{F}, P)$, *the conditional expectation* $E[X|\mathcal{G}]$ *exists and it is uniquely determined almost surely* (*abbreviated a.s.*) *and we denote it by* $E[X|\mathcal{G}]$. For, assume that $X \in L^1(\Omega, P)$. We may define the real-valued measure μ on $(\Omega, \mathcal{G})$ as $\mu(\Lambda) := \int_\Lambda X\, dP$, $\Lambda \in \mathcal{G}$. It follows that μ is absolutely continuous with respect to the restriction of P to $\mathcal{G}$ and by the Radon-Nikodym theorem there exists $f \in L^1(\Omega, \mathcal{G}, P|_\mathcal{G})$ such that $\mu(\Lambda) = \int_\Lambda f\, dP$ for all $\Lambda \in \mathcal{G}$. Hence $f = E[X|\mathcal{G}]$.

(A.3.1). Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. A real-valued random variable X is $\mathcal{G}$-measurable if and only if $E[X|\mathcal{G}] = X$. In addition we have $E(E[X|\mathcal{G}]) = E(X)$. If $\mathcal{G}'$ is a second sub-σ-algebra, $\mathcal{G}' \subset \mathcal{G}$, then

$$E[E[X|\mathcal{G}]|\mathcal{G}'] = E[X|\mathcal{G}'] \,.$$

(A.4). **Kernel.** A *kernel* on $(E, \mathcal{B})$ is a map $N : p\mathcal{B} \to p\mathcal{B}$ such that $N0 = 0$ and for each sequence $(f_n)_n \subset p\mathcal{B}$ we have $N(\sum_n f_n) = \sum_n N f_n$.

The kernel N is called *Markovian* (respectively *sub-Markovian*) provided that $N1 = 1$ (resp. $N1 \leq 1$).

(i) If N is a kernel on $(E, \mathcal{B})$ then for each $x \in E$ the map $f \mapsto Nf(x)$ is a measure on $(E, \mathcal{B})$ denoted by N_x or $N(x, \cdot)$, i.e: $Nf(x) = \int f \, dN_x$, $N(1_A)(x) = N(x, A) = N_x(A)$ for all $A \in \mathcal{B}$.

(ii) If $(N_k)_k$ is a sequence of kernels then $\sum_k N_k$ is a kernel and $N_1 \circ N_2$ is also a kernel on $(E, \mathcal{B})$.

Example. If $g : E \times E \to \overline{\mathbb{R}}_+$ is $\mathcal{B} \times \mathcal{B}$-measurable and λ is a σ-finite measure on $(E, \mathcal{B})$, then we define

$$Gf(x) := \int_E g(x, y) f(y) \lambda(dy) \qquad \text{for all } \ x \in E, \ f \in p\mathcal{B} \,.$$

By Fubini theorem it follows that G is a kernel on $(E, \mathcal{B})$.

Let $E = \mathbb{R}^d$, λ be the Lebesgue measure on $\mathbb{R}^d$, $v : \mathbb{R}^d \to \overline{\mathbb{R}}_+$ a Borel measurable function, and set $g(x, y) := v(x - y)$. Then

$$Gf(x) = \int_{\mathbb{R}^d} f(y) v(x - y) \lambda(dy) = v * f(x) \quad , \quad x \in \mathbb{R}^d \,.$$

G is called *convolution kernel*.

(A.5). **Proofs of several results from Section 2.**

Proof of Proposition 2.5 (cf. the proof of Proposition 5.1 from [9]). By the monotone class theorem (see (A.2) in Appendix), we may assume without loss of generality that f is continuous with compact support.

Let $S_n : \Omega \to D_n$, $n \in \mathbb{N}$, defined by

$$S_n(\omega) := \begin{cases} i2^{-n} & \text{on } \ [(i-1)2^{-n} \leq S < i2^{-n}], \ i \in \mathbb{N} \,, \\ \infty & \text{on } \ [S = \infty] \end{cases}$$

and $D_n := \{i2^{-n}; i = 0, 1, \cdots\} \cup \{\infty\}$. It is easily seen that $S_n \searrow S$ as $n \nearrow \infty$. S_n is a stopping time because if $0 < t \leq 2^{-n}$ then $[S_n < t] = \emptyset$, and if $2^{-n} < t < \infty$ and i is the largest natural number such that $i2^{-n} < t$, then $[S_n < t] = [S < i2^{-n}] \in \mathcal{F}_{i2^{-n}} \subset \mathcal{F}_t$. The sequence $(S_n)_n$ is called the *canonical approximating sequence for* S.

Fix $n \in \mathbb{N}$ and for $i \in \mathbb{N}$ define $A_i := [S_n = i2^{-n}]$ and $s_i := i2^{-n}$. We know that $A_i \in \mathcal{F}_{s_i}$ for all $i \in \mathbb{N}$. By (2.5) (the Markov property)

$$E^x(f \circ X_{S_n + t} \,; \, S < \infty) = \sum_{i=1}^{\infty} E^x(f \circ X_{s_i + t} \,; A_i) = \sum_{i=1}^{\infty} E^x(E^{X_{s_i}}(f \circ X_t); A_i) =$$

$$= E^x(E^{X_{S_n}}(f \circ X_t); S < \infty) = E^x(P_t f(X_{S_n}); S < \infty) \,.$$

By Proposition 2.1 (ii) we know that $P_t f$ is a continuous function on $\mathbb{R}^d$. Letting n tend to infinity and since $X_{S_n} \to X_S$ (because $t \mapsto X_t(\omega)$ is continuous), the claimed relation holds.

$\square$

Proof of Proposition 2.6 (cf. the proof of Proposition 3.3 from [9]). Let $(S_n)_n$ be the canonical approximation sequence for S (from the proof of Proposition 2.5). Fix $n \in \mathbb{N}$, $t > 0$, $A \in \mathcal{B}(\mathbb{R}^d)$, and let $m \in \mathbb{N}$ be maximal such that $ms^{-n} < t$. Then $[X_{S_n} \in A] \cap [S_n < t] = \bigcup_{i=1}^{m} [(i-1)2^{-n} \le S < i\,2^{-n}, X_{i\,2^{-n}} \in B] \in \mathcal{F}_t$. Consequently for all n X_{S_n} is $\mathcal{F}^+_{S_n}$-measurable on $[S < \infty]$ and therefore $X_S = \lim_n X_{S_n}$ is also $\mathcal{F}^+_{S_n}$-measurable on $[S < \infty]$ for all n. By Proposition 2.4 (v) we conclude X_S is $\mathcal{F}^+_S$-measurable.

$\square$

Proof of Corollary 2.7 (cf. the proof of Corollary 5.2 from [9]). Clearly, it is sufficient to consider the case $Z = 1_A$, $A \in \mathcal{F}^+_S$. Define

$$\widetilde{S} = \begin{cases} S & \text{on } A \\ \infty & \text{on } A^c \,. \end{cases}$$

Then $\widetilde{S}$ is a stopping time, since for every $t > 0$ $[\widetilde{S} < t] = A \cap [S < t] \in \mathcal{F}_t$. By the strong Markov property (Proposition 2.5) we have

$$E^x(Z \cdot f \circ X_{S+t}; S < \infty) = E^x(f \circ X_{\widetilde{S}+t}; \widetilde{S} < \infty) =$$
$$= E^x(E^{X_{\tilde{S}}}(f \circ X_t); \widetilde{S} < \infty) = E^x(Z \cdot E^{X_S}(f \circ X_t); S < \infty) \,.$$

$\square$

Proof of Proposition 2.8 (cf. the proof of Proposition 5.3 from [9]). By the monotone class theorem (see (A.2)) it suffices to consider $Y := \prod_{j=1}^{n} f_j \circ X_{t_j}$, where $f_1, \cdots, f_n \in bp\mathcal{B}(\mathbb{R}^d)$ and $0 \le t_1 < \cdots < t_n < \infty$. We argue by induction on n. For $n = 1$ the equality holds by the strong Markov property (Proposition 2.5). Suppose that the statement holds for some n, then by Proposition 2.4 (i) $S + t_n$ is a stopping time. We know from Proposition 2.6 that X_S is $\mathcal{F}^+_S$-measurable and we have $X_t \circ \theta_S = X_{S+t}$ on $[S < \infty]$, $t \ge 0$. It follows that

$$f_j \circ X_{t_j} \circ \theta_S = f_j \circ X_{S+t_j} \in bp\mathcal{F}^+_{S+t_j} \subset bp\mathcal{F}^+_{S+t_n} \quad , \quad 1 \le j \le n \,.$$

By Corollary 2.7

$$E^x(\prod_{j=1}^{n+1} f_j \circ X_{t_j} \circ \theta_S) = E^x(\prod_{j=1}^{n} f_j \circ X_{S+t_j} \cdot f_{n+1} \circ X_{t_{n+1}-t_n} \circ \theta_{S+t_n}) =$$

$$= E^x(\prod_{j=1}^{n} f_j \circ X_{S+t_j} \cdot E^{X_{S+t_n}}(f_{n+1} \circ X_{t_{n+1}-t_n})) \,.$$

The last term has the form $E^x(\prod_{j=1}^{n} g_j \circ X_{t_j} \circ \theta_S)$ with $g_j = f_j$ for every $1 \le j \le n-1$ and $g_n(y) = f_n(y) E^y(f_{n+1} \circ X_{t_{n+1}-t_n})$. Therefore, by the induction hypothesis and using (2.5) (the Markov property) we have:

$$E^x(\prod_{j=1}^{n} f_j \circ X_{t_j} \circ \theta_S) = E^x(E^{X_S} \prod_{j=1}^{n} g_j \circ X_{t_j}) =$$

$$E^x(E^{X_S}(\prod_{j=1}^{n} f_j \circ X_{t_j} \cdot E^{X_{t_n}}(f_{n+1} \circ X_{t_{n+1}-t_n}))) = E^x(E^{X_S}(\prod_{j=1}^{n+1} f_j \circ X_{t_j})).$$

□

Proof of Theorem 2.9 (cf. the proof of Theorem 5.4 from [9]). We argue as in the proof of Corollary 2.7. Consider the case $Z = 1_A$, $A \in \mathcal{F}_S^+$, and by Proposition 2.8

$$E^x(Z \cdot Y \circ \theta_S; S < \infty) = E^x(Y \circ \theta_{\widetilde{S}}; \widetilde{S} < \infty) =$$
$$= E^x(E^{X_{\tilde{S}}}(Y); \widetilde{S} < \infty) = E^x(Z \cdot E^{X_S}(Y); S < \infty).$$

□

References

[1] L. Beznea & M. Boboc, *Potential theory and right processes*, Springer Series, *Mathematics and Its Applications*, 572, Kluwer, Dordrecht, 2004.

[2] L. Beznea, A. Cornea & M. Röckner, *Potential theory of infinite dimensional Lévy processes*, J. Funct. Anal., 261(2011), 2845–2876.

[3] J. Bliedtner & W. Hansen, *Potential theory - An analytic and probabilistic approach to balayage*, Springer Verlag, Berlin, 1986.

[4] R.M. Blumenthal & R.K. Getoor, *Markov processes and potential theory*, Academic Press, New York, 1968.

[5] C. Constantinescu & A. Cornea, *Potential theory on harmonic spaces*, Springer Verlag, 1972.

[6] A. Cornea, em Résolution du problème de Dirichlet et comportement des solutions à la frontière à l'aide des fonctions de contrôle, C. R. Acad. Sci. Paris, Sér. I Math., 320(1995), 159–164.

[7] A. Cornea, *Applications of controlled convergence in analysis*, *Analysis and Topology*, World Sci. Publishing, 1998, 257–275.

[8] C. Dellacherie, & P.A. Meyer, *Probabilités et potentiel*, Chapters I–IV, IX–XI, XII–XVI, Hermann, Paris, 1975, 1983, 1987.

[9] W. Hansen, *Brownian motion and potential theory*, Lecture Notes, Prague, 2005; `http://www.karlin.mff.cuni.cz/ hansen/lecture/praha05.pdf` .

[10] J. Lukeš, J. Malý, I. Netuka & J. Spurný, *Integral representation theory: applications to convexity, Banach spaces and potential theory*, de Gruyter, 2010.

[11] M. Sharpe, *General theory of Markov processes*, Academic Press, Boston, 1988.