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Interlacing properties of Laguerre zeros and some applications. A survey

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Abstract[1]. Let $w_\alpha(x) = e^{-x^\beta} x^\alpha$, $\alpha > -1$, $\beta > 1/2$ be a Generalized Laguerre weight, and denote by $\{p_m(w_\alpha)\}_m$ the corresponding sequence of orthonormal polynomials. The starting point is that the polynomial $Q_{2m+1} = p_{m+1}(w_\alpha)p_m(w_{\alpha+1})$ has simple zeros and also well distributed in some sense. In view of this property two different applications are described: the *extended interpolation polynomial* $L_{2m+2}(w_\alpha, w_{\alpha+1}, f)$, defined as the Lagrange polynomial interpolating a given function f at the zeros of Q_{2m+1} and on the extra points a_m, being a_m the Maskar-Rackmanoff-Saff number w.r.t. w_α. For this process will be estimati the Lebesgue constants in some weighted uniform spaces [31]. The second application deals with the approximation of the Hilbert transform

$$\int_0^{+\infty} \frac{f(x)w_\alpha(x)}{x-t}\,dx \quad , \quad t > 0\,,$$

by a suitable Lagrange interpolating polynomial [32].

1. Introduction

This survey deals with some applications of the interlacing property of the zeros of Generalized Laguerre polynomials. Let $w_\alpha(x) = e^{-x^\beta} x^\alpha$, $\alpha > -1$, $\beta > 1/2$ and $w_{\alpha+1}(x) = x w_\alpha(x)$ two Generalized Laguerre weights, and let $\{p_m(w)\}_m$, $\{p_m(w_{\alpha+1})\}_m$ be the corresponding sequences of orthonormal polynomials. In [31] it was proved that the zeros of $p_m(w_{\alpha+1})$ interlace those of $p_{m+1}(w_\alpha)$ and that the zeros of the polynomial $Q_{2m+1} := p_m(w_{\alpha+1})p_{m+1}(w_\alpha)$ are sufficiently far among them. This property allows to different applications. The first is the approximation of functions f by *extended interpolating polynomials*, i.e. by the Lagrange polynomial $\mathcal{L}_{2m+1}(w_\alpha, w_{\alpha+1}, f)$ interpolating f at the zeros $\{z_i\}_{i=1}^{2m+1}$ of Q_{2m+1}

$$\mathcal{L}_{2m+1}(w_\alpha, w_{\alpha+1}, f; z_i) = f(z_i), \quad i = 1, 2, \ldots, 2m+1\,.$$

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Since $\mathcal{L}_{2m+1}(w_\alpha, w_{\alpha+1}, f)$ can be expressed as

$$\mathcal{L}_{2m+1}(w_\alpha, w_{\alpha+1}, f) = p_m(w_{\alpha+1})\mathcal{L}_{m+1}\left(w_\alpha, \frac{f}{p_m(w_{\alpha+1})}\right) +$$

$$+p_{m+1}(w_\alpha)\,\mathcal{L}_m\left(w_{\alpha+1}, \frac{f}{p_{m+1}(w_\alpha)}\right),$$

extended interpolation can be used to "extend" a previous interpolation process, for instance $\mathcal{L}_{m+1}(w_\alpha, f)$, reusing the previous $m+1$ function evaluations and obtaining a new interpolating polynomial of degree $2m+1$ with only new $m+1$ computations of f. A second advantage produced by extended processes, deals with the construction of "high" degree Lagrange polynomials by two half-degree Lagrange polynomials. By this way some difficulties in computing the zeros of "large" degree orthogonal polynomials are shifted.

With $\sigma_\delta(x) = e^{-x^\beta}x^\delta$, $\delta \geq 0$, let C_σ be the space of functions f such that $f\sigma_\delta$ is continuous in $[0, \infty)$. Functions in C_σ can grow exponentially at infinity, with a possibly algebraic singularity in 0. We start showing how a "bad" distance between the interpolation knots induces an algebraic divergence of the Lebesgue constants in C_σ, whereas the minimal order is $\log m$ (see [37],[38]). However, although the zeros of Q_{2m+1} are well distanced, the extended interpolation sequence $\{\mathcal{L}_{2m+1}(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\}_m$ has a bad behavior in C_σ, since for any choice of δ, α

$$\|\mathcal{L}_{2m+1}(w_\alpha, w_{\alpha+1})\sigma_\delta\|_\infty = \sup_{\|f\sigma_\delta\|_\infty=1} \|\mathcal{L}_{2m+1}(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\|_\infty \sim m^\tau, \quad \tau > 0\,.$$

In order to obtain the $\log m$-Lebesgue constants, in this paper we collect some modified interpolation processes essentially based on the zeros of Q_{2m+1} and introduced in [31], [33]. A first approach follows from an idea introduced in [28] (see also [22]), namely considering only a finite section of the sequence interpolating a finite part of f, i.e. $\{\mathcal{L}_{2m+1}(w_\alpha, w_{\alpha+1}, f\chi_j)\chi_j\sigma_\delta\}$ where χ_j is the characteristic function of the interval $(0, z_j)$, for a special choice of the index $j = j(m)$. In this case there are necessary and sufficient conditions under which the "truncated" sequence approximates f like the best approximation sequence in C_σ, except the $\log m$ factor.
However $\{\mathcal{L}_{2m+1}(w_\alpha, w_{\alpha+1}, f\chi_j)\chi_j\sigma_\delta\}$ is not a polynomial sequence and this can be a limit in some contexts where it is crucial to preserve the polynomial nature of the approximating sequence (see for instance Gaussian quadrature rule and /or methods to approximate the solutions of integral equations by polynomials).

So, following an idea in [24] (see also [35]), we consider here the Lagrange polynomial $L_{2m+2}(w_\alpha, w_{\alpha+1}, f)$ interpolating a given $f \in C_\sigma$ on the zeros of $Q_{2m+1}(x)$ and on the special knot a_{m+1}, where $a_{m+1} = a_{m+1}(\sqrt{w_\alpha})$ is the Mhaskar- Rakhmanov -Saff number w.r.t. w_α.

Also in this case necessary and sufficient conditions hold, under which the sequence $\{L_{2m+2}(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\}_m$ approximates f like the best approximation sequence in C_σ, except the $\log m$ factor.
A similar result can be obtained considering the sequence $\{L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f\chi_j)\}_m$ with the considerable advantage of a reduced computational effort and avoiding possible overflows when f increases exponentially. Moreover, in the case the parameter α and δ are both fixed (i.e. for a given function $f \in C_\sigma$, the zeros of the interpolating polynomial are fixed), an useful tool is the Lagrange polynomial $L^*_{2m+2,s}(w_\alpha, w_{\alpha+1}, f)$ interpolating $f\chi_j$ at the zeros of $Q_{2m+1}(x)(a_{m+1} - x)$ and

on s additional knots $t_i, i = 1, \dots, s$ added in a neighborhood of the origin. Also in this case we prove that under a suitable assumption on s, δ, α, optimal Lebesgue constants can be obtained.

All the results described till now have been obtained for extended interpolation processes essentially based on the zeros $\tilde{Q}_{2m+1} = p_m(w_\alpha)p_{m+1}(w_\alpha)$.

The second main application of the interlacing property deals with the approximation of the Hilbert transform on the semiaxis. Let $H(fw_\alpha; t)$ be the weighted Hilbert transform of a given function f

$$H(fw_\alpha; t) = \int_0^{+\infty} \frac{f(x)}{x-t} w_\alpha(x)\, dx \quad , \quad t > 0 \tag{1.1}$$

where $w_\alpha(x) = e^{-x^\beta} x^\alpha$, $\alpha > -1$, $\beta > 1/2$ provided the integral exists as a principal value. The Hilbert transform arises in many fields of the applied sciences and also in singular integral equations of Cauchy type (see [14],[29]).

As far as the methods based on the zeros of orthogonal polynomials are concerned, these consist of quadrature rules, like Gaussian-type quadrature rules or product integration rules. A drawback of the product rules is the heavy effort in computing their coefficients, while instability phenomena arise using Gaussian rules, for values of t close to the Gaussian knots [6]. This last problem has been overcome by suitable Gaussian rules, modified in some sense in order to approximate Cauchy principal value integrals or weakly singular integrals (see [22], [5], [6], [7], [30], [25], [8]). However, any of these quadrature rule have to be applied for any fixed value of t. In the present paper, following an idea introduced in [23], we propose to approximate the function $H(fw_\alpha)$ by a suitable Lagrange interpolating polynomial based on Laguerre zeros.

The method is based on the following idea: we start from

$$H(fw_\alpha; t) = \mathcal{F}(fw_\alpha; t) + f(t)H(w_\alpha; t)\,,$$

$$\mathcal{F}(fw_\alpha; t) = \int_0^{\infty} \frac{f(x) - f(t)}{x-t} w_\alpha(x)\, dx\,,$$

and after determining the conditions under which $\mathcal{F}(fw_\alpha)$ belongs to a weighted uniform space, we approximate $\mathcal{F}(fw_\alpha)$ by the truncated Lagrange polynomial $L_m(\mathcal{F}(fw_\alpha))$ (see [20]). Since in the general case the computation of $\mathcal{F}(fw_\alpha)$ at the interpolation knots cannot be exactly performed, we approximate them by using the truncated Gauss-Laguerre rule (see [22]). In order to obtain a convergent procedure, the choice of the interpolation knots and the degree approximation in the Gaussian rule have to be carefully performed. Furthermore the interpolation knots and Gaussian knots have to be chosen sufficiently far among them to avoid possible numerical cancellation phenomena. This goal is achieved by selecting as interpolation nodes, the zeros of suitable Laguerre polynomials.[31].

The plan of the paper is the following: next section contains some preliminary results about the interlacing property and the distance of the zeros. Section 3 and 4 are devoted to extended interpolation processes, Section 5 contains some numerical evidence while in Section 6 are given the proof related to the interpolation. Section 7 includes the method to approximate the Hilbert transform and the corresponding estimate of the error, while in Section 8 some numerical tests are proposed. Finally Section 9 contains the proofs related to the approximation of the Hilbert transform.

2. Notations and basic results

In the sequel $\mathcal{C}$ will denote any positive constant which can be different in different formulas. Moreover $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ will be used to mean the constant $\mathcal{C}$ is independent of $a, b, \dots$. The notation $A \sim B$, where A and B are positive quantities depending on some parameters, will be used if and only if $(A/B)^{\pm 1} \leq \mathcal{C}$, with $\mathcal{C}$ positive constant independent of the above parameters.
Throughout the paper θ will denote a fixed real number, with $0 < \theta < 1$, which can be different in different formulas and $\mathbb{P}_m$ will be the space of all algebraic polynomials of degree at most m.

Consider the weight

$$w_\alpha(x) = e^{-x^\beta} x^\alpha, \quad \alpha > -1 \quad , \quad \beta > \frac{1}{2} \tag{2.2}$$

and let $\{p_m(w_\alpha)\}_m$ be the corresponding sequence of orthonormal polynomials having positive leading coefficients, i.e.

$$p_m(w_\alpha, x) = \gamma_m(w_\alpha) x^m + \text{ terms of lower degree} \quad , \quad \gamma_m(w_\alpha) > 0 \, .$$

By $\{x_{m,k}\}_{k=1}^m$ will be denoted the zeros of $p_m(w_\alpha)$ in increasing order

$$x_{m,1} < x_{m,2} < \cdots < x_{m,m} \, .$$

In what follows we shall denote by $a_m := a_m(\sqrt{w_\alpha})$ the Mhaskar- Rakhmanov -Saff number (shortly M-R-S number) w.r.t. w_α, defined as the smallest positive number satisfying

$$\max_{x \in \mathbb{R}} |P_m(x) \sqrt{w_\alpha(x)}| = \max_{x \in [0, \; a_m(\sqrt{w_\alpha})]} |P_m(x) \sqrt{w_\alpha(x)}| \quad , \quad \forall \, P_m \in \mathbb{P}_m \, ,$$

and having the following expression

$$a_m = a_m(\sqrt{w_\alpha}) = \left[\frac{2^{2\beta} (\Gamma\,(\beta))^2}{\Gamma(2\beta)} \right]^{1/\beta} \left(1 + \frac{2\alpha + 1}{8m} \right)^{1/\beta} m^{1/\beta} \, , \tag{2.3}$$

where Γ denotes the Gamma function. Therefore $a_m(\sqrt{w_\alpha}) \sim m^{1/\beta}$. We observe that in view of (2.3), if we consider two generalized Laguerre weights having the same exponential part e^{-bx^β}, in case with different constants b, then the corresponding M-R-S numbers differ for a constant. Therefore, in what follows we use only a_m for the Mhaskar-Rakhmanov-Saff number of any generalized Laguerre weight.

As proved in [16],[13] the zeros of $p_m(w_\alpha)$ lie in the range $(0, a_m(\sqrt{w_\alpha})$ and, setting $\Delta x_{m,k} = x_{m,k+1} - x_{m,k}$, the following estimate holds:

$$\Delta x_{m,k} \sim \frac{\sqrt{a_m}}{m} \sqrt{x_{m,k}} \, \frac{1}{\sqrt{1 - \frac{x_{m,k}}{a_m} + \frac{1}{m^{2/3}}}} \quad , \quad k = 1, 2, \dots, m-1 \, , \tag{2.4}$$

It is well known that the zeros $\{x_{m+1,k}\}_{k=1}^{m+1}$ of $p_{m+1}(w_\alpha)$ interlace the zeros $\{x_{m,k}\}_{k=1}^m$ of $p_m(w_\alpha)$, i.e.

$$x_{m+1,k} < x_{m,k} < x_{m+1,k+1}, \quad k = 1, 2, \dots, m \, .$$

Moreover, in a restricted range $(0, \theta a_m)$, where $0 < \theta < 1$ is fixed, the zeros $\{\tilde{z}_i\}_{i=1}^{2m+1}$ of $\tilde{Q}_{2m+1} := p_{m+1}(w_\alpha) p_m(w_\alpha)$ are also sufficiently far among them. To be more precise, let $\tilde{z}_{2i-1} = x_{m+1,i}, i = 1, 2, \dots, m+1, \; \tilde{z}_{2i} = x_{m,i}, \; i = 1, 2, \dots, m$ and denote by $\tilde{z}_j$ the knot defined as

$$\tilde{z}_j = \tilde{z}_{j(m)} = \min \{ \tilde{z}_k \, : \, \tilde{z}_k \geq \theta a_{m+1} \quad , \quad k = 1, 2, \dots, 2m+1 \} \, . \tag{2.5}$$

Then, in the subset $(0, \theta a_m)$, being $0 < \theta < 1$ fixed, the following estimate holds

$$\Delta \tilde{z}_k = \tilde{z}_{k+1} - \tilde{z}_k \sim \frac{\sqrt{a_{m+1}}}{m} \sqrt{\tilde{z}_{k+1}} \quad , \quad k = 1, 2, \ldots, j \,, \tag{2.6}$$

uniformly in $m \in \mathbb{N}$. In the special case $\beta = 1$, (2.6) was proved in [3].

Note that the distance between two consecutive zeros of $\tilde{Q}_{2m+1}$ is comparable with those of consecutive zeros of $p_m(w_\alpha)$ in $(0, \theta a_m)$. Indeed, by (2.4) it can be easily deduced

$$x_{m,k+1} - x_{m,k} \sim \frac{\sqrt{a_m}}{m} \sqrt{x_{m,k}} \quad , \quad k = 1, 2, \ldots, j \,,$$

being

$$x_{m,j} = \min \{ x_{m,k} \, : \, x_{m,k} \geq \theta a_m \quad , \quad k = 1, 2, \ldots, m \} \,. \tag{2.7}$$

Now we show that the interlacing of the zeros of orthogonal polynomials also holds for different but suitable related weight functions, and that in some cases the distance is "good" too.

To be more precise, setting $w_{\alpha+1}(x) = x w_\alpha(x)$, let $\{y_{m,k}\}_{k=1}^m$ be the zeros of the corresponding m-th orthonormal polynomial $p_m(w_{\alpha+1})$.
Set z_i, $i = 1, 2, \ldots, 2m+1$ be the zeros of $Q_{2m+1} := p_{m+1}(w) p_m(w_{\alpha+1})$, being $z_{2i-1} = x_{m+1,i}, i = 1, 2, \ldots, m+1$, $z_{2i} = y_i$, $i = 1, 2, \ldots, m$ and define

$$z_j = z_{j(m)} = \min \{ z_k \, : \, z_k \geq \theta a_{m+1} \quad , \quad k = 1, 2, \ldots, 2m+1 \} \,. \tag{2.8}$$

The following proposition holds [31]:

Proposition 2.1. *The zeros of $p_{m+1}(w_\alpha, x)$ interlace with those of $p_m(w_{\alpha+1})$, i.e.*

$$x_{m+1,k} < y_{m,k} < x_{m+1,k+1} \quad , \quad k = 1, 2, \ldots, m \,. \tag{2.9}$$

Denoted by z_j the knot defined in (2.8)*, we have*

$$\Delta z_k = z_{k+1} - z_k \sim \frac{\sqrt{a_{m+1}}}{m} \sqrt{z_k} \quad , \quad k = 1, 2, \ldots, j \,, \tag{2.10}$$

uniformly in $m \in \mathbb{N}$.

In the special case $\beta = 1$, the interlacing property (2.9) was proved in [9] by using a different approach.

Note that (2.10) is comparable with the distance between two consecutive zeros of $p_m(w_{\alpha+1})$.

Till now we have proved that there exist infinite triangular matrices

$$\begin{aligned} &\tilde{\mathcal{Z}} := \{ \tilde{z}_i, \ i = 1, 2, \ldots, m, \ m \in \mathbb{N} \} \,, \\ &\{\tilde{z}_i\}_{i=1}^{2m+1} \ \text{ zeros of } \ \tilde{Q}_{2m+1} = p_m(w_\alpha) p_{m+1}(w_\alpha) \\ &\mathcal{Z} := \{ z_i, \ i = 1, 2, \ldots, m, \ m \in \mathbb{N} \} \,, \\ &\{z_i\}_{i=1}^{2m+1} \ \text{ zeros of } \ Q_{2m+1} = p_m(w_{\alpha+1}) p_{m+1}(w_\alpha) \end{aligned} \tag{2.11}$$

such that for each of them, the $n-$th row contains n knots which are distinct and sufficiently far among them.

Now we want to prove that a "good" distance (roughly speaking, it means that any two consecutive zeros are sufficiently far among them) is a necessary (but not a sufficient) condition in order to construct optimal interpolation processes. In order to precise what is the meaning of "optimal" interpolating process, we recall the definition of Lebesgue constant.

For a given sequence of polynomials $\{q_r\}_{r\in\mathbb{N}}$ such that each polynomial q_m has m distinct zeros $\xi_{m,k} \quad k=1,\dots,m \in \mathbb{R}^+$, being $\xi_{m,1}<\xi_{m,2}<\dots<\xi_{m,m}$, let us define the infinite triangular matrix $\mathcal{X}=\{\xi_{m,i}, \quad i=1,2,\dots,m, \quad m\in\mathcal{N}\}$. Moreover, denote by $\mathcal{L}_m(\mathcal{X},g)$ the Lagrange polynomial interpolating a given function g at the elements of the m-th row of $\mathcal{X}$, i.e.

$$\mathcal{L}_m(\mathcal{X},g)\in\mathbb{P}_{m-1} : \mathcal{L}_m(\mathcal{X},g,\xi_{m,i})=g(\xi_{m,i}) \quad , \quad i=1,2,\dots,m\,.$$

With $\sigma_\delta(x)=e^{-x^\beta}x^\delta$, $\delta\geq 0$, $\beta>1/2$, let us introduce the space of functions

$$C_\sigma=\left\{f\in C^0(\mathbb{R}^+),\ \lim_{x\to 0^+}|f(x)|\sigma_\delta(x)=0=\lim_{x\to\infty}|f(x)|\sigma_\delta(x)\right\},$$

equipped with the norm $\|f\|_{C_\sigma}=\sup_{x\geq 0}|f(x)|\sigma_\delta(x)$. Functions in C_σ can have an exponential growth at infinity and, for $\delta>0$, they can have an algebraic singularity at the origin.

The $m-$th Lebesgue constant in C_σ is defined as the norm of the operator $\mathcal{L}_m(\mathcal{X})$ in C_σ, i.e.

$$\|\mathcal{L}_m(\mathcal{X})\|_{C_\sigma}=\sup_{\|g\sigma_\delta\|=1}\|\mathcal{L}_m(\mathcal{X},g)\sigma_\delta\| \quad , \quad m=1,2,\dots \tag{2.12}$$

We remark that we have to assume $\beta>1/2$ in order to assure the density of the polynomials in the space C_σ [18]. For this reason, from now this hypothesis is assumed to be true in what follows.

Now we recall the definition of error of the best approximation of f in C_σ

$$E_m(f)_{\sigma_\delta}=\inf_{P\in\mathbb{P}_m}\|(f-P)\sigma_\delta\|_\infty\,.$$

We start from a result of P. Vértesi on the behavior of the sequence $\{\|\mathcal{L}_m(\mathcal{X})\|_{C_\sigma}\}_m$. Indeed, in a more general context, in [37],[38] he proved the following result:

For any matrix $\mathcal{X}$ of knots defined in $(0,+\infty)$ one has

$$\|\mathcal{L}_m(\mathcal{X})\|_{C_\sigma}\geq\mathcal{C}\log m\,.$$

This estimate plays the same role of the classical Faber result in the case of the finite interval.

Then we focus our attention to the construction of knot's matrices such that

$$\|\mathcal{L}_m(\mathcal{X})\|_{C_\sigma}\leq\mathcal{C}\log m\,.$$

These matrices will be said *optimal.*

In order to look for optimal matrices, we recall the following result which binds in some sense the good order of Lebesgue constants to the good distance between consecutive interpolation knots [31]:

Proposition 2.2. *If for m sufficiently large (say $m>m_0$), there exists $k:=k(m)$ s. t.*

$$\Delta\xi_{m,k}\leq\left(\frac{\sqrt{a_m}}{m}\right)^{\eta+1}\sqrt{\xi_{m,k}} \quad , \quad \eta>0 \tag{2.13}$$

then

$$\|\mathcal{L}_m(\mathcal{X})\|_{C_\sigma} \geq \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^\eta, \tag{2.14}$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

An analogous result on finite intervals can be found in [36].

In view of (2.4) and (2.6), according to Proposition 2.2, the matrices $\tilde{\mathcal{Z}}$ and $\mathcal{Z}$ can be taken into account to obtain optimal Lebesgue constants.

3. Extended Lagrange interpolation on the matrix $\tilde{\mathcal{Z}}$

Denote by $\mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f)$ the Lagrange polynomial interpolating a given function f at $\{\tilde{z}_i\}_{i=1}^{2m+1}$. This polynomial is called *extended Lagrange interpolating polynomial.* It can take the following expression:

$$\begin{aligned} \mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f; x) &= \sum_{k=1}^{2m+1} \ell_{2m+1,k}(x) f(\tilde{z}_k)\,, \\ \ell_{2m+1,k}(x) &= \frac{\tilde{Q}_{2m+1}(x)}{\tilde{Q}'_{m+1}(\tilde{z}_k)(x-\tilde{z}_k)}\,. \end{aligned} \tag{3.15}$$

As we have said, a "good" distance is a necessary condition in order to obtain corresponding Lebesgue constants having a logarithmic behavior. However, as we go to describe, this condition is not sufficient.

Proposition 3.1 ([33])**.** *For any choice of* α, $\delta \geq 0$ *and* $\beta > 1/2$, *there exists a positive* τ *s.t.*

$$\|\mathcal{L}_{2m+1}(w_\alpha, w_\alpha)\|_{C_\sigma} = \sup_{\|f\sigma_\delta\|_\infty = 1} \|\mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty \geq \mathcal{C} m^\tau\,, \tag{3.16}$$

with $0 < \mathcal{C} \neq \mathcal{C}(m)$.

In view of the previous Proposition there exist choices of α, δ such that the Lebesgue constants in C_σ have an algebraic growth.

Nevertheless the system of knots made up of the zeros of $\{\tilde{Q}_n\}_n$ can be proposed to obtain optimal Lebesgue constants too, slightly changing the interpolation process, according to some different approaches that we go to describe.

3.1. Interpolation with the additional knot a_m.

For a given function f denote by $L_{2m+1}(w_\alpha, w_\alpha, f)$ the Lagrange polynomial interpolating f at the zeros of $\tilde{Q}_{2m+1}(x)(a_{m+1} - x)$, being a_{m+1} the M-R-S number w.r.t. w_α.

$L_{2m+2}(w_\alpha, w_\alpha, f)$ is a polynomial of degree $2m+1$ which can be represented as

$$L_{2m+2}(w_\alpha, w_\alpha, f; x) = \sum_{k=1}^{2m+2} \bar{l}_{2m+2,k}(x) f(\tilde{z}_k)\,, \tag{3.17}$$

where

$$\bar{l}_{2m+2,k}(x) = \ell_{2m+1,k}(x)\,\frac{(a_{m+1}-x)}{(a_{m+1}-\tilde{z}_k)}\ ,\quad k = 1, 2, \dots, 2m+1\,,$$

$$\bar{l}_{2m+2,2m+2}(x) = \frac{\tilde{Q}_{2m+1}(x)}{\tilde{Q}_{2m+1}(a_{m+1})}\ ,\quad \ell_{2m+1,k}(x) = \frac{\tilde{Q}_{2m+1}(x)}{\tilde{Q}_{2m+1,k}(\tilde{z}_k)(x-\tilde{z}_k)}\,.$$

The idea of adding an extra point was firstly suggested by Szabados [35] in the case of a Freud weight on the real line. Indeed, similarly to that happens in the Freud weight case (see [18], [19]), the factors $(a_m - x)/(a_m - x_{m,j})$ influence substantially the behavior of the Lebesgue constants. Indeed, recalling that the Laguerre polynomial in a neighborhood of a_m is estimated as follows

$$|\tilde{Q}_{2m+1}(w_\alpha, x)| \sqrt{w_\alpha(x)} \le \mathcal{C} \left(\left| 1 - \frac{x}{a_{m+1}} \right| + m^{-2/3} \right)^{-1/2}, \qquad a_{m+1}(1-\varepsilon) \le x \le a_{m+1}(1+\delta),$$

where $\delta, \varepsilon > 0$, the factor $(a_{m+1} - x)$ damp the growth of the polynomial in the range $a_{m+1} \le x \le a_{m+1}(1+\delta)$, $\delta > 0$ (see fig. 1).

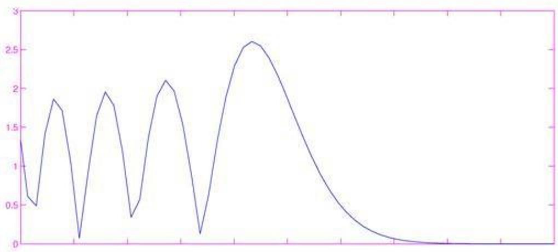

FIGURE 1. $p_m(w_\alpha, x)\sqrt{w_\alpha(x)}$, $\beta = 1, \alpha = 0, m = 20$

We are able to prove that under suitable conditions involving the interpolation weight w_α and the weight σ_δ of the space functions, the corresponding Lebesgue constants have a logarithmic divergence and that $L_{2m+2}(w_\alpha, w_\alpha, f)$ approximates $f \in C_\sigma$ like the best approximation of this space, except the factor $\log m$:

Theorem 3.1. *For any function $f \in C_\sigma$, with $\delta > 0$,*

$$\|L_{2m+2}(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty \le \mathcal{C}\|f\sigma_\delta\|_\infty \log m \tag{3.18}$$

with $0 < \mathcal{C} \ne \mathcal{C}(m, f)$, if and only if

$$\frac{1}{2} \le \delta - \alpha \le \frac{3}{2}. \tag{3.19}$$

Moreover

$$\|[f - L_{2m+2}(w_\alpha, w_\alpha, f)]\sigma_\delta\|_\infty \le \mathcal{C} E_{2m+1}(f)_{\sigma_\delta} \log m \tag{3.20}$$

where $0 < \mathcal{C} \ne \mathcal{C}(m, f)$.

In particular for smoother functions, say $f \in W^\infty_{\delta,r}$, being

$$W^\infty_{\delta,r} = \left\{ f \in C_\sigma \, : \, \|f^{(r)}\varphi^r\sigma_\delta\|_\infty < \infty \right\} \quad , \quad r \ge 1 \quad , \quad \varphi(x) = \sqrt{x}$$

a Sobolev-type space, equipped with the norm

$$\|f\|_{W^\infty_{\delta,r}} = \|f\sigma_\delta\|_\infty + \|f^{(r)}\varphi^r\sigma_\delta\|_\infty,$$

it will be useful the following estimate [27]

$$E_m(f)_{\sigma_\delta} \le \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^r \|f^{(r)}\varphi^r\sigma_\delta\|_\infty \quad , \quad \mathcal{C} \ne \mathcal{C}(m, f). \tag{3.21}$$

Assuming now that the parameters α, δ are both fixed, the assumption in (3.19) could be never satisfied. For instance, consider the function

$$f(x) = \frac{\arctan(x)^{5/2}}{(1+x^3)^2}\, e^{x^3/3} \in W_5(\sigma_\delta) \quad , \quad \sigma_\delta(x) = e^{-x^3} \tag{3.22}$$

and suppose that $\alpha = 0$, i.e. $w_\alpha(x) = e^{-x^3}$. In this case (3.19) is not satisfied. In this case it can be useful to make use of the "additional knots method", i.e. to modify the previous interpolation process by adding some interpolation knots in a neighbourhood of 0. Indeed the Lagrange polynomial based on the previous knots $\{z_k\}_{k=1}^{2m+1} \bigcup a_{m+1}$ and on the additional points $\{t_i\}_{i=1}^{s}$, under suitable assumptions, is an optimal interpolation process again.

To be more precise, let $\{t_i\}_{i=1}^{s}$, s simple knots added in the range $(0, x_{m+1,1})$, for instance $t_i = (i/(s+1))\, x_{m+1,1}$, $i = 1, 2, \ldots, s$ and let $B_s(x) = \prod_{i=1}^{s}(x - t_i)$. Denote by $L_{2m+1,s}(w_\alpha, w_\alpha, f)$ the Lagrange polynomial interpolating f at the zeros of $\tilde{Q}_{2m+1}(x)B_s(x)(a_{m+1} - x)$ i.e.

$$\begin{aligned} L_{2m+2,s}(w_\alpha, w_\alpha, f, \tilde{z}_k) &= f(\tilde{z}_k) \quad , \quad k = 1, 2, \ldots, 2m+1\,, \\ L_{2m+2,s}(w_\alpha, w_\alpha, f, a_{m+1}) &= f(a_{m+1})\,, \\ L_{2m+2,s}(w_\alpha, w_\alpha, f, t_i) &= f(t_i) \quad , \quad i = 1, 2, \ldots, s\,. \end{aligned} \tag{3.23}$$

An expression for this polynomial is

$$\begin{aligned} L_{2m+2,s}(w_\alpha, w_\alpha, f; x) = &\sum_{i=1}^{s} \frac{\tilde{Q}_{2m+1}(x)(a_{m+1} - x)B_s(x)}{\tilde{Q}_{2m+1}(t_i)(a_{m+1} - t_i)B_s'(t_i)}\, \frac{f(t_i)}{(x - t_i)} + \\ &+ \sum_{k=1}^{2m+1} \frac{\tilde{Q}_{2m+1}(x)(a_{m+1} - x)B_s(x)}{\tilde{Q}_{2m+1}'(\tilde{z}_k)(a_{m+1} - \tilde{z}_k)B_s(\tilde{z}_k)}\, \frac{f(\tilde{z}_k)}{(x - \tilde{z}_k)} + \\ &+ \frac{\tilde{Q}_{2m+1}(x)B_s(x)}{\tilde{Q}_{2m+1}(a_{m+1})B_s(a_{m+1})}\, f(a_{m+1}) \end{aligned} \tag{3.24}$$

By the same arguments used in the proof of Theorem 3.5 in [24], it is no hard to prove the following result:

Theorem 3.2. *For any function $f \in C_\sigma$, if there exists an integer s such that*

$$\frac{1}{2} \le \delta - \alpha + s \le \frac{3}{2}\,, \tag{3.25}$$

then we have

$$\|L_{2m+2,s}(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty \le \mathcal{C}\|f\sigma_\delta\|_\infty \log m\,, \tag{3.26}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$.

The introduction of the additional points "close" to the endpoint 0 in the case of the classical Laguerre weight is referable to [24].

In view of the previous result, we can approximate the function

$$f(x) = \frac{\arctan(x)^{5/2}}{(1+x^3)^2}\, e^{x^3/3} \in W_5(\sigma_\delta) \quad , \quad \sigma_\delta(x) = e^{-x^3} \tag{3.27}$$

by the Lagrange polynomial $L_{2m+2,1}(w_\alpha, w_\alpha, f)$ with $\alpha = 0$. (See Example 4.)

3.2. **Truncated sequences.** Another approach which produces a "good" extended interpolation process is the Lagrange polynomial based on the zeros of $\tilde{Q}_{2m+1}(x)$ and interpolating only a finite section of the function f. To be more precise, with $j = j(m)$ defined in (2.5), we approximate the truncated function $f_j = f(1 - \Psi_j)$, being

$$\Psi(x) = \begin{cases} 0 & , \text{ if } x \leq 0 \\ \\ 1 & , \text{ if } x \geq 1 \end{cases} \quad \text{and} \quad \Psi_j(x) = \Psi\left(\frac{x - \tilde{z}_j}{\tilde{z}_{j+1} - \tilde{z}_j}\right)$$

f_j has the same smoothness of f. Moreover, it coincides with f in the interval $(0, \tilde{z}_j]$, it is identically null for $x \in [\tilde{z}_{j+1}, +\infty)$, and these two "pieces" are smoothly linked by the function Ψ_j in the interval $(\tilde{z}_j, \tilde{z}_{j+1})$.

So we consider the Lagrange polynomial of the truncated function f_j,

$$\mathcal{L}^*_{2m+1}(w_\alpha, w_\alpha, f) := \mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f_j)\,, \tag{3.28}$$

i.e.

$$\mathcal{L}^*_{2m+1}(w_\alpha, w_\alpha, f; x) = \sum_{k=1}^{j} \ell_{2m+1,k}(x) f(\tilde{z}_k)\,, \tag{3.29}$$

being $\ell_{2m+1,k}(x)$ defined in (3.15). Obviously, $\mathcal{L}^*_{2m+1}(w_\alpha, w_\alpha, f)$ is a polynomial of degree $2m$ such that $\mathcal{L}^*_{2m+1}(w_\alpha, w_\alpha, f; \tilde{z}_k) = 0$, for $k > j$.

So, denoting by $\chi_{m,\theta}$ the characteristic function of the segment $(0, \tilde{z}_j)$, we will consider the "truncated" sequence $\{\chi_{m,\theta}\mathcal{L}_m(w_\alpha, f_j)\}_m$ of Lagrange polynomials interpolating only a finite section of the function f. It makes sense since [28], [27](see also [25])

$$\|[f - f_j]\sigma_\delta\|_\infty \leq E_M(f)_{\sigma_\delta} + \mathcal{C}e^{-Am}\|f\sigma_\delta\|_\infty\,, \tag{3.30}$$

with[2] $M = \left[(\theta/(1+\theta))\; 2m\right]^{1/\beta} \sim m^{1/\beta} \sim m$, i.e. the neglected part of f behaves like the error of best approximation $E_M(f)_{\sigma_\delta}$ being M a proper fraction of $2m$ depending on θ.

Then, for this modified Lagrange process, we are able to prove that under suitable conditions on the weights w_α and σ_δ, $\mathcal{L}^*_{2m+1}(w_\alpha, w_\alpha, f)$ is an optimal interpolation process in C_σ:

Theorem 3.3. *For any function $f \in C_\sigma$, with $\delta > 0$,*

$$\|\mathcal{L}^*_{2m+1}(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty \leq \mathcal{C}\|f\sigma_\delta\|_\infty \log m \tag{3.31}$$

with $0 < \mathcal{C} \neq \mathcal{C}(m, f)$, if and only if

$$\frac{1}{2} \leq \delta - \alpha \leq \frac{3}{2}\,. \tag{3.32}$$

Moreover

$$\|[f - \mathcal{L}^*_{2m+1}(w_\alpha, w_\alpha, f)]\sigma_\delta\|_\infty \leq \mathcal{C}\left\{E_{\bar{M}}(f)_{\sigma_\delta} \log m + e^{-Am}\|f\sigma_\delta\|_\infty\right\} \tag{3.33}$$

where $\bar{M} = \left[2m\; (\theta/(1+\theta))^\beta\right]$, $0 < \mathcal{C} \neq \mathcal{C}(m, f)$, $0 < A \neq A(m, f)$.

---

[2] $[a]$ denotes the largest integer smaller than or equal to $a \in \mathbb{R}^+$.

Truncated sequences are a "remedium" in some sense on the growth of the polynomial $\tilde{Q}_{2m+1}$ in a neighborhood of a_m, since in the restricted range $(a_m/m^2, \theta a_m)$, with $0 < \theta < 1$, the following estimate holds

$$|\tilde{Q}_{2m+1}(x)|w_\alpha(x) \le \frac{\mathcal{C}}{\sqrt{a_m x}} \quad , \quad \mathcal{C}\,\frac{a_m}{m^2} \le x \le \theta a_m\,.$$

However, $\{\chi_{m,\theta}\mathcal{L}_{2m+1}(w_\alpha, f_j)\}_m$ is not a polynomial sequence and $\chi_{m,\theta}\mathcal{L}_m(w_\alpha)$ does not project C_σ onto $\mathbb{P}_{m-1}$ as well as required in some applications like the numerical treatment of functional equations (see [21]).

In order to overcome this problem, in [16] (see also [20][26]) the authors introduced and studied a polynomial sequence interpolating a finite section of the function. The same procedure was applied to extended Lagrange interpolation processes in [31], [33].

3.3. **Polynomial sequences of a finite section of f.** To be more precise, for any fixed $\theta \in (0,1)$, let $\tilde{z}_j$ be the zero of $\tilde{Q}_{2m+1}$ defined in (2.5) and $f_{j,\theta} := f\chi_{m,\theta}$. Then, the interpolating polynomial $\bar{L}^{**}_{2m+2}(w_\alpha, f)$ is defined as

$$L^*_{2m+2}(w_\alpha, w_\alpha, f; x) := \sum_{k=1}^{j} \bar{l}_{2m+2,k}(x) f(\tilde{z}_k)\,, \tag{3.34}$$

where

$$\bar{l}_{2m+2,k}(x) = \tilde{\ell}_{2m+1,k}(x)\frac{(a_{m+1}-x)}{(a_{m+1}-\tilde{z}_k)} \quad , \quad k = 1, 2, \dots, 2m+1\,,$$
$$\bar{l}_{2m+2,2m+2}(x) = \frac{\tilde{Q}_{2m+1}(x)}{\tilde{Q}_{2m+1}(a_{m+1})}\,.$$

Obviously, $L^*_{2m+2}(w_\alpha, w_\alpha, f)$ is a polynomial of degree $2m+1$ such that $L^*_{2m+2}(w_\alpha, w_\alpha, f; a_m) = 0 = L^*_{2m+2}(w_\alpha, w_\alpha, f; \tilde{z}_k)$, for $k > j$.

The Lagrange operator $L_{2m+2}(w_\alpha, w_\alpha)$ projects C_u on $\mathbb{P}_{2m+1}$, while $L^*_{2m+2}(w_\alpha, w_\alpha)$ does not. However, letting

$$\mathcal{P}^*_{2m+1} = \{q \in \mathbb{P}_{2m+1} \,:\, q(\tilde{z}_i) = q(a_{m+1}) = 0 \quad , \quad \tilde{z}_i > \tilde{z}_j\} \subset \mathbb{P}_{2m+1}\,,$$

with $\tilde{z}_j$ defined in (2.5), we have $L^*_{2m+2}(w_\alpha, w_\alpha)$ is a projector of C_u on $\mathcal{P}^*_{2m+1}$. Moreover, $\bigcup_m \mathcal{P}^*_m$ is dense in C_u. Indeed, setting

$$\tilde{E}_{2m+1}(f)_{\sigma_\delta} := \inf_{Q \in \mathcal{P}^*_{2m+1}} \|(f-Q)\sigma_\delta\|_\infty\,.$$

from a more general in [26], next estimate follows

Lemma 3.1. *For any function $f \in C_u$,*

$$\tilde{E}_{2m}(f)_{\sigma_\delta} \le \mathcal{C}\left\{E_M(f)_{u_\gamma} + e^{-Am}\|f\sigma_\delta\|_\infty\right\}\,, \tag{3.35}$$

where $M = \left[2m\,(\theta/(1+\theta))^\beta\right]$ and the constants $0 < A \ne A(m,f)$, $0 < \mathcal{C} \ne \mathcal{C}(m,f)$.

In view of (3.35), $\tilde{E}_{2m}(f)_{\sigma_\delta}$ can be estimated by the best approximation error $E_M(f)_{\sigma_\delta}$, where M is a proper fraction of $2m$.

We are able to prove that under suitable relations between the weights w_α and σ, the corresponding Lebesgue constants grow logarithmically:

Theorem 3.4. *For any function $f \in C_u$, with $\delta > 0$,*

$$\|L^*_{2m+2}(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty \leq \mathcal{C}\|f\sigma_\delta\|_\infty \log m \tag{3.36}$$

with $0 < \mathcal{C} \neq \mathcal{C}(m, f)$, if and only if

$$\frac{1}{2} \leq \delta - \alpha \leq \frac{3}{2}\,. \tag{3.37}$$

Moreover

$$\|[f - L^*_{2m+2}(w_\alpha, w_\alpha, f)]\sigma_\delta\|_\infty \leq \mathcal{C}\tilde{E}_{2m}(f)_{\sigma_\delta} \log m \tag{3.38}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$, $0 < A \neq A(m, f)$.

Remark 3.1. In view of (3.35), one has

$$\|[f - L^*_{2m+2}(w_\alpha, w_\alpha, f)]\sigma_\delta\|_\infty \leq \mathcal{C}\left\{E_M(f)_{u_\gamma} + e^{-Am}\|f\sigma_\delta\|_\infty\right\}\,, \tag{3.39}$$

where $M = \left[2m\,(\theta/(1+\theta))^\beta\right]$ and the constants $0 < A \neq A(m, f)$, $0 < \mathcal{C} \neq \mathcal{C}(m, f)$.

Remark 3.2. Truncated Lagrange polynomial sequences were successfully applied in quadrature by introducing truncated Gaussian rules and truncated product integration formulae, which are more convenient and faster convergent (see [4],[22], [20], [25], [26]).

We conclude this section showing empirically how the number of the interpolation knots involved in truncated processes depends on θ.

Defined

$$N_m(a, b) = \text{ Number of zeros of } p_m(w_\alpha) \text{ in } (a, b)$$

for any $\theta \in (0, 1)$ let

$$v_m(\theta) = \frac{N_m(0, \theta a_m)}{m}$$

and

$$\tilde{v}_m(\theta) = \frac{1}{N-1}\sum_{m=2}^{N} v_m(\theta) \quad , \quad N = 2048\,.$$

In the case $\alpha = 0$, the values obtained for different choices of the parameter β are plotted in the figure 2. As we can see, for increasing values of $\beta > 1$ the number of zeros in $(0, \theta a_m)$ decreases.

We remark that some procedures are in progress in order to compute only j zeros of $p_m(w_\alpha)$ (see [15]).

4. Extended interpolation on the matrix $\mathcal{Z}$

Now we go to state the main results obtained in [31] about extended Lagrange interpolation processes essentially based on the knots of the matrix $\mathcal{Z}$. Denote by $\mathcal{L}_{2m+1}(w_\alpha, w_{\alpha+1}, f)$ the Lagrange polynomial interpolating a given function f at the zeros $\{z_i\}_{i=1}^{2m+1}$ of Q_{2m+1}, let z_j be defined in (2.8) and let $\chi_{m,\theta}$ be the characteristic function of the segment $(0, z_j)$. Let $L_{2m+2}(w_\alpha, w_{\alpha+1}, f)$ be the Lagrange polynomial approximating a function f at the zeros of $Q_{2m+1}(x)(a_{m+1} - x)$. Here we shall consider the Lagrange process $L_{2m+2}(w_\alpha, w_{\alpha+1})$ of the truncated function $f\chi_{m,\theta}$ associated to a given $f \in C_\sigma$, i.e. we will study

$$L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f) := L_{2m+2}(w_\alpha, w_{\alpha+1}, f\chi_{m,\theta})\,. \tag{4.40}$$

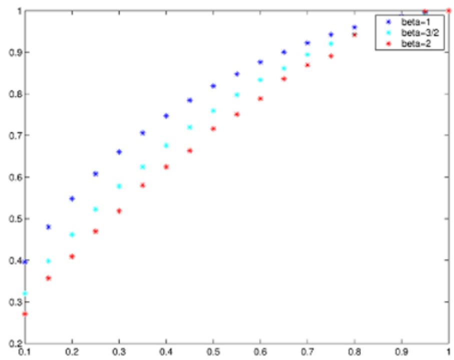

FIGURE 2. $\tilde{v}_m(\theta)$ for $\theta \in (0,1)$

Also in this case the polynomial sequence $\{L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f)\}_m$, can be used to approximate successfully any function $f \in C_\sigma$. Indeed, the following result holds [31]

Theorem 4.1. *For any function $f \in C_\sigma$, with $\delta > 0$,*

$$\|L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\|_\infty \le \mathcal{C}\|f\sigma_\delta\|_\infty \log m \tag{4.41}$$

with $0 < \mathcal{C} \neq \mathcal{C}(m,f)$, if and only if

$$1 \le \delta - \alpha \le 2\,. \tag{4.42}$$

Moreover

$$\|[f - L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f)]\sigma_\delta\|_\infty \le \mathcal{C}\left\{E_M(f)_{\sigma_\delta} \log m + e^{-Am}\|f\sigma_\delta\|_\infty\right\} \tag{4.43}$$

where $M = \left[2m\,(\theta/1+\theta))^\beta\right] \sim m$, $0 < \mathcal{C} \neq \mathcal{C}(m,f)$, $0 < A \neq A(m,f)$.

In the case that for a given function $f \in C_\sigma$ the interpolation knots cannot be chosen and the parameters δ and α don't satisfy (4.42), we can adopt the additional nodes method [31]. Let $t_i, i = 1, \dots, s$ some simple knots added in the range $[0, z_1]$, for instance $t_i = (i/(s+1))\,x_{m+1,1}, \quad i = 1, 2, \dots, s$ and let $B_s(x) = \prod_{i=1}^{s}(x - t_i)$. Denote by $L_{2m+2,s}(w_\alpha, w_{\alpha+1}, f)$ the Lagrange polynomial interpolating f at the zeros of $Q_{2m+1}(x)B_s(x)(a_{m+1} - x)$, and define $L^*_{2m+2,s}(w_\alpha, w_{\alpha+1}, f) = L_{2m+2,s}(w_\alpha, w_{\alpha+1}, f\chi_{m,\theta})$. An expression for this polynomial is

$$L^*_{2m+2,s}(w_\alpha, w_{\alpha+1}, f; x) = \sum_{i=1}^{s} \frac{Q_{2m+1}(x)(a_{m+1} - x)B_s(x)}{Q_{2m+1}(t_i)(a_{m+1} - t_i)B'_s(t_i)}\,\frac{f(t_i)}{(x - t_i)} +$$

$$+\sum_{k=1}^{j} \frac{Q_{2m+1}(x)(a_{m+1} - x)B_s(x)}{Q'_{2m+1}(z_k)(a_{m+1} - z_k)B_s(z_k)}\,\frac{f(x_k)}{(x - z_k)} \tag{4.44}$$

The following result holds

Theorem 4.2. *For any function $f \in C_\sigma$, if there exists an integer s such that*

$$1 \le \delta - \alpha + s \le 2\,, \tag{4.45}$$

then we have

$$\|L^*_{2m+2,s}(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\|_\infty \le \mathcal{C}\|f\sigma_\delta\|_\infty \log m\,, \tag{4.46}$$

where $0 < \mathcal{C} \ne \mathcal{C}(m,f)$*. Moreover,*

$$\|[f - L^*_{2m+2,s}(w_\alpha, w_{\alpha+1}, f)]\sigma_\delta\|_\infty \le \mathcal{C}\left\{E_M(f)_{\sigma_\delta}\log m + e^{-Am}\|f\sigma_\delta\|_\infty\right\}\,, \tag{4.47}$$

where $M = \left[2m\,(\theta/(1+\theta))^\beta\right] \sim m$*,* $0 < \mathcal{C} \ne \mathcal{C}(m,f)$*,* $0 < A \ne A(m,f)$*.*

In particular, in the case $f \in C_\sigma$ with $\delta = 0$, it does not exist α s.t. (4.42) holds true. Nevertheless, in this last case we can again construct an interpolant sequence with Lebesgue constants going like $\log m$. Indeed, let us consider the sequence $\{L^*_{2m+2,1}(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\}_m$, with $t_1 = 0$, i.e.

$$\begin{aligned} L^*_{2m+2,1}(w_\alpha, w_{\alpha+1}, f; x) &= \frac{Q_{2m+1}(x)(a_{m+1}-x)}{Q_{2m+1}(0)a_{m+1}}\, f(0) + \\ &+ \sum_{k=1}^{j} \frac{Q_{2m+1}(x)(a_{m+1}-x)x}{Q'_{2m+1}(z_k)(a_{m+1}-z_k)z_k}\,\frac{f(x_k)}{(x-z_k)} \end{aligned} \tag{4.48}$$

In this case we have the following

Corollary 4.4. *For any function* $f \in C_\sigma$ *with* $\delta = 0$ *and for any* $\alpha \le 0$

$$\|L^*_{2m+2,1}(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\|_\infty \le \mathcal{C}\|f\sigma_\delta\|_\infty \log m \quad , \quad 0 < \mathcal{C} \ne \mathcal{C}(m,f) \tag{4.49}$$

and

$$\|[f - L^*_{2m+2,1}(w_\alpha, w_{\alpha+1}, f)]\sigma_\delta\|_\infty \le \mathcal{C}\left\{E_M(f)_{\sigma_\delta}\log m + e^{-Am}\|f\sigma_\delta\|_\infty\right\} \tag{4.50}$$

where $M = \left[2m\,(\theta/(1+\theta))^\beta\right] \sim m$*,* $0 < \mathcal{C} \ne \mathcal{C}(m,f)$*,* $0 < A \ne A(m,f)$*.*

5. Numerical tests I

The following examples can be found in [31] and are useful to show the performance of the interpolation processes previously described. We compare the results obtained by extended Lagrange polynomials with those produced by the Lagrange polynomial based on the zeros of $p_{2m+1}(\rho, x)(a_m - x)$, where $\rho(x) = e^{-2x^\beta}x^\lambda$, $a_m = a_m(\rho)$. More precisely, denote by ξ_i, $i = 1, 2, \ldots, 2m+1$ the zeros of $p_{2m+1}(\rho)$, define

$$h = h(m) = \min_{1 \le i \le 2m+1}\{\xi_i \ge a_m\theta\} \tag{5.51}$$

and let $L_{2m+2}(\rho, f)$ be the Lagrange polynomial interpolating f at $\{\xi_i\}_{i=1}^{2m+2}$, being $\xi_{2m+2} = a_m$. Setting $L^*_{2m+2}(\rho, f) := L_{2m+2}(\rho, f\chi_h)$, where χ_h is the characteristic function of the segment $(0, \xi_h)$, in [20] it was proved that for any $f \in C_\sigma$

$$\begin{gathered} \frac{1}{4} \le \delta - \frac{\lambda}{2} \le \frac{5}{4} \implies \\ \|L^*_{2m+2}(\rho, f)\sigma_\delta\|_\infty \le \mathcal{C}\|f\sigma_\delta\|_\infty \log m \;,\; 0 < \mathcal{C} \ne \mathcal{C}(m,f)\,. \end{gathered} \tag{5.52}$$

In addition, in the case $\lambda = \delta = 0$, in [11] it was proved that for the Lagrange process $L_{2m+1,1}(\rho)$ for approximating $f \in C_\sigma$ at the knots $\{\xi_i\}_{i=1}^{2m+2} \bigcup \{0\}$,

$$\frac{1}{4} \le \delta - \frac{\lambda}{2} + s \le \frac{5}{4} \implies \tag{5.53}$$
$$\|L_{2m+1,1}(\rho, f)\sigma_\delta\|_\infty \le \mathcal{C}\|f\sigma_\delta\|_\infty \log m \;,\quad 0 < \mathcal{C} \ne \mathcal{C}(m, f)\,.$$

We remark that the same estimate can be easily deduced if we replace $L_{2m+1,1}(\rho, f)$ with $L^*_{2m+1,1}(\rho, f)$. We use the following notations

$$D_m = \left\{\frac{\xi_i + \xi_{i+1}}{2}, \quad i = 1, 2, \ldots, 2m+1\right\} \;,\; \|g\|_{\infty,D_m} = \max_{x \in D_m} |g(x)|\,. \tag{5.54}$$

Setting

$$\bar{z}_i = z_i \quad,\quad i = 1, 2, \ldots, 2m+1 \quad,\quad \bar{z}_{2m+2} = a_m\,,$$

let

$$F_m = \left\{\frac{\bar{z}_i + \bar{z}_{i+1}}{2},\; i = 1, 2, \ldots, 2m+1\right\}\,. \tag{5.55}$$

Any table contains the number $j = j(m)$ (being j the integer defined in (2.8)) of knots used in extended interpolation polynomial $L^*_{2m+2}(w_\alpha, w_{\alpha+1}w, f)$ and the corresponding maximum error committed in the set F_m. Similarly the column $h = h(m)$ contains the number of knots used in $L^*_{2m+2}(\rho, f)$ and the corresponding errors taken in the set D_m. We recall that to compute the interpolation knots in $L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f)$ we need the zeros of two almost one half degree polynomials w.r.t. the zeros involved in $L^*_{2m+2}(\rho, f)$.
The computation of the Lagrange polynomials are performed in double machine precision 2.2204×10^{-16}.

Example 1.

$$f_1(x) = \sin(x)e^{x/2} \quad \sigma_\delta(x) = xe^{-x} \quad f_1 \in C_\sigma \quad \forall\, r \ge 1\,. \tag{5.56}$$

We choose as interpolation weights $\rho(x) = e^{-2x}x^{1/2}$ and $w_{-1/2}(x) = e^{-x}x^{-1/2}$, respectively, so the assumptions in (5.52) and (3.19) are both fulfilled.

$2m+2$	h	$\|[f_1 - L^*_{2m+2}(\rho, f_1)]\sigma_\delta\|_{\infty,D_m}$	j	$\|[f_1 - L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f_1)]\sigma_\delta\|_{\infty,F_m}$
22	22	2.2 e-4	22	3.8e-4
32	32	1.9e-6	32	2.5e-6
42	42	2.6e-8	42	4.5e-8
52	49	1.5e-10	49	2.6e-10
62	55	2.0e-12	55	4.1e-12
72	60	1.7e-14	60	3.0e-14
82	65	6.6e-16	65	9.8e-16

In this example the function is smooth and with $m = 81$ we obtain the machine precision.

Example 2.

$$f_2(x) = |x-5|^{11/3}e^{x^2/2} \in W^\infty_{3,3}, \quad \sigma_\delta(x) = x^3e^{-x^2}\,. \tag{5.57}$$

We choose $\rho(x) = e^{-2x}x^4$ and $w_{3/2}(x) = e^{-x}x^{3/2}$, so that (3.19) and (5.52) are both satisfied.

$2m+2$	h	$\|[f_2 - L^*_{2m+2}(\rho, f_2)]\sigma_\delta\|_{\infty,D_m}$	j	$\|[f_2 - L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f_2)]\sigma_\delta\|_{\infty,F_m}$
16	16	2.7e-5	16	2.4e-5
32	27	1.2e-6	28	2.3e-7
64	51	5.1e-6	51	3.3e-7
120	90	4.8e-8	90	3.5e-8
302	178	9.0e-9	178	4.1e-9
400	236	1.5e-10	236	2.3e-10
798	—	—	377	3.3e-11

The zeros of $p_m(\rho)$ are computable up to $m = 399$. So $L^*_{400}(\rho, f)$ is the maximum degree Lagrange polynomial that we can consider. In this case, using the zeros of $p_{399}(w)$ and $p_{398}(\bar{w})$ we can construct the polynomial $L^*_{798}(w_\alpha, w_{\alpha+1}, f)$ interpolating f at the zeros of $p_{399}(w, x)p_{398}(\bar{w}, x)(a_{399} - x)$.

Example 3.

$$f_3(x) = \frac{\log(1+x)}{(1+x^2)^6}\, e^{x^2} \in W_5(u_\gamma) \quad , \quad \sigma_\delta(x) = x^3 e^{-x^2} \tag{5.58}$$

According to (5.52) and (3.19) in Theorem 3.1, we choose $\rho(x) = e^{-2x}x^4$ and $w_{3/2}(x) = e^{-x}x^{3/2}$.

$2m+2$	h	$\|[f_3 - L^*_{2m+2}(\rho, f_3)]\sigma_\delta\|_{\infty,D_m}$	j	$\|[f_3 - L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f_3)]\sigma_\delta\|_{\infty,F_m}$
8	6	5.0e-4	6	4.8e-4
16	12	1.8e-5	12	2.0e-5
32	25	2.2e-6	26	1.1e-6
48	38	5.3e-7	38	3.8e-7
64	51	2.0e-7	51	1.6e-8
202	152	5.0e-9	162	1.2e-9
302	227	9.9e-10	242	2.4e-10
400	381	8.1e-10	281	2.35e-11
422	—	—	390	9.6e-12
502	—	—	445	7.8e-13

Also in this test by using extended interpolation, we can almost double the maximum degree Lagrange polynomial, since we are able to construct $L^*_{502}(w_\alpha, w_{\alpha+1}, f)$ which interpolates f at the zeros of $p_{251}(w_\alpha, x)p_{250}(w_{\alpha+1}, x)(a_{251} - x)$.

Example 4.

$$f_4(x) = \frac{\arctan(x)^{5/2}}{(1+x^3)^2}\, e^{x^3/3} \in W^\infty_{0,5} \quad , \quad \sigma_\delta(x) = e^{-x^3} \tag{5.59}$$

In this case for $\alpha = 0$, i.e. $w_0(x) = e^{-x^3}$ (3.19) is not satisfied. Therefore we can supply by using the Lagrange polynomial with an additional knot defined in (4.48).

$2m+2$	h	$\Vert[f_4 - L^*_{2m+2}(\rho, f_4)]\sigma_\delta\Vert_{\infty,D_m}$	j	$\Vert[f_4 - L^*_{2m+2,1}(w_\alpha, w_{\alpha+1}, f_4)]\sigma_\delta\Vert_{\infty,F_m}$
8	8	9.7e-4	8	1.5e-3
16	16	4.1e-6	16	2.1e-5
32	32	1.4e-7	31	8.1e-8
50	50	2.1e-8	40	5.0e-8
100	88	6.5e-10	81	6.5e-10V
202	190	5.8e-11	164	3.5e-11
302	195	1.1e-11	193	6.5e-12
400	257	3.3e-12	256	2.0e-12
600	—	—	330	3.2e-13

5.1. **Computational details.** We recall the three-term recurrence relation for the orthogonal polynomials w.r.t. the weight w_α.

$$p_{-1}(w_\alpha, x) = 0 \quad , \quad p_0(w_\alpha, x) = \left(\int_0^\infty w_\alpha(x)\,dx\right)^{-1/2}$$

$$b_{n+1}p_{n+1}(w_\alpha; x) = (x - e_n)p_n(w_\alpha, x) - b_n p_{n-1}(w_\alpha, x) \tag{5.60}$$

$$b_n = \frac{\gamma_{n-1}(w_\alpha)}{\gamma_n(w_\alpha)} \qquad e_n = \int_0^\infty x p_n^2(w_\alpha, x) w_\alpha(x)\,dx\,.$$

Although the coefficients $\{b_k\}_k, \{e_k\}_k$ are not always known, there exist efficient numerical procedures to calculate them [2]. The computation of the zeros of generalized Laguerre polynomials with parameter $\beta \neq 1$, requires an higher computational effort. Indeed, when $\beta \neq 1$ the coefficients in the three term recurrence relation for the polynomials $\{p_m(w_\alpha)\}_m$ are not always known. However there exists the Mathematica Package OrthogonalPolynomials [2] to compute these zeros by using "high" variable precision.

6. The proofs: first part

Now we collect some polynomial inequalities deduced in [26] by a change of variable in analogous estimates in [13]. Let $x \in [x_{m,1}, x_{m,m}]$ and $d = d(x) \in \{1, \ldots, m\}$ be an index of a zero of $p_m(w_\alpha)$ closest to x. Then, for some positive constant $\mathcal{C} \neq \mathcal{C}(m, x, d)$, we have

$$\frac{1}{\mathcal{C}}\left(\frac{x - x_{m,d}}{x_d - x_{d\pm1}}\right)^2 \leq$$

$$\leq p_m^2(w_\alpha, x)e^{-x^\beta}\left(x + \frac{a_m}{m^2}\right)^{\alpha+(1/2)}\sqrt{|a_m - x| + a_m m^{-2/3}} \leq \tag{6.61}$$

$$\leq \mathcal{C}\left(\frac{x - x_d}{x_{m,d} - x_{m,d\pm1}}\right)^2 .$$

and for a fixed real number $0 < \delta < 1$,

$$|p_m(w_\alpha, x)|\sqrt{w_\alpha(x)} \leq \frac{\mathcal{C}}{\sqrt[4]{x}\,\sqrt[4]{|a_m - x| + a_m m^{-2/3}}}\,, \tag{6.62}$$

$$\frac{a_m}{m^2} \leq x \leq a_m(1 + \delta)\,.$$

In particular, for a fixed $0<\theta<1$

$$|p_m(w_\alpha,x)|\sqrt{w_\alpha(x)}\le \mathcal{C}\,\frac{1}{\sqrt[4]{a_m x}}\quad,\quad \frac{a_m}{m^2}\le x\le \theta a_m\,. \tag{6.63}$$

Moreover, for $k=1,2,\dots,m$ and $\Delta x_{m,k}=x_{m,k+1}-x_{m,k}$

$$\frac{1}{|p'_m(w_\alpha,x_{m,k})|\sqrt{w_\alpha(x_{m,k})}}\sim \Delta x_{m,k}\,\sqrt[4]{a_m x_{m,k}}\,\sqrt[4]{\left|1-\frac{x_{m,k}}{a_m}\right|+m^{-2/3}} \tag{6.64}$$

for any polynomial $P_m\in\mathbb{P}_m$, the Bernstein inequality [13] [27],

$$\begin{aligned}&\max_{x\ge 0}|P'_m(x)|\sqrt{w_\alpha(x)}\,\sqrt{x}\le\\ &\le \mathcal{C}\,\frac{m}{\sqrt{a_m}}\,\max_{x\ge 0}|P_m(x)\sqrt{w_\alpha(x)}|\quad,\quad \mathcal{C}\ne\mathcal{C}(m,P_m)\end{aligned} \tag{6.65}$$

and the Remez-type inequality [27]

$$\max_{x\ge 0}|P_m(x)\sqrt{w_\alpha(x)}|\le \mathcal{C}\max_{x\ge a_m/m^2}|P_m(x)\sqrt{w_\alpha(x)}|\,. \tag{6.66}$$

Finally we recall that for any polynomial $P_m\in\mathbb{P}_m$, the following inequality holds [27]

$$\max_{x\ge a_m(1+\delta)}|P_m(x)|\sqrt{w_\alpha(x)}\le \mathcal{C}\,e^{-Am}\max_{x\le a_m}|P_m(x)|\sqrt{w_\alpha(x)} \tag{6.67}$$

where $\mathcal{C}\ne\mathcal{C}(m)$, $A\ne A(m)$.

In the next will be useful the following

Lemma 6.1.

$$\frac{1}{|\tilde Q'_{2m+1}(\tilde z_k)|\sigma_\delta(\tilde z_k)}\le \mathcal{C}\,\frac{\sqrt{a_m-\tilde z_k}}{\tilde z_k^{\delta-\alpha-1}}\,\Delta\tilde z_k\quad,\quad \tilde z_k<\tilde z_j\quad,\quad \mathcal{C}\ne\mathcal{C}(m)\,. \tag{6.68}$$

Proof. Using (6.61)

$$\begin{aligned}&\frac{1}{|p_m(w_\alpha,x_{m+1,k})|\sqrt{w_\alpha(x_{m+1,k})}}\le\\ &\le \mathcal{C}\,\sqrt[4]{x_{m+1,k}(a_m-x_{m+1,k})}\quad,\quad x_{m+1,k}\le z_j\,,\end{aligned} \tag{6.69}$$

so, by (6.64), it follows

$$\frac{1}{|Q'_{2m+1}(x_{m+1,k})|\sigma_\delta(x_{m+1,k})}\le \mathcal{C}\,\frac{\sqrt{a_m-x_{m+1,k}}}{x_{m+1,k}^{\gamma-\alpha-(1/2)}}\,\Delta x_{m+1,k}\quad,\quad x_{m+1,k}\le z_j\,.$$

An analogous estimate holds replacing $x_{m+1,k}$ with $x_{m,k}$. Using then $\Delta x_{m+1,k}\sim\Delta x_{m,k}\sim\Delta\tilde z_k$, $\tilde z_k<z_j$, the Lemma follows.

□

Lemma 6.2. *For $x\in(\tilde z_1,\tilde z_{2m+1})$ and denoted with $\tilde z_d$ the zero of $\tilde Q_{2m+1}$ closest to x, we have*

$$\frac{|\tilde Q_{2m+1}(x)|}{|\tilde Q'_{2m+1}(z_d)(x-z_d)|}\,\frac{\sigma_\delta(x)}{\sigma_\delta(\tilde z_d)}\le\mathcal{C}\quad,\quad \mathcal{C}\ne\mathcal{C}(m,x)\,. \tag{6.70}$$

Proof. Denoted by $x_{m+1,d}$ a zero of $p_{m+1}(w_\alpha)$ closest to $x \in [x_{m+1,1}, x_{m+1,m+1}]$, in [20] it was proved,

$$\left| \frac{p_{m+1}(w_\alpha, x)}{p'_{m+1}(w_\alpha, x_{m+1,d})(x - x_{m+1,d})} \right| \frac{\sqrt{w_\alpha(x)}}{\sqrt{w_\alpha(x_{m+1,d})}} \sim 1 \, .$$

Therefore, assuming $\tilde{z}_d$ is a zero of $p_{m+1}(w_\alpha)$, we have

$$\frac{|\tilde{Q}_{2m+1}(x)|}{|\tilde{Q}'_{2m+1}(\tilde{z}_d)(x - \tilde{z}_d)|} \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_d)} \leq \mathcal{C} \frac{|p_m(w_\alpha, x)|}{|p_m(w_\alpha, \tilde{z}_d)|} \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_d)} \frac{\sqrt{w_\alpha(\tilde{z}_d)}}{\sqrt{w_\alpha(x)}}$$

and using (6.69) and (6.62),

$$\frac{|\tilde{Q}_{2m+1}(x)|}{|\tilde{Q}'_{2m+1}(\tilde{z}_d)(x - \tilde{z}_d)|} \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_d)} \leq \mathcal{C} \left(\frac{x}{\tilde{z}_d} \right)^{\delta - \alpha - 1/4} \leq \mathcal{C} \, ,$$

since $x \sim \tilde{z}_d$.

□

Lemma 6.3. *Let* $\{\tilde{z}_k\}_{k=1}^{2m+1}$ *the zeros of* $\tilde{Q}_{2m+1}$ *and denote with* $\tilde{z}_d$ *a zero closest to* x*,* $\Delta \tilde{z}_k = \tilde{z}_{k+1} - \tilde{z}_k$*. Assuming* $0 \leq \rho, \sigma \leq 1$*, for* $x \in (0, a_m)$ *and for* m *sufficiently large, we have*

$$\sum_{\substack{k=1 \\ k \neq d}}^{2m+1} \frac{\Delta \tilde{z}_k}{|x - \tilde{z}_k|} \frac{(a_m - x)^\rho}{(a_m - \tilde{z}_k)^\rho} \frac{x^\sigma}{\tilde{z}_k^\sigma} \leq \mathcal{C} \log m \tag{6.71}$$

where $\mathcal{C} \neq C(m, x)$*.*

We omit the proof of the previous Lemma since it can be easily obtained following the same arguments used in [24] to prove Lemma 4.1, p. 36.

Proof of Theorem 4.1. First we prove the sufficient condition. By (6.66) we have

$$\|L^*_{2m+2}(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty \leq$$

$$\leq \mathcal{C} \max_{\mathcal{C}\, a_m/m^2 \leq x \leq a_m} |L^*_{2m+2}(w_\alpha, w_\alpha, f; x)\sigma_\delta(x)| \tag{6.72}$$

and by (3.34)

$$\|L^*_{2m+2}(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty \leq$$

$$\leq \mathcal{C}\|f\sigma_\delta\|_\infty \max_{\mathcal{C}a_m/m^2 \leq x \leq a_m} \sum_{k=1}^{j} \left| \frac{\tilde{Q}_{2m+1}(x)(a_{m+1} - x)}{\tilde{Q}'_{2m+1}(\tilde{z}_k)(a_{m+1} - \tilde{z}_k)(x - \tilde{z}_k)} \right| \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_k)} =:$$

$$=: \mathcal{C}\|f\sigma_\delta\|_\infty \max_{\mathcal{C}a_m/m^2 \leq x \leq a_m} \tilde{\Sigma}(x) \tag{6.73}$$

By (6.63)

$$|\tilde{Q}_{2m+1}(x)|\, \sigma_\delta(x) \leq \mathcal{C} \frac{x^{\delta - \alpha - (1/2)}}{\sqrt{a_m}} \quad , \quad \frac{a_m}{m^2} \leq x \leq \theta a_m \, , \tag{6.74}$$

and recalling (6.68) we have

$$\tilde{\Sigma}(x) \le \mathcal{C} \sum_{k=1,k\neq d}^{j} \frac{\Delta \tilde{z}_k}{|x-\tilde{z}_k|} \frac{\sqrt{a_m - x}}{\sqrt{a_m - \tilde{z}_k}} \frac{x^{\delta-\alpha-(1/2)}}{\tilde{z}_k^{\delta-\alpha-(1/2)}} + \\ + \left| \frac{\tilde{Q}_{2m+1}(x)(a_{m+1}-x)}{\tilde{Q}'_{2m+1}(\tilde{z}_d)(x-\tilde{z}_d)(a_{m+1}-\tilde{z}_d)} \right| \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_d)} .$$

By Lemma 6.3 under the assumption $0 \le \delta - \alpha - 1/2 \le 1$, using Lemma 6.2 and taking into account $(a_{m+1}-x) \sim (a_{m+1}-\tilde{z}_d)$, we get

$$\tilde{\Sigma}(x) \le \mathcal{C} \log m \, . \tag{6.75}$$

Combining last inequality with (6.73), (4.46) follows.

We omit the proof of the necessary part, since it follows with a slight change in the proof of Theorem 3.1 in [31].

□

Proof of Proposition 2.1. Consider the Fourier expansion of $xp_m(w_{\alpha+1}, x)$ in the system $\{p_j(w_\alpha)\}_{j=0}^{\infty}$:

$$xp_m(w_{\alpha+1}, x) = d_m p_m(w_\alpha, x) + d_{m+1} p_{m+1}(w_\alpha, x) \, ,$$

$$d_i = \int_0^\infty xp_m(w_{\alpha+1}, x) p_i(w_\alpha, x) w_\alpha(x) \, dx$$

and for $x = x_{m+1,k}$, $k = 1, 2, \ldots, m+1$

$$x_{m+1,k} \, p_m(w_{\alpha+1}, x_{m+1,k}) = \frac{\gamma_m(w_\alpha)}{\gamma_m(w_{\alpha+1})} \, p_m(w_\alpha, x_{m+1,k}) \, . \tag{6.76}$$

Similarly proceeding , by the Fourier expansion of $p_{m+1}(w_\alpha, x)$ in the system $\{p_j(w_{\alpha+1})\}_j$,

$$p_{m+1}(w_\alpha, x) = c_m p_m(w_{\alpha+1}, x) + c_{m+1} p_{m+1}(w_{\alpha+1}, x) \, ,$$

$$c_i = \int_0^\infty p_{m+1}(w_\alpha, x) p_i(w_{\alpha+1}, x) w_{\alpha+1}(x) \, dx$$

and for $x = y_{m,k}$, $k = 1, 2, \ldots, m$, we have

$$p_{m+1}(w_\alpha, y_{m,k}) = \frac{\gamma_{m+1}(w_\alpha)}{\gamma_{m+1}(w_{\alpha+1})} \, p_{m+1}(w_{\alpha+1}, y_{m,k}) \, . \tag{6.77}$$

By the three-term recurrence relation (5.60)

$$p_{m+1}(w_{\alpha+1}, y_{m,k}) = -\frac{\gamma_{m-1}(w_{\alpha+1})\gamma_{m+1}(w_{\alpha+1})}{\gamma_m^2(w_{\alpha+1})} \, p_{m-1}(w_{\alpha+1}, y_{m,k})$$

and replacing in (6.77)

$$p_{m+1}(w_\alpha, y_{m,k}) = -\frac{\gamma_{m-1}(w_{\alpha+1})\gamma_{m+1}(w_\alpha)}{\gamma_m^2(w_{\alpha+1})} \, p_{m-1}(w_{\alpha+1}, y_{m,k}) \, . \tag{6.78}$$

By (6.76) and (6.78) it follows that $Q_{2m+1} = p_{m+1}(w_\alpha) p_m(w_{\alpha+1})$ has simple zeros. Moreover,

$$Q'_{2m+1}(x_{m+1,k}) > 0 \quad , \quad Q'_{2m+1}(y_{m,k}) < 0 \, , \tag{6.79}$$

by which it follows that the zeros of $p_{m+1}(w_\alpha, x)$ interlace with those of $p_m(w_{\alpha+1})$. Now we prove (2.10). Using $x_{m+1,k+1} - y_{m,k} < x_{m+1,k+1} - x_{m+1,k}$, and $y_{m,k} - x_{m+1,k} < y_{m,k+1} - y_{m,k}$, by (2.4) it follows

$$\Delta z_k \leq \mathcal{C}\, \frac{\sqrt{a_m}}{m}\, \sqrt{z_k} \quad , \quad z_k \leq \theta a_m\,. \tag{6.80}$$

To prove the converse inequality in (2.10), we first prove

$$x_{m+1,k+1} - y_{m,k} \geq \mathcal{C}\, \frac{\sqrt{a_m}}{m}\, \sqrt{x_{k+1}} \quad , \quad k = 1, 2, \ldots, j\,, \tag{6.81}$$

with $\mathcal{C} \neq \mathcal{C}(m)$. By (6.79) we have

$$0 < Q'_{2m+1}(x_{m+1,k+1}) - Q'_{2m+1}(y_{m,k}) = (x_{m+1,k+1} - y_{m,k}) Q''_{2m+1}(\xi_k)\,,$$

where $\xi_k \in (y_{m,k}, x_{m+1,k+1})$. Therefore

$$\frac{1}{(x_{m+1,k+1} - y_{m,k})} \leq \frac{Q''_{2m+1}(\xi_k)}{Q'_{2m+1}(x_{m+1,k+1})}\,. \tag{6.82}$$

By (6.65) and (6.62) it follows

$$|p'_m(w_\alpha, \xi_k)|\, \sqrt{w_\alpha(\xi_k)} \leq \mathcal{C}\, \frac{m}{\sqrt{a_m \xi_k}}\, \frac{1}{\sqrt[4]{\xi_k(a_m - \xi_k)}}$$

and therefore

$$|Q''_{2m+1}(\xi_k)|\, \sqrt{w_\alpha(\xi_k)\bar{w}(\xi_k)} \leq \mathcal{C} |p'_m(w_\alpha, \xi_k) p'_{m+1}(\bar{w}, \xi_k)|\, \sqrt{w_\alpha(\xi_k)\bar{w}(\xi_k)} \leq$$
$$\leq \mathcal{C}\, \frac{m^2}{a_m \xi_k}\, \frac{1}{\sqrt{\xi_k(a_m - \xi_k)}}\,.$$

By (6.61)

$$\frac{1}{|p_m(\bar{w}, x_{m+1,k+1})|\, \sqrt{\bar{w}(x_{m+1,k+1})}} \leq \mathcal{C}\, \sqrt[4]{x_{m+1,k+1}(a_m - x_{m+1,k+1})} \tag{6.83}$$

and by (6.64) we get

$$\begin{aligned} &\frac{1}{Q'_{2m+1}(x_{m+1,k+1})\, \sqrt{w_\alpha(x_{m+1,k+1})\bar{w}(x_{m+1,k+1})}} \leq \\ &\leq \mathcal{C}\, \sqrt{x_{m+1,k+1}(a_m - x_{m+1,k+1})} \Delta x_{m+1,k+1}\,. \end{aligned} \tag{6.84}$$

Using $\sqrt{w_\alpha(x_{m+1,k+1})\bar{w}(x_{m+1,k+1})} \sim \sqrt{w_\alpha(\xi_k)\bar{w}(\xi_k)}$, it follows

$$\frac{|Q''_{2m+1}(\xi_k)|}{Q'_{2m+1}(x_{m+1,k+1})} \leq \mathcal{C}\, \frac{m^2}{a_m \xi}\, \Delta x_{m+1,k+1}\,. \tag{6.85}$$

Since by (2.4)

$$\Delta x_{m+1,k+1} \sim \frac{\sqrt{a_m}}{m}\, \sqrt{x_{m+1,k+1}} \quad , \quad k \leq j\,,$$

it follows

$$\frac{1}{(x_{m+1,k+1} - y_{m,k})} \leq \mathcal{C}\, \frac{m}{\sqrt{a_m x_{m+1,k+1}}}\,. \tag{6.86}$$

Since the estimate

$$\frac{1}{(y_{m,k} - x_{m+1,k})} \leq \mathcal{C}\, \frac{m}{\sqrt{a_m x_{m+1,k}}} \tag{6.87}$$

follows by similar arguments, the Proposition is completely proved.

□

Lemma 6.4. *Let* $Q_{2m+1} = p_{m+1}(w_\alpha)p_m(w_{\alpha+1})$ *and* $z_k, k = 1, 2, \dots, 2m+1$ *the zeros of* Q_{2m+1}

$$\frac{1}{|Q'_{2m+1}(z_k)|\sigma_\delta(z_k)} \le \mathcal{C}\,\frac{\sqrt{a_m - z_k}}{z_k^{\delta-\alpha-1}}\,\Delta z_k \quad , \quad z_k < z_j \quad , \quad \mathcal{C} \ne \mathcal{C}(m) \tag{6.88}$$

Proof. Using (6.61)

$$\begin{aligned}&\frac{1}{|p_m(w_{\alpha+1}, x_{m+1,k})|\sqrt{w_{\alpha+1}(x_{m+1,k})}} \le \\ &\le \mathcal{C}\sqrt[4]{x_{m+1,k}(a_m - x_{m+1,k})} \quad , \quad x_{m+1,k} \le z_j\,,\end{aligned} \tag{6.89}$$

so, by (6.64), it follows

$$\frac{1}{|Q'_{2m+1}(x_{m+1,k})|\sigma_\delta(x_{m+1,k})} \le \mathcal{C}\,\frac{\sqrt{a_m - x_{m+1,k}}}{x_{m+1,k}^{\delta-\alpha-1}}\,\Delta x_{m+1,k} \quad , \quad x_k \le z_j\,.$$

An analogous estimate holds replacing $x_{m+1,k}$ with $y_{m,k}$. Using then $\Delta x_{m+1,k} \sim \Delta y_{m,k} \sim \Delta z_k, \quad x_{m+1,k}, y_{m,k}, z_k < z_j$, the Lemma follows.

□

Lemma 6.5. *For* $x \in (z_1, z_{2m+1})$ *and denoted with* z_d *the zero of* Q_{2m+1} *closest to* x*, we have*

$$\frac{|Q_{2m+1}(x)|}{|Q'_{2m+1}(z_d)(x - z_d)|}\,\frac{\sigma_\delta(x)}{\sigma_\delta(z_d)} \le \mathcal{C} \quad , \quad \mathcal{C} \ne \mathcal{C}(m, x)\,. \tag{6.90}$$

Proof. Denoted by x_d a zero of $p_{m+1}(w_\alpha)$ closest to $x \in [x_{m+1,1}, x_{m+1,m+1}]$, in [20] it was proved,

$$\left|\frac{p_{m+1}(w_\alpha, x)}{p'_{m+1}(w_\alpha, x_{m+1,d})(x - x_{m+1,d})}\right|\,\frac{\sqrt{w_\alpha(x)}}{\sqrt{w_\alpha(x_{m+1,d})}} \sim 1\,.$$

Therefore, assuming z_d is a zero of $p_{m+1}(w_\alpha)$, we have

$$\frac{|Q_{2m+1}(x)|}{|Q'_{2m+1}(z_d)(x - z_d)|}\,\frac{\sigma_\delta(x)}{\sigma_\delta(z_d)} \le \mathcal{C}\,\frac{|p_m(w_{\alpha+1}, x)|}{|p_m(w_{\alpha+1}, z_d)|}\,\frac{\sigma_\delta(x)}{\sigma_\delta(z_d)}\,\frac{\sqrt{w_\alpha(z_d)}}{\sqrt{w_\alpha(x)}}$$

and using (6.69) and (6.62),

$$\frac{|Q_{2m+1}(x)|}{|Q'_{2m+1}(z_d)(x - z_d)|}\,\frac{\sigma_\delta(x)}{\sigma_\delta(z_d)} \le \mathcal{C}\left(\frac{x}{z_d}\right)^{\delta-\alpha-3/4} \le \mathcal{C}\,,$$

since $x \sim z_d$.

□

Proof of Theorem 4.1. First we prove the sufficient condition. By (6.66) we have

$$\begin{aligned}&\|L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\|_\infty \le \\ &\le \mathcal{C}\max_{\mathcal{C}a_m/m^2 \le x \le a_m} |L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f; x)\sigma_\delta(x)|\end{aligned} \tag{6.91}$$

and using (3.34)

$$\|L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\|_\infty \le$$

$$\le \mathcal{C}\|f\sigma_\delta\|_\infty \max_{\mathcal{C}\, a_m/m^2 \le x \le a_m} \sum_{k=1}^{j} \left| \frac{Q_{2m+1}(x)(a_{m+1}-x)}{Q'_{2m+1}(z_k)(a_{m+1}-z_k)(x-z_k)} \right| \frac{\sigma_\delta(x)}{\sigma_\delta(z_k)} =:$$

$$=: \mathcal{C}\|f\sigma_\delta\|_\infty \max_{\mathcal{C}\, a_m/m^2 \le x \le a_m} \Sigma(x) \tag{6.92}$$

By (6.63)

$$|Q_{2m+1}(x)|\sigma_\delta(x) \le \mathcal{C}\, \frac{x^{\delta-\alpha-1}}{\sqrt{a_m}} \quad , \quad \frac{a_m}{m^2} \le x \le \theta a_m\,, \tag{6.93}$$

and recalling (6.88) we have

$$\Sigma(x) \le \mathcal{C} \sum_{k=1, k\neq d}^{j} \frac{\Delta z_k}{|x-z_k|} \frac{\sqrt{a_m-x}}{\sqrt{a_m-z_k}} \frac{x^{\delta-\alpha-1}}{z_k^{\delta-\alpha-1}} +$$

$$+ \left| \frac{Q_{2m+1}(x)(a_{m+1}-x)}{Q'_{2m+1}(z_d)(x-z_d)(a_{m+1}-z_d)} \right| \frac{\sigma_\delta(x)}{\sigma_\delta(z_d)}\,.$$

By Lemma 6.3 under the assumption $0 \le \delta-\alpha-1 \le 1$, using Lemma 6.5 and taking into account $(a_m - x) \sim (a_m - z_d)$, we get

$$\Sigma(x) \le \mathcal{C} \log m\,. \tag{6.94}$$

Combining last inequality with (6.92), (3.18) follows.

Now we prove that (4.41) implies (4.42). Let g a linear piecewise function such that

$$g(y_{m,k}) = 0 \;\; , \;\; k = 1,2,\dots,m \quad , \quad g(a_{m+1}) = 0$$

and

$$g(x_{m+1,k}) = \frac{\operatorname{sgn}\left(p'_{m+1}(w_\alpha, x_{m+1,k})p_m(w_{\alpha+1}, x_k)(x-x_{m+1,k})\right)}{\sigma_\delta(x_{m+1,k})} \quad , \;\; k = 1,2,\dots,j$$

and

$$g(x_{m+1,k}) = 0 \quad , \quad k > j\,.$$

So $\|g\sigma_\delta\|_\infty = 1$, and

$$\|L^*_{2m+2}(w_\alpha, w_{\alpha+1}, g)\sigma_\delta\|_\infty \ge$$

$$\ge \sum_{k=2}^{j} \frac{|p_{m+1}(w_\alpha, \bar{x})p_m(w_{\alpha+1}, \bar{x})|(a_m-\bar{x})}{|p'_{m+1}(w_\alpha, x_{m+1,k})p_m(w_{\alpha+1}, x_{m+1,k})|(x_{m+1,k}-\bar{x})(a_m - x_{m+1,k})} \times$$

$$\times \frac{\sigma_\delta(\bar{x})}{\sigma_\delta(x_{m+1,k})}$$

where $\bar{x} = x_{m+1,1}/2 \sim a_m/m^2$. Using (6.61) we have

$$(a_m - \bar{x})|p_{m+1}(w_\alpha, \bar{x})p_m(w_{\alpha+1}, \bar{x})|\sigma_\delta(\bar{x}) \ge \mathcal{C}\bar{x}^{\delta-\alpha-1}\sqrt{a_m-\bar{x}} \tag{6.95}$$

and by (6.64) and (6.61) it follows

$$|p'_{m+1}(w_\alpha, x_{m+1,k})p_m(w_{\alpha+1}, x_{m+1,k})|(a_m - x_{m+1,k})\sigma_\delta(x_{m+1,k}) \leq \tag{6.96}$$
$$\leq \sqrt{a_m - x_{m+1,k}}\ \frac{x_{m+1,k}^{\delta-\alpha-1}}{\Delta x_{m+1,k}}\ .$$

Using $a_m - \bar{x} > a_m - x_{m+1,k}$ and $\bar{x} - x_{m+1,k} \leq x_{m+1,k}$ we have

$$\|L^*_{2m+2}(w_\alpha, w_{\alpha+1}, g)\sigma_\delta\|_\infty \geq \sum_{k=2}^{m+1} \frac{\Delta x_{m+1,k}\bar{x}^{\delta-\alpha-1}}{x_{m+1,k}^{\delta-\alpha}} \geq$$
$$\geq \left(\frac{a_m}{m^2}\right)^{\delta-\alpha-1} \int_{x_2}^{a_m} \frac{dt}{t^{\delta-\alpha}} \geq m^{-2(\delta-\alpha-1)}\ .$$

Since (4.41) holds,

$$\mathcal{C}\log m \geq \|L^*_{2m+2}(w_\alpha, w_{\alpha+1}, g)\sigma_\delta\|_\infty \geq \mathcal{C}m^{-2(\delta-\alpha-1)}$$

it follows $\delta - \alpha - 1 \geq 0$.

To prove that (4.41) implies right hand condition in (4.42), consider the same function g previously defined and let $\tilde{x} = \theta a_m$. We have

$$\|L^*_{2m+2}(w_\alpha, w_{\alpha+1}, g)\sigma_\delta\|_\infty \geq$$

$$\geq \sum_{k=1}^{j-1} \frac{|p_{m+1}(w_\alpha, \tilde{x})p_m(w_{\alpha+1}, \tilde{x})|(a_m - \tilde{x})}{|p'_{m+1}(w_\alpha, x_{m+1,k})p_m(w_{\alpha+1}, x_{m+1,k})|(x_{m+1,k} - \tilde{x})(a_m - x_{m+1,k})} \times$$
$$\times \frac{\sigma_\delta(\tilde{x})}{\sigma_\delta(x_{m+1,k})}\ .$$

Using (6.61)

$$(a_m - \tilde{x})|p_{m+1}(w_\alpha, \tilde{x})p_m(w_{\alpha+1}, \tilde{x})|\sigma_\delta(\bar{x}) \geq \mathcal{C}\tilde{x}^{\delta-\alpha-1}\sqrt{a_m - \tilde{x}}$$

and using (6.96) again and $a_m - x_{m+1,k} < a_m$, $\tilde{x} - x_{m+1,k} \leq \tilde{x}$, $a_m - x_{m+1,k} > (1-\theta)a_m$ we have

$$\|L^*_{2m+2}(w_\alpha, w_{\alpha+1}, g)\sigma_\delta\|_\infty \geq \mathcal{C}\ \frac{\sqrt{a_m - \tilde{x}}}{a_m}\ \tilde{x}^{\delta-\alpha-2} \sum_{k=1}^{j-1} \frac{\sqrt{a_m - x_{m+1,k}}\,\Delta x_{m+1,k}}{x_{m+1,k}^{\delta-\alpha-1}} \geq$$
$$\geq \mathcal{C}\tilde{x}^{\delta-\alpha-2} \int_{x_{m+1,1}}^{\theta a_m} \frac{dt}{t^{\delta-\alpha-1}}\ .$$

Since (4.41) holds,

$$\mathcal{C}\log m \geq \|L^*_{2m+2}(w_\alpha, w_{\alpha+1}, g)\sigma_\delta\|_\infty \geq \mathcal{C}\left(m^{-2}\right)^{-\delta+\alpha+2}$$

it follows $\delta - \alpha - 1 \leq 1$.

Let us prove (3.20). Let $P \in \mathcal{P}^*_{2m+1}$. Since $L^*_{2m+2}(w_\alpha, w_{\alpha+1}) : f \in C_u \to \mathcal{P}^*_{2m+1}$ is a projector,

$$\|[f - L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f)]\sigma_\delta\|_\infty \leq \|(f-P)\sigma_\delta\|_\infty + \|L^*_{2m+2}(w_\alpha, w_{\alpha+1}, f-P)\sigma_\delta\|_\infty\ .$$

Using then (3.35) and (4.41), the proof is complete.

□

Proof of Theorem 4.2. By (6.66) and (4.44) we have

$$\|L^*_{2m+2,s}(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\|_\infty \le$$

$$\le \mathcal{C}\|f\sigma_\delta\|_\infty \max_{\mathcal{C}\, a_m/m^2 \le x \le a_m} \left\{ \sum_{i=1}^{s} \frac{|Q_{2m+1}(x)(a_{m+1}-x)B_s(x)|}{|Q_{2m+1}(t_i)(a_{m+1}-t_i)B_s'(t_i)(x-t_i)|} \frac{\sigma_\delta(x)}{\sigma_\delta(t_i)} + \right.$$

$$\left. + \sum_{k=1}^{j} \frac{|Q_{2m+1}(x)(a_{m+1}-x)B_s(x)|}{|Q'_{2m+1}(z_k)(a_{m+1}-z_k)B_s(z_k)(x-z_k)|} \frac{\sigma_\delta(x)}{\sigma_\delta(z_k)} \right\} =:$$

$$=: \mathcal{C}\|f\sigma_\delta\|_\infty \max_{\mathcal{C}\, a_m/m^2 \le x \le a_m} \{C_1(x) + C_2(x)\} \tag{6.97}$$

By (6.61)

$$|p_m(w_\alpha, t_i)|e^{-t_i^\beta/2} \left(\frac{a_m}{m^2}\right)^{\alpha/2+1/4} \sqrt[4]{a_m - t_i} \ge$$

$$\ge \mathcal{C} \left| \frac{t_i - x_{m+1,1}}{x_{m+1,1} - x_{m+1,2}} \right| \ge \mathcal{C} \;\;,\;\; i = 1, \ldots, s$$

and consequently, since $t_i \sim a_m/m^2$,

$$|Q_{2m+1}(t_i)|\sigma_\delta(t_i) \ge \frac{\mathcal{C} t_i^\delta}{\sqrt{a_m - t_i}} \left(\frac{m^2}{a_m}\right)^{\alpha+1} \ge \frac{\mathcal{C}}{\sqrt{a_m - t_i}} \left(\frac{m^2}{a_m}\right)^{\alpha+1-\delta} .$$

Using

$$\left|\frac{B_s(x)}{B_s'(t_i)}\right| \le \prod_{j=1, j\ne i}^{s} \left|\frac{x - t_j}{t_i - t_j}\right| \le \mathcal{C} \left(\frac{m^2 x}{a_m}\right)^{s-1}$$

$\sqrt{a_m - x} \le \sqrt{a_m - t_i}$, and (6.93), we have

$$C_1(x) \le \mathcal{C} x^{\delta-\alpha-2+s} \left(\frac{m^2 x}{a_m}\right)^{s-2+\delta-\alpha}$$

and taking into account the assumption $\delta - \alpha - 2 + s \le 0$ and $x \ge a_m/m^2$, it follows

$$C_1(x) \le \mathcal{C} \,. \tag{6.98}$$

By (6.93)- (6.68) again and using

$$|B_s(x)| = \prod_{i=1}^{s} |x - t_i| \le (x - t_1)^s \le \mathcal{C} x^s$$

$$|B_s(z_k)| \ge (z_k - t_s)^s \sim z_k^s$$

we have

$$C_2(x) \quad \le \mathcal{C} \sum_{k=1, k\ne d}^{j} \frac{\Delta z_k}{|x - z_k|} \frac{\sqrt{a_m - x}}{\sqrt{a_m - z_k}} \frac{x^{\delta-\alpha-1+s}}{z_k^{\delta-\alpha-1+s}} +$$

$$+ \|l_{m+1,d}(w_\alpha, x) \,| \frac{|p_m(w_\alpha, x)B_s(x)|(a_m - x)}{|B_s(z_d)p_m(w_{\alpha+1}, z_d)|(a_m - z_d)} \frac{\sigma_\delta(x)}{\sigma_\delta(z_d)} .$$

Using Lemma 6.3, under the assumption $0 \le \delta - \alpha - 1 + s \le 1$, by $B_s(x) \sim B_s(x_d)$ and Lemma 6.2, it follows

$$\mathcal{C}_2(x) \le \mathcal{C} \log m \,. \tag{6.99}$$

Combining (6.97)–(6.99), (4.46) follows.

□

7. Approximation of the Hilbert transform

Let $H(\mathcal{G};t)$ be the Hilbert transform of a given function $\mathcal{G}$

$$H(\mathcal{G};t)=\int_0^{+\infty}\frac{\mathcal{G}(x)}{x-t}\,dx \quad , \quad t>0\,,$$

provided the integral exists as a principal value. The Hilbert transform arises in many fields of the applied sciences and also in singular integral equations of Cauchy type (see [14],[29]). In [32], we introduced a numerical method in order to approximate

$$H(fw_\alpha;t)=\int_0^{+\infty}\frac{f(x)}{x-t}\,w_\alpha(x)\,dx \quad , \quad t>0 \tag{7.100}$$

where $w_\alpha(x)=e^{-x^\beta}x^\alpha$, $\alpha>-1$, $\beta>1/2$ is a generalized Laguerre weight and the function f can be singular at the origin, having an exponential growth at infinity.

As far as the methods based on the zeros of orthogonal polynomials are concerned, these consist of quadrature rules, like Gaussian-type quadrature rules or product integration rules. A drawback of the product rules is the heavy effort in computing their coefficients, while instability phenomena arise using Gaussian rules, for values of t close to the Gaussian knots [6]. This last problem has been overcome by suitable Gaussian rules, modified in some sense in order to approximate Cauchy principal value integrals or weakly singular integrals (see [22], [5], [6], [7], [30], [25], [8]). However, any of these quadrature rule have to be applied for any fixed value of t. In the present paper, following an idea introduced in [23], we propose to approximate the function $H(fw_\alpha)$ by a suitable Lagrange interpolating polynomial based on Laguerre zeros.

The method is based on the following idea: we start from

$$H(fw_\alpha;t)=\mathcal{F}(fw_\alpha;t)+f(t)H(w_\alpha;t)\ \ ,\ \ \mathcal{F}(fw_\alpha;t)=\int_0^{\infty}\frac{f(x)-f(t)}{x-t}\,w_\alpha(x)\,dx\,,$$

and after determining the conditions under which $\mathcal{F}(fw_\alpha)$ belongs to a weighted uniform space, we approximate $\mathcal{F}(fw_\alpha)$ by the truncated Lagrange polynomial $L_m(\mathcal{F}(fw_\alpha))$ (see [20]). Since in the general case the computation of $\mathcal{F}(fw_\alpha)$ at the interpolation knots cannot be exactly performed, we approximate them by using the truncated Gauss-Laguerre rule (see [22]). In order to obtain a convergent procedure, the choice of the interpolation knots and the degree approximation in the Gaussian rule have to be carefully performed. Furthermore the interpolation knots and Gaussian knots have to be chosen sufficiently far among them to avoid possible numerical cancellation phenomena. This goal is achieved by selecting as interpolation nodes, the zeros of suitable Laguerre polynomials [31].

7.1. **Functional spaces.**

With $X\subseteq[0,\infty)$, $1\le p<+\infty$ let $L^p(X)$ be defined in the usual way. Introducing the weight

$$u(x)=e^{-x^\beta/2}x^\gamma \quad , \quad \gamma>-\frac{1}{p} \quad , \quad \beta>\frac{1}{2}\,,$$

let $L^p_u(X)$ be the space of measurable functions f s.t. $fu \in L^p(X)$, equipped with the norm

$$\|f\|_{L^p_u(X)} = \|fu\|_p = \left(\int_0^{+\infty} |f(x)u(x)|^p\, dx\right)^{1/p} . \tag{7.101}$$

Moreover with $\gamma \geq 0$, let $L^u_\infty([0,+\infty)) =: C_u$ be the space of functions

$$C_u = \left\{ f \,:\, fu \in C^0(\mathbb{R}^+),\ \lim_{x\to 0^+} f(x)u(x) = 0 = \lim_{x\to+\infty} f(x)u(x) \right\} ,$$

with the norm $\|f\|_{C_u} = \sup_{x\geq 0} |f(x)|u(x)$.

By

$$E_m(f)_{u,p} = \inf_{P\in\mathbb{P}_m} \|(f-P)u\|_p$$

we denote the error of L^p- weighted approximation of f by algebraic polynomials of degree $\leq m$.

We recall the Sobolev space

$$W_s(u) = \left\{ f \in C_u \,:\, f^{(s-1)} \in AC(\mathbb{R}^+),\ \|f^{(s)}\varphi^s u\|_\infty < \infty \right\} \quad , \quad \varphi(x) = \sqrt{x}\ ,\ s \geq 1\,,$$

where $AC(\mathbb{R}^+)$ denotes the set of the functions which are absolutely continuous on every closed subset of $\mathbb{R}^+$, equipped with the norm

$$\|f\|_{W_s(u)} = \|fu\|_\infty + \|f^{(s)}\varphi^s u\|_\infty\,. \tag{7.102}$$

The following estimate holds true [27]

$$E_m(f)_{u,\infty} \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s \|f^{(s)}\varphi^s u\|_\infty \quad , \quad \mathcal{C} \neq \mathcal{C}(m,f)\,. \tag{7.103}$$

7.2. **Lagrange interpolation.** We need to recall some results about Lagrange interpolation processes. Let $\rho(x) = e^{-x^\beta} x^\eta, \rho > -1, \beta > 1/2$ and let $\{\tau_k\}_{k=1}^n$ be the zeros of $p_n(\rho)$. The polynomial $L_{n+1}(\rho, g)$ denotes the Lagrange polynomial interpolating a given function g at the zeros of $p_n(\rho,x)(a_n - x)$, where a_n is the M-R-S number w.r.t ρ, i.e.

$$L_{n+1}(\rho, g; \tau_i) = g(\tau_i) \quad , \quad i = 1,2,\ldots,n \quad , \quad L_{n+1}(\rho,g;a_n) = g(a_n)\,.$$

For any fixed $0 < \theta < 1$, define

$$\tau_{j^*} = \tau_{j^*(n)} = \min\left\{\tau_k \,:\, \tau_k \geq \theta a_n,\ k = 1,2,\ldots,n\right\} . \tag{7.104}$$

and setting χ_{j^*} the characteristic function of the segment $(0, \tau_{j^*})$, in [20] the authors introduced the Lagrange polynomial

$$L^*_{n+1}(\rho, g) := L_{n+1}(\rho, g\chi_{j^*})\,. \tag{7.105}$$

The polynomial $L^*_{n+1}(\rho, g)$ belongs to a subspace of $\mathbb{P}_n$, namely $\mathcal{P}^*_n$, with

$$\mathcal{P}^*_n = \{q \in \mathbb{P}_n \,:\, q(\tau_i) = q(a_n) = 0,\ \xi_i > \tau_{j^*}\} \subset \mathbb{P}_n \tag{7.106}$$

and $L^*_{n+1}(\rho)$ projects C_u onto $\mathcal{P}^*_n$.

About the behaviour of the Lebesgue constants in C_u they prove the following

Theorem 7.1 ([20]). *For any $g \in C_u$, under the assumption*

$$\frac{\eta}{2} + \frac{1}{4} \leq \gamma \leq \frac{\eta}{2} + \frac{5}{4}\,, \tag{7.107}$$

we have

$$\|L^*_{n+1}(\rho, g)u\|_\infty \leq \mathcal{C}\|gu\|_\infty \log n\,, \tag{7.108}$$

where $\mathcal{C} \neq \mathcal{C}(n, f)$.

Nevertheless, if the parameters η, γ do not satisfy the assumption in the previous theorem, it is possible to modify the previous process making use of the *method of additional knots.* Let $t_i, i = 1, \dots, \nu$ be some simple knots added in the range $[0, \tau_1)$, for instance $t_i = (i/(\nu+1))\tau_1$, $i = 1, 2, \dots, \nu$ and let $B_\nu(x) = \prod_{i=1}^{\nu}(x - t_i)$. Denote by $L_{n+1,\nu}(\rho, f)$ the Lagrange polynomial interpolating f at the zeros of $p_n(\rho, x)B_\nu(x)(a_n - x)$, and define $L^*_{n+1,\tau}(\rho, f) := L_{n+1,\nu}(\rho, f\chi_{j^*})$. We are able to determine suitable assumptions on the parameters γ, ν, η under which the above Lagrange process has optimal Lebesgue constants again.

Theorem 7.2. *For any* $g \in C_u$*, under the assumption*

$$\frac{\eta}{2} + \frac{1}{4} - \nu \leq \gamma \leq \frac{\eta}{2} + \frac{5}{4} - \nu \tag{7.109}$$

we have

$$\|L^*_{n+1,\nu}(\rho, g)u\|_\infty \leq \mathcal{C}\|gu\|_\infty \log n\,, \tag{7.110}$$

where $\mathcal{C} \neq \mathcal{C}(n, f)$.

The idea of interpolating a truncation f_j of the function f was introduced in [22], where f_j is obtained as a link between the function f with zero by a smooth function, having f_j the same smoothness of f.

In the case $\nu = 1, t_1 = 0$, Theorem 7.2 was proved in [11].

Theorems 7.1 and 7.2 deal with the construction of optimal Lagrange interpolation processes, i.e. with Lebesgue constants sequences behaving like $\log m$. Furthermore, both of the two sequences $\{L^*_m(\rho, f)u\}_m$ and $\{L^*_{m,\nu}(\rho, f)u\}_m$, offers some advantages: a reduced computational effort and the possibility of constructing higher degree approximation for functions f with an exponential growth, avoiding possible overflow drawbacks.

8. The method

Let $t \in (0, a_m\theta)$. We start from the relation

$$H(fw_\alpha; t) = \mathcal{F}(fw_\alpha; t) + f(t)H(w_\alpha; t)$$

where

$$\mathcal{F}(fw_\alpha; t) = \int_0^\infty \frac{f(x) - f(t)}{x - t}\, w_\alpha(x)\, dx\,.$$

Referring to the Section "Numerical tests II" the discussion on the evaluation of $H(w_\alpha; t)$, we focus the attention in approximating the function $\mathcal{F}(fw_\alpha)$. Let $\mathcal{L}_R(\mathcal{F}(fw_\alpha))$ be the Lagrange polynomial interpolating $\mathcal{F}(fw_\alpha)$ at the knots $\{\xi_1, \dots, \xi_R\}$, i.e.

$$\mathcal{L}_R(\mathcal{F}(fw_\alpha); t) = \sum_{k=1}^{R} \ell_k(t)\mathcal{F}(fw_\alpha; \xi_k), \quad \ell_k(t) = \prod_{i=1, i\neq k}^{R} \frac{z - \xi_i}{\xi_k - \xi_i}\,.$$

In the general case the quantities $\mathcal{F}(fw_\alpha; \xi_k)$, $k = 1, \dots, R$ cannot be computed exactly and so we will use a suitable Gaussian rule to approximate them. Indeed, for any fixed $0 < \theta < 1$, define the zero $x_{N,j}(w_\alpha)$ s.t.

$$x_{N,j}(w_\alpha) = x_{N,j(N)}(w_\alpha) = \tag{8.111}$$
$$= \min\{x_{N,k}(w_\alpha) \,:\, x_{N,k}(w_\alpha) \geq \theta a_N,\ k = 1, 2, \dots, N\},$$

and replace $\mathcal{F}(fw_\alpha; z_k)$ with

$$\mathcal{F}_N(fw_\alpha;\xi_k) = \sum_{i=1}^{j(N)} \lambda_{N,i}(w_\alpha)\, \frac{f(x_{N,i}(w_\alpha)) - f(\xi_k)}{x_{N,i}(w_\alpha) - \xi_k} \quad , \quad k = 1, \dots, R$$

where $\{\lambda_{N,k}(w_\alpha)\}_{k=1}^N$ are the Christoffel numbers w.r.t. the weight w_α.

Therefore we approximate $H(fw_\alpha)$ by

$$H_R(f;t) := L_R(\mathcal{F}_N(fw_\alpha);t) + f(t)H(w_\alpha;t)\,.$$

Now we have to select the interpolation knots far enough from the Gaussian nodes in order to avoid numerical instability in computing the denominators $(x_{N,i}(w_\alpha) - \xi_k)^{-1}$. In the meantime this choice have to guarantee an optimal interpolation process in the space C_u, i.e. such that the corresponding Lebesgue constants sequence has a logarithmic behaviour.

Regarding the good distance between the knots, by Proposition 2.1 it follows that the minimal distance between $x_{N+1,k}(w_\alpha)$ and $x_{N,k}(w_{\alpha+1})$ behaves like the distance between two consecutive zeros of $p_N(w_{\alpha+1})$ (see [31],[13]), the knots $\{x_{N+1,k}(w_\alpha)\}_{k=1}^{N+1}, \{x_{N,k}(w_{\alpha+1})\}_{k=1}^N$ are sufficiently far among them.

In view of the results presented in the previous Section about Lagrange interpolation processes, we are now able to select either the interpolation knots and the degree N of the truncated Gauss-Laguerre rule. We assume $-1/4 < \alpha < 1$. Indeed the cases $\alpha \geq 1$ can be easily treated setting $f(x)x^{[\alpha]}$ instead of f and with weight $\widetilde{w_\alpha}(x) = e^{-x^\beta}x^{\alpha-[\alpha]}$.

We distinguish now two different cases.

$\boxed{-1/4 < \alpha \leq 0}$. By Proposition 2.1, the zeros of the polynomial $p_{m+1}(w_\alpha)p_m(w_{\alpha+1})$ are sufficiently far apart. Therefore we choose as Gaussian knots the zeros $\{x_{m+1,i}(w_\alpha)\}_{i=1}^{m+1}$ of the polynomial $p_{m+1}(w_\alpha)$. Moreover, from Theorem 7.2 with $\tau = 1$, $\eta = \alpha + 1$, $t_1 = (1/2)\, x_{m,1}(w_{\alpha+1})$ we select as interpolation knots

$$\{x_{m,i}(w_{\alpha+1})\}_{i=1}^m \cup \{a_m\} \cup \{t_1\}\,. \tag{A}$$

So we will have

$$H(fw_\alpha;t) = H_m^{(A)}(f;t) := L^*_{m+1,1}(w_{\alpha+1}, \mathcal{F}_{m+1}(fw_\alpha);t) + f(t)H(w_\alpha;t)\,. \tag{8.112}$$

$\boxed{\alpha > 0}$. In this case we choose as Gaussian knots the zeros $\{x_{m,i}(w_\alpha)\}_{i=1}^m$ of the polynomial $p_m(w_\alpha)$. Moreover, from Theorem 7.1 with $\eta = \alpha - 1$, we select as interpolation knots

$$\{x_{m+1,i}(w_{\alpha-1})\}_{i=1}^{m+1} \cup \{a_m\}\,. \tag{B}$$

So we will have

$$H(fw_\alpha;t) = H_m^{(B)}(f;t) := L^*_{m+2}(w_{\alpha-1}, \mathcal{F}_m(fw_\alpha);t) + f(t)H(w_\alpha;t)\,. \tag{8.113}$$

About the stability of the proposed procedure, we are able to prove the following

Theorem 8.1. *Let $f \in C_u$. Under the assumption*

$$\frac{\alpha}{2} - \frac{1}{4} \le \gamma < \alpha + \frac{1}{4}\,, \tag{8.114}$$

we have

$$\sup_{t>0} |L^*_{m+2}(w_{\alpha-1}, \mathcal{F}_m(fw_\alpha); t)| u(t) \le \mathcal{C} \|fu\|_\infty \log m \quad , \quad \alpha > 0 \tag{8.115}$$

$$\sup_{t>0} |L^*_{m+1,1}(w_{\alpha+1}, \mathcal{F}_{m+1}(fw_\alpha); t)| u(t) \le \mathcal{C} \|fu\|_\infty \log m \;, \quad -\frac{1}{4} < \alpha \le 0 \tag{8.116}$$

where $0 < \mathcal{C} \ne \mathcal{C}(m, f)$.

About the error estimate, we are able to prove the following

Theorem 8.2. *Let $f \in W^\infty_{s+1}(u)$, and assume $t \in (0, \theta a_m)$, and*

$$\frac{\alpha}{2} - \frac{1}{4} \le \gamma < \alpha + \frac{1}{4}\,. \tag{8.117}$$

For $-1/4 < \alpha \le 0$

$$|H(fw_\alpha; t) - H_m^{(A)}(fw_\alpha; t)| u(t) \le \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^s \|f^{(s+1)} \varphi^{s+1} u\|_\infty \log m \tag{8.118}$$

and for $0 < \alpha < 1$

$$|H(fw_\alpha; t) - H_m^{(B)}(fw_\alpha; t)| u(t) \le \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^s \|f^{(s+1)} \varphi^{s+1} u\|_\infty \log m\,, \tag{8.119}$$

where $0 < \mathcal{C} \ne \mathcal{C}(m, f)$.

As we can see the method is easy to carry out, since it uses tools like the Gaussian rule and Lagrange interpolation, both of them w.r.t. generalized Laguerre polynomials. We point out that the coefficients in the three-term recurrence relation for the sequence $\{p_m(w)\}_m$ are not always known, even if there exist efficient numerical procedures to calculate them [2].

9. Numerical tests II

First of all we show how to compute the Hilbert transform of the weight $w_\alpha(x) = e^{-x^\beta} x^\alpha$, $\alpha > -1$, $\beta > 1/2$ for some choices of the the parameter β.

In the case $\boxed{\beta = 1}$, i.e. for the classical Laguerre weight, it is [34, p.325 n.16]

$$\int_0^{+\infty} \frac{e^{-x}}{x - t}\, dx = -e^{-t} Ei(t) \quad , \quad \alpha = 0$$

$$\int_0^{+\infty} \frac{e^{-x} x^\alpha}{x - t}\, dx = -\pi t^\alpha e^{-t} \cot((1+\alpha)\pi) + \Gamma(\alpha)\, {}_1F_1(1, 1-\alpha, -t) \;, \quad \alpha \ne 0 \tag{9.120}$$

where $Ei(t)$ is the exponential integral function and ${}_1F_1(a, b, x)$ is the Confluent Hypergeometric function.

Let $\boxed{\beta \in \mathbb{N}}$. We will use for $\beta > 1$

$$\int_0^{+\infty} \frac{e^{-x^\beta} x^\alpha}{x-t}\,dx = \frac{1}{\beta}\int_0^{+\infty} \frac{y^{(\alpha+1)/\beta-1}}{y^{1/\beta}-t}\,e^{-y}\,dy =$$

$$= \frac{1}{\beta}\sum_{k=0}^{\beta-1} t^{\beta-1-k}\int_0^{+\infty} \frac{y^{(\alpha+1+k)/\beta\,-1}}{y-t^\beta}\,e^{-y}\,dy\,,$$

and for $\alpha \neq 0$ by (9.120)

$$\begin{aligned}\int_0^{+\infty} \frac{e^{-x^\beta} x^\alpha}{x-t}\,dx &= \frac{1}{\beta}\sum_{k=0}^{\beta-1} t^{\beta-1-k}\left\{-\pi t^{\alpha+1+k-\beta} e^{-t^\beta}\cot\Big(\big(\frac{\alpha+1+k}{\beta}\big)\pi\Big)+\right.\\ &\left.+\Gamma\left(\frac{\alpha+1+k}{\beta}-1\right){}_1F_1\big(1,2-\frac{\alpha+1+k}{\beta},-t^\beta\big)\right\}.\end{aligned}$$

In the case $\alpha = 0$ the exponent $(\alpha+1+k)/\beta - 1 = 0$ for $k = \beta - 1$ and then we have

$$\begin{aligned}\int_0^{+\infty} \frac{e^{-x^\beta}}{x-t}\,dx &= \frac{1}{\beta}\sum_{k=0}^{\beta-2} t^{\beta-1-k}\left\{-\pi t^{\alpha+1+k-\beta} e^{-t^\beta}\cot\Big(\big(\frac{\alpha+1+k}{\beta}\big)\pi\Big)+\right.\\ &\left.+\Gamma\left(\frac{\alpha+1+k}{\beta}-1\right){}_1F_1\big(1,2-\frac{\alpha+1+k}{\beta},-t^\beta\big)\right\}-\\ &-\frac{1}{2}\,e^{-t^\beta}\,Ei(t^\beta)\,.\end{aligned}$$

Consider now $\boxed{\beta = p/q}$, with $p, q \in \mathbb{N}$, $\quad q > 1$

$$\int_0^{+\infty} \frac{e^{-x^{p/q}} x^\alpha}{x-t}\,dx = \frac{q}{p}\int_0^{+\infty} \frac{e^{-y} y^{(\alpha+1)q/p\,-1}}{y^{q/p}-t}\,dy = \frac{q}{p}\sum_{k=0}^{p-1} t^{p-1-k}\int_0^{+\infty} \frac{e^{-y} y^{\delta_k}}{y^q - t^p}\,dy$$

and

$$\begin{aligned}\int_0^{+\infty} \frac{e^{-x^{p/q}} x^\alpha}{x-t}\,dx &= \frac{q}{p}\sum_{k=0}^{p-1} \frac{t^{(p-1-k)p/q}}{q t^{(q-1)p/q}}\left\{-\int_0^{+\infty} \frac{e^{-y} y^{\delta_k}}{y-t^{p/q}}\,dy+\right.\\ &\left.+\int_0^{+\infty} e^{-y} y^{\delta_k}\,\frac{\sum_{j=0}^{q-2} b_j\, y^j\, t^{(b_j-1)p/q}}{\sum_{j=0}^{q-1} y^j\, t^{b_j p/q}}\,dy\right\}\end{aligned} \tag{9.121}$$

where $b_j = (q-j-1), j = 0, 1, \ldots, q-2, \delta_k = (\alpha+1+k)q/(p-1)$. The Cauchy principal value of the integrals in (9.121) can be computed using (9.120) again. About the regular integrals in (9.121), in some cases they can be expressed in terms of special functions again. For instance, with $q = 2$, we can use the Stiltjies transform [10, p.217, n.(17)] to obtain

$$\int_0^{+\infty} e^{-y} y^{\delta_k}\,\frac{b_0}{y+t^{\frac{p}{2}}}\,dy = \Gamma(1+\delta_k)\,t^{\delta_k\,p/2}\,e^{t^{p/2}}\,\Gamma\left(-\delta_k, t^{p/2}\right),$$

where $\Gamma(a,b)$ is the incomplete Gamma function. In the general case we observe that the poles of the rational functions are sufficiently far from the integration interval and the integrand function is very smooth. Therefore, the integrals can be computed successfully by a truncated Gaussian rule.

Now we show the performance of our method by some numerical examples. Let j the integer defined as

$$x_{n,j}(\sigma) = \min_{1\le i\le n} \{x_{n,k}(\sigma) \,:\, x_{n,k}(\sigma) \ge \theta a_n\} \,, \tag{9.122}$$

where

$$\begin{aligned} n &= m & \sigma &= w_{\alpha+1} \quad , \quad \text{if we use } H_m^{(A)}(fw) \\ n &= m+1 & \sigma &= w_{\alpha-1} \quad , \quad \text{if we use } H_m^{(B)}(fw) \end{aligned}$$

Example 1.

$$\int_0^{+\infty} \frac{\cos(\log(1+x))}{x-t}\, e^{-x}\, dx \,,$$

$$w_\alpha(x) = e^{-x} \quad , \quad u(x) = e^{-x/2} \quad , \quad f \in W_s(u) \quad , \quad \forall\, s \ge 1\,.$$

In this case $\alpha = 0$, and we use $H_m^{(A)}(fw_\alpha)$ to approximate $H(fw_\alpha)$. Since $\gamma = 0$, there are fulfilled the assumptions in order to (8.118) holds true.

m	j	$t = 0.1$	$t = 1.5$	t=5
20	13	1.259	-0.7498	-0.0815
60	37	1.259397	-0.7498893	-0.081515
80	49	1.2593971	-0.74988935	-0.0815154
90	55	1.25939717	-0.749889351	-0.081515457
110	68	1.2593971718	-0.7498893518	-0.0815154576
190	116	1.259397171841	-0.7498893518853	-0.081515457671
210	128	1.2593971718412	-0.74988935188533	-0.081515457671181

The function f is smooth and the machine precision is attained with $m = 210$, but $j = 128$.

Example 2.

$$\int_0^{+\infty} \frac{\sinh(x/4)|x-0.5|^7}{x-t}\, \sqrt{x}e^{-x}\, dx \,,$$

$$w_\alpha(x) = \sqrt{x}\, e^{-x} \quad , \quad u(x) = e^{-x/2} \quad , \quad f \in W_6(u)\,.$$

In this case $\alpha = 0.5$, $\gamma = 0$ and we use $H_m^{(B)}(fw_\alpha)$ to approximate $H(fw_\alpha)$. The estimate (8.119) holds true since the hypothesis (8.114) is fulfilled. Here f has an exponential growth and it is less regular then the previous example. However, the machine precision is attained with $j = 142$.

m	j	$t = 0.1$	$t = 1.5$	t=5
32	30	5.37e+3	6.371e+3	1.229e+4
50	46	5.3775062e+3	6.3713515e+3	1.2292521e+4
64	58	5.37750622e+3	6.37135157e+3	1.22925217e+4
128	114	5.3775062230e+3	6.371351571e+3	1.229252173e+4
175	133	5.37750622304e+3	6.3713515719e+3	1.22925217399e+4
256	142	5.377506223011475e+3	6.371351571909615e+3	1.229252173992784e+4

Example 3.

$$\int_0^{+\infty} \frac{\sin(x)}{x-t}\, e^{-x^3}\, dx\,,$$

$$w_\alpha(x) = e^{-x^3} \quad , \quad u(x) = e^{-x^3/3} \quad , \quad f \in W_s(u) \quad , \quad \forall\, s \geq 1\,.$$

Here $\alpha = 0, \gamma = 0$ and we use $H_m^{(A)}(fw_\alpha)$ to approximate $H(fw_\alpha)$ inside $(0, \theta a_m)$, where in this case $a_m \sim m^{1/3}$. Therefore, for these selections of m, and $\theta = 0.6$ we have to choose $t < 1.51$

m	j	$t = 0.1$	$t = 0.5$	t=1.5
16	10	1.02	0.51	-0.6
32	20	1.026	0.516	-0.648
64	40	1.02669	0.516715	-0.648546
128	79	1.026694395	0.516715317	-0.64854614
175	120	1.026694395671	0.5167153177437	-0.648546146289
256	158	1.02669439567133	0.51671531774378	-0.64854614628959

Example 4.

$$\int_0^{+\infty} \frac{e^{-x^2}}{(x-t)(1+x^4)^8}\, dx\,,$$

$$w(x) = e^{-x^2} \quad , \quad u(x) = e^{-x^2/2} \quad , \quad f \in W_s(u) \quad , \quad \forall\, s \geq 1$$

m	j	$t = 0.1$	$t = 1$	t=1.5
16	11	-0.3	-0.74	-0.40
32	21	-0.34	-0.745	-0.4096
64	30	-0.34818	-0.7457	-0.4096838
128	103	-3.4818430	-0.745766	-0.40963867
256	120	-3.481843089	-0.7457660923	-0.4096386720

Here $\alpha = 0, \gamma = 0$ and we use $H_m^{(A)}(fw_\alpha)$ to approximate $H(fw_\alpha)$. We point out that the value of the seminorm in the error expression cannot be overlooked. Indeed, for $s = 10$, it is $\|f^{(10)}\varphi^{10}u\|_\infty \sim 10^8$ and this large value justify a slow approximation rate.

All the computations, except those regarding zeros and Christoffel numbers with $\beta \neq 1$, were performed in double machine precision 2.2204×10^{-16}. In the case

$\beta \neq 1$, the coefficients in the three term recurrence relation for the polynomials $\{p_m(w)\}_m$ are not always known. Therefore the computation of zeros and Christoffel numbers can be performed by using the package "*OrthogonalPolynomials*" in MATHEMATICA [2]. This procedure uses "high" variable precision and requires an higher computational effort. It seems that some difficulties appear for $m > 400$, probably due to almost $500-$digits precision required in computing the coefficients of the recurrence relation.

10. The proofs: second part

First we need some auxiliary results and notations.

Lemma 10.1. *With $w_\alpha = e^{-x^\beta} x^\alpha$, $-1/4 < \alpha < 1$, $\beta > 1/2$*

$$\|H(w_\alpha)\|_2 \leq \mathcal{C}\,, \tag{10.123}$$

with $\mathcal{C} \neq \mathcal{C}(t)$.

Proof. We start from

$$H(w_\alpha;t) = \int_0^{2t} \frac{w_\alpha(x)}{x-t}\,dx + \int_{2t}^{+\infty} \frac{w_\alpha(x)}{x-t}\,dx =: I_1(t) + I_2(t)\,. \tag{10.124}$$

For any $\alpha > -1$

$$I_1(t) \leq \int_0^{2t} \frac{dx}{x-t} = 0\,. \tag{10.125}$$

Now let us assume $t > 1$. For any $\alpha > -1$

$$I_2(t) \leq e^{-t^\beta} \int_{2t}^{+\infty} \frac{e^{-x^\beta/2} x^\alpha}{x-t}\,dx =$$

$$= w_\alpha(t) \int_2^{\infty} \frac{e^{-(ty)^\beta/2}\, y^\alpha}{y-1}\,dy \leq w_\alpha(t) \int_2^{\infty} \frac{e^{-y^\beta/2}\, y^\alpha}{y-1}\,dy \leq \mathcal{C} w_\alpha(t)\,.$$

Consider now $0 < t \leq 1$ and $\alpha > 0$. Using $x - t > x/2$,

$$I_2(t) \leq 2e^{-t^\beta} \int_0^{+\infty} e^{-x^\beta/2} x^{\alpha-1}\,dx \leq \mathcal{C} e^{-t^\beta}\,.$$

Finally for be $0 < t \leq 1$ and $-1/4 < \alpha < 0$, using $x - t > x/2$,

$$I_2(t) \leq 2^{\alpha+3/4} e^{-t^\beta} t^{\alpha-1/4} \int_{2t}^{+\infty} \frac{e^{-x^\beta/2}}{x^{3/4}}\,dx \leq \mathcal{C}\, \frac{w_\alpha(t)}{\sqrt{\varphi(t)}}\,. \tag{10.126}$$

Combining estimates (10.125)–(10.126) with 10.124, we have

$$\|H(w)\|_2 \leq \mathcal{C} \left\| \frac{w_\alpha}{\sqrt{\varphi}} \right\|_2$$

and under the assumption $\alpha > -1/4$, the Lemma follows. □

For $1 \leq p \leq +\infty$ we introduce the following Zygmund-type spaces,

$$Z_s^p(u) = \left\{ f \in L_u^p([0,+\infty)) \,:\, \|f\|_{Z_s^p(u)} := \|fu\|_p + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s} < \infty,\ r > s \right\},$$

where, setting $\varphi(x) = \sqrt{x}$,

$$\Omega^r_\varphi(f,t)_{u,p} = \sup_{0<h\le t} \|(\bar\Delta^r_{h\varphi} f)u\|_{L^p(I_{rh})}\,,$$

$I_{rh} = [8(rh)^2, \mathcal{C}h^*]$, $\mathcal{C}$ an arbitrary constant, $h^* = 1/h^{2/(2\beta-1)}$ and

$$\bar\Delta^r_{h\varphi} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f\left(x + \left(\frac{r}{2} - i\right) h\sqrt{x}\right)\,.$$

In the next we need two additional results. The first is a weaker version of the Jackson inequality, the second is the so called Salem-Stechkin inequality.

Theorem 10.2 ([27])**.** *Let $f \in L^p_u, 1 \le p \le \infty$, such that*

$$\int_0^{\sqrt{a_m}/m} \frac{\Omega^r_\varphi(f;t)_{u,p}}{t}\, dt < \infty\,,$$

$r \ge 1$. Then we have

$$E_m(f)_{u,p} \le \mathcal{C} \int_0^{\sqrt{a_m}/m} \frac{\Omega^r_\varphi(f;t)_{u,p}}{t}\, dt\,,$$

with $m > r$ and $\mathcal{C} \ne \mathcal{C}(m,f)$.

Theorem 10.3 ([27])**.** *Let $f \in L^p_u, 1 \le p \le \infty$. Then for every $m, r \in \mathbb{N}$, with $m > r \ge 1$ we have*

$$\Omega^r_\varphi(f;t)_{u,p} \le \mathcal{C}t^r \sum_{i=0}^{[1/t]} \left(\frac{1+i}{\sqrt{a_i}}\right)^r \frac{E_i(f)_{u,p}}{1+i}\,, \tag{10.127}$$

with $m > r$ and $0 < \mathcal{C} \ne \mathcal{C}(m,f)$ and $\mathcal{C} = \mathcal{C}(r)$.

Lemma 10.4. *For any $f \in Z^\infty_{s+1}(u)$, under the assumption $0 \le \gamma < \alpha + 1/4$, the function $\mathcal{F}(fw_\alpha) \in Z^\infty_s(u)$.*

Proof. First we prove

$$E_{m-1}(\mathcal{F}(fw_\alpha))_{u,2} \le \mathcal{C}E_m(f)_{u,\infty}\,. \tag{10.128}$$

Consider the polynomial

$$Q_{m-1}(t) = \int_0^{+\infty} \frac{P_m(x) - P_m(t)}{x-t}\, w_\alpha(x)\, dx\,, \tag{10.129}$$

where $P_m \in \mathbb{P}_m$ is a polynomial of quasi best approximation of f in C_u, i.e. such that (see [27]),

$$\|(f - P_m)u\|_\infty \le \mathcal{C}E_m(f)_{u,\infty} \quad , \quad \mathcal{C} > 1\,. \tag{10.130}$$

Setting $r_m(f) = f - P_m$, we get

$$\|[\mathcal{F}(fw_\alpha) - Q_{m-1}]u\|_2 \le \|H(r_m(f)w_\alpha)u\|_2 + \|r_m(f)uH(w)\|_2\,. \tag{10.131}$$

Using Lemma 10.1, $\|H(w_\alpha)\|_2 \le \mathcal{C}$ since $\alpha > -1/4$, and

$$\begin{aligned}\|[\mathcal{F}(fw_\alpha) - Q_{m-1}]u\|_2 &\le \|H(r_m(f)w_\alpha)u\|_2 + \|r_m(f)u\|_\infty \|H(w)\|_2 \le \\ &\le \|H(r_m(f)w_\alpha)u\|_2 + \mathcal{C}E_m(f)_{u,\infty}\,.\end{aligned}$$

Setting $\varphi(x) = \sqrt{x}$, and recalling [1, p.115]

$$\left\|\frac{1}{\sqrt{\varphi}} H(g\varphi)\right\|_2 = \pi\|g\sqrt{\varphi}\|_2, \quad \forall g \in L^2(\mathbb{R}^+)\,,$$

we have, since $u(t)\sqrt{\varphi(t)} \le \mathcal{C}$,

$$\|H(r_m(f)w_\alpha)u\|_2 \le \left\| \frac{1}{\sqrt{\varphi}} H(r_m(f)w_\alpha) \right\|_2 =$$

$$= \pi \left\| \frac{r_m(f)w}{\sqrt{\varphi}} \right\|_2 \le \mathcal{C} \left\| \frac{w}{u\sqrt{\varphi}} \right\|_2 E_m(f)_{u,\infty} \le \mathcal{C} E_m(f)_{u,\infty}\,,$$

where last inequality holds since $\left\| w/u\sqrt{\varphi} \right\|_2 \le \mathcal{C}$ under the assumption $\alpha-\gamma+1/4 > 0$. Therefore

$$\|[\mathcal{F}(fw_\alpha) - Q_{m-1}]u\|_2 \le \mathcal{C} E_m(f)_{u,\infty}\,, \tag{10.132}$$

and (10.128) follows.

By (10.128) and using the Jackson theorem

$$E_{m-1}(\mathcal{F}(fw_\alpha))_{u,2} \le \mathcal{C} E_m(f)_{u,\infty} \le \mathcal{C} \int_0^{\sqrt{a_m}/m} \frac{\Omega_\varphi^r(f;t)_{u,\infty}}{t}\,dt$$

and under the assumption $f \in Z_{s+1}^\infty(u)$, it follows

$$\begin{aligned} E_{m-1}(\mathcal{F}(fw_\alpha))_{u,2} &\le \mathcal{C} \sup_{t>0} \frac{\Omega_\varphi^r(f;t)_{u,\infty}}{t^{s+1}} \int_0^{\sqrt{a_m}/m} t^s\,dt \le \\ &\le \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^{s+1} \|f\|_{Z_{s+1}^\infty(u)}\,. \end{aligned} \tag{10.133}$$

Now we prove $\mathcal{F}(fw_\alpha) \in Z_{s+1}^2(u)$, i.e.

$$\sup_{t>0} \frac{\Omega_\varphi^r(\mathcal{F}(fw_\alpha);t)_{u,2}}{t^{s+1}} < \infty\,. \tag{10.134}$$

Using the Salem-Stechkin inequality (10.127) and estimates (10.133) and $a_m(\sqrt{w_\alpha}) \sim m^{1/\beta}$, we have

$$\begin{aligned} \frac{\Omega_\varphi^r(\mathcal{F}(fw_\alpha);t)_{u,2}}{t^{s+1}} &\le \mathcal{C}\|f\|_{Z_{s+1}^\infty(u)} t^{r-s-1} \sum_{i=1}^{[1/t]} \left(\frac{\sqrt{a_i}}{i} \right)^{s+1-r} \frac{1}{i+1} \\ &\le \mathcal{C}\|f\|_{Z_{s+1}^\infty(u)} t^{r-s-1} \sum_{i=1}^{[1/t]} i^{(1/2\beta-1)(s+1-r)-1} \end{aligned}$$

with $\beta > 1/2$ and $r > s+1$. Then we have

$$\frac{\Omega_\varphi^r(\mathcal{F}(fw_\alpha);t)_{u,2}}{t^{s+1}} \le \mathcal{C} t^{r-s-1} \int_0^{[1/t]} x^{(1/2\beta-1)(s+1-r)-1}\,dx \le \mathcal{C}\,.$$

Using then Theorem 4.5 in [27], with $g \in L_u^2$, under the assumption

$$\int_0^1 \frac{\Omega_\varphi^r(g;t)_{u,2}}{t^2}\,dt < +\infty\,,$$

we have

$$\|gu\|_\infty \le \mathcal{C} \left(\|gu\|_2 + \int_0^1 \frac{\Omega_\varphi^r(g;t)_{u,2}}{t^2}\,dt \right),$$

and

$$\left.\begin{array}{l} E_m(g)_{u,\infty} \\ \Omega_\varphi^r\left(g; \dfrac{\sqrt{a_m}}{m}\right)_{u,\infty} \end{array}\right\} \leq \mathcal{C} \int_0^{\sqrt{a_m}/m} \frac{\Omega_\varphi^r(g;t)_{u,2}}{t^2}\, dt\,. \tag{10.135}$$

Therefore, by two last inequalities it follows $Z_{s+1}^2(u) \subset Z_s^\infty(u)$. Thus $\mathcal{F}(fw_\alpha) \in Z_{s+1}^2(u) \subset Z_s^\infty(u)$ and the Lemma is completely proved.

□

Lemma 10.5. *For any $f \in Z_{s+1}^\infty(u)$, under the assumption $0 \leq \gamma < \alpha + 1/4$,*

$$E_m(\mathcal{F}(fw_\alpha))_{u,\infty} \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m}\right)^s \|f\|_{Z_{s+1}^\infty(u)}\,, \tag{10.136}$$

$$E_m(\mathcal{F}_m(fw_\alpha))_{u,\infty} \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m}\right)^s \|f\|_{Z_{s+1}^\infty(u)}\,, \tag{10.137}$$

where $\mathcal{C}$ is independent on f, m.

Proof. By Lemma 10.4 under the assumption $0 \leq \gamma < \alpha + 1/4$, $f \in Z_{s+1}^\infty(u) \Longrightarrow \mathcal{F}(fw) \in Z_{s+1}^2(u)$ and therefore by (10.135), we have

$$E_{m-1}(\mathcal{F}(fw_\alpha))_{u,\infty} \leq \mathcal{C} \int_0^{\sqrt{a_m}/m} \frac{\Omega_\varphi^r(\mathcal{F}(fw_\alpha);t)_{u,2}}{t^2}\, dt\,. \tag{10.138}$$

Using then inequality (10.127) and (10.128)

$$\begin{aligned} \Omega_\varphi^r(\mathcal{F}(fw_\alpha);t)_{u,2} &\leq \mathcal{C} t^r \sum_{i=0}^{[1/t]} \left(\frac{i+1}{\sqrt{a_i}}\right)^r \frac{E_i(f)_{u,\infty}}{i+1} \leq \\ &\leq \mathcal{C} \|f\|_{Z_{s+1}^\infty(u)} t^r \sum_{i=1}^{[1/t]} \left(\frac{\sqrt{a_i}}{i}\right)^{s+1-r} \frac{1}{i} \\ &\leq \mathcal{C} \|f\|_{Z_{s+1}^\infty(u)} t^r \sum_{i=1}^{[1/t]} i^{\left((1/2\beta)-1\right)(s+1-r)-1} \end{aligned}$$

with $\beta > 1/2$ and $r > s+1$. Then we have

$$\begin{aligned} \Omega_\varphi^r(\mathcal{F}(fw_\alpha);t)_{u,2} \leq \mathcal{C} \|f\|_{Z_{s+1}^\infty(u)} t^r \int_0^{[1/t]} x^{\left((1/2\beta)-1\right)(s+1-r)-1}\, dx \leq \\ \leq \mathcal{C} \|f\|_{Z_{s+1}^\infty(u)} t^{s+1}\,. \end{aligned}$$

Therefore

$$\begin{aligned} E_{m-1}(\mathcal{F}(fw_\alpha))_{u,\infty} &\leq \mathcal{C} \|f\|_{Z_{s+1}^\infty(u)} \int_0^{\sqrt{a_m}/m} t^{s-1}\, dt \leq \\ &\leq \left(\frac{\sqrt{a_m}}{m}\right)^s \|f\|_{Z_{s+1}^\infty(u)}\,. \end{aligned} \tag{10.139}$$

and (10.136) follows. We omit the proof of (10.137) since it is very similar.

□

We slightly change the notation in the following two lemmas, setting $x_k := x_{m,k}(w_\alpha), k = 1, 2, \dots, m$.

In order to prove next lemma, we recall the Posse-Markov-Stieltjes inequalities [12, p.33]. For any function g s.t. $g^{(k)}(x) \geq 0$, $k = 0, 1, \dots 2m-1, m > 1$, for $x \in (0, x_d)$, $d = 2, 3, \dots, m$, then we have

$$\sum_{k=1}^{d-1} \lambda_{m,k}(w_\alpha) g(x_k) \leq \int_0^{x_d} g(x) w_\alpha(x)\, dx \leq \sum_{k=1}^{d} \lambda_{m,k}(w_\alpha) g(x_k)\,. \tag{10.140}$$

For any function g s.t. $(-1)^k g^{(k)}(x) \geq 0$, $k = 0, 1, \dots 2m-1, m > 1$, for $x \in (x_d, +\infty)$, $d = 1, 2, \dots, m-1$, then we have

$$\sum_{k=d+1}^{m} \lambda_{m,k}(w_\alpha) g(x_k) \leq \int_{x_d}^{+\infty} g(x) w_\alpha(x)\, dx \leq \sum_{k=d}^{m} \lambda_{m,k}(w_\alpha) g(x_k)\,. \tag{10.141}$$

Lemma 10.6. *Let $\rho(x) = e^{-x^\beta} x^\delta$, $\delta > -1/4$, $\beta > 1/2$ and let x_d be a zero of $p_m(w_\alpha)$ closest to t, i.e. $x_{d-1} < t < x_{d+1}$. We have*

$$\int_0^{+\infty} \left| \int_0^{x_{d-1}} \frac{\rho(x)}{x-t}\, dx \right|^2 dt \leq \mathcal{C} \tag{10.142}$$

$$\int_0^{+\infty} \left| \int_{x_{d+1}}^{+\infty} \frac{\rho(x)}{x-t}\, dx \right|^2 \leq \mathcal{C} \tag{10.143}$$

with $\mathcal{C} \neq \mathcal{C}(t)$.

Proof. We prove (10.142). Start from

$$\left| \int_0^{x_{d-1}} \frac{\rho(x)}{x-t}\, dx \right| \leq |H(\rho; t)| + \left| \int_{x_{d-1}}^{+\infty} \frac{\rho(x)}{x-t}\, dx \right| =:$$

$$=: |H(\rho; t)| + |H(\rho\mu; t)|\,, \tag{10.144}$$

where $\mu(t)$ is the characteristic function of the interval $(x_{d-1}, +\infty)$.

Using Lemma 10.1, taking into account the assumption $\delta > -1/4$, (10.142) follows. We omit the proof of (10.143) since it follows by similar arguments used in the proof of (10.143).

□

Lemma 10.7. *For any $f \in Z^\infty_{s+1}(u)$, under the assumption $0 \leq \gamma < \alpha + 1/4$,*

$$\mathcal{F}_m(f w_\alpha; t) = \sum_{k=1}^{j} \lambda_{m,k}(w_\alpha) \frac{f(x_k) - f_j(t)}{x_k - t} \in Z^\infty_s(u)\,.$$

Proof. First we prove

$$E_{m-1}(\mathcal{F}_m(f w_\alpha))_{u,2} \leq \mathcal{C} E_m(f)_{u,\infty} \tag{10.145}$$

Let $P_m \in \mathbb{P}_m$ a polynomial of quasi best approximation of f and consider the polynomial

$$Q_{m-1}(t) = \sum_{k=1}^{j} \lambda_{m,k}(w_\alpha) \frac{P_m(x_k) - P_m(t)}{x_k - t}\,. \tag{10.146}$$

Setting $r_m(f) = f - P_m$,

$$|\mathcal{F}_m(fw_\alpha;t) - Q_{m-1}(t)|u(t) = \left|\sum_{k=1}^{j} \lambda_{m,k}(w_\alpha)\,\frac{r_m(f,x_k) - r_m(f,t)}{x_k - t}\right| u(t) \le$$

$$\le \|r_m(f)u\|_\infty \left\{\sum_{k=1}^{j} \frac{\lambda_{m,k}(w_\alpha)}{u(x_k)|x_k - t|} + \sum_{k=1}^{j} \frac{\lambda_{m,k}(w_\alpha)}{|x_k - t|}\right\} =:$$

$$=: \|r_m(f)u\|_\infty \{S_1(t) + S_2(t)\}\,, \tag{10.147}$$

and therefore

$$\|(\mathcal{F}_m(fw_\alpha) - Q_{m-1})u\|_2 \le \|r_m(f)u\|_\infty \left(\|S_1\|_2 + \|S_2\|_2\right)\,. \tag{10.148}$$

Consider first S_1. Using

$$\lambda_{m,k}(w_\alpha) \sim \Delta x_k w_\alpha(x_k), \quad \Delta x_k = x_{k+1} - x_k, \quad k = 1, 2, \dots, m\,, \tag{10.149}$$

we get

$$S_1(t) = \sum_{k=1}^{j} \frac{w_\alpha(x_k)\Delta x_k}{u(x_k)|x_k - t|} = \left\{\sum_{k=1}^{d-2} + \sum_{k=d-1}^{d+1} + \sum_{k=d+2}^{j}\right\} \frac{w_\alpha(x_k)\Delta x_k}{u(x_k)|x_k - t|}\,,$$

where x_d is a zero of $p_m(w_\alpha)$ closest to t.
We have

$$\sum_{k=1}^{d-2} \frac{w_\alpha(x_k)\Delta x_k}{u(x_k)(t - x_k)} \le \mathcal{C} \int_0^{x_{d-1}} \frac{w_\alpha(x)}{u(x)(t - x)}\, dx\,. \tag{10.150}$$

Similarly we obtain

$$\sum_{k=d+2}^{j} \frac{w_\alpha(x_k)\Delta x_k}{u(x_k)(x_k - t)} \le \mathcal{C} \int_{x_{d+1}}^{x_j} \frac{w_\alpha(x)}{u(x)(x - t)}\, dx \tag{10.151}$$

Moreover,

$$\frac{\lambda_{m,d}(w_\alpha)}{u(x_d)|t - x_d|} \sim \frac{\Delta x_d w_\alpha(x_d)}{u(x_d)|t - x_d|} \le \mathcal{C}\frac{w_\alpha(t)}{u(t)}\,. \tag{10.152}$$

The same estimate holds for the terms of indices $d-1$ and $d+1$.
Combining (10.150)–(10.152), we have

$$\begin{aligned} \|S_1\|_2 \le \mathcal{C} \Bigg[\int_0^{+\infty} \Bigg[&\left(\int_0^{x_{d-1}} \frac{w_\alpha(x)}{u(x)(t - x)}\, dx\right)^2 + \\ &+ \left(\int_{x_{d+1}}^{x_j} \frac{w_\alpha(x)}{u(x)(x - t)}\, dx\right)^2 + \left(\frac{w_\alpha(t)}{u(t)}\right)^2\Bigg]\, dt\Bigg]^{1/2} \le \mathcal{C}\,, \end{aligned} \tag{10.153}$$

where the last bound follows by virtue of Lemma 10.6, which holds true under the assumption $\alpha - \gamma + 1/4 > 0$.

Consider S_2.

$$S_2(t)=\sum_{k=1}^{d-2}\frac{\lambda_{m,k}(w_\alpha)}{t-x_k}+\sum_{k=d+2}^{j}\frac{\lambda_{m,k}(w_\alpha)}{x_k-t}+\sum_{k=d-1}^{d+1}\frac{\lambda_{m,k}(w_\alpha)}{|t-x_k|}\,. \tag{10.154}$$

Using (10.140)–(10.141),

$$\sum_{k=1}^{d-2}\frac{\lambda_{m,k}(w_\alpha)}{t-x_k}\le\frac{\lambda_{m,d-2}(w_\alpha)}{t-x_{d-2}}-\int_0^{x_{d-1}}\frac{w(x)}{x-t}\,dx \tag{10.155}$$

$$\sum_{k=d+2}^{m}\frac{\lambda_{m,k}(w_\alpha)}{x_k-t}\le\frac{\lambda_{m,d-2}(w_\alpha)}{x_{d-2}-t}+\int_{x_{d+2}}^{+\infty}\frac{w(x)}{x-t}\,dx\quad .$$

By (10.149) and taking into account $\Delta x_d(w_\alpha)\sim|t-x_d|$, we get

$$\frac{\lambda_{m,d}(w_\alpha)}{|t-x_d|}\sim\frac{\Delta x_d w_\alpha(x_d)}{|t-x_d|}\le\mathcal{C}w_\alpha(t) \tag{10.156}$$

The same estimate holds for the terms indexed by $d-2$ to $d+2$. Then, combining (10.155)–(10.156) in (10.154), and using Lemma 10.6,

$$\begin{aligned}\|S_2\|_2\le\mathcal{C}\left[\int_0^{+\infty}\left[\left(\int_0^{x_{d-1}}\frac{w_\alpha(x)}{(t-x)}\,dx\right)^2+\right.\right.\\ \left.\left.+\left(\int_{x_{d+1}}^{x_j}\frac{w_\alpha(x)}{(x-t)}\,dx\right)^2+w_\alpha(t)^2\right]dt\right]^{1/2}\le\mathcal{C}\,.\end{aligned} \tag{10.157}$$

By (10.153), (10.157) and (10.148) it follows

$$\|[\mathcal{F}_m(fw_\alpha)-Q_{m-1}]u\|_2\le\mathcal{C}E_m(f)_{u,\infty} \tag{10.158}$$

and taking the infimum of $Q_{m-1}\in\mathbb{P}_{m-1}$, (10.145) follows.
Following the same arguments used in the proof of the Lemma 10.4, the assertion is completely proved.

□

We omit the proof of Theorem 7.2 since it can be easily obtained by the same arguments used in the proof of Theorem 3.5 in [24, p.34]. (See also [31])

Proof of Theorem 8.1 We prove (8.115). By Theorem 7.1 under the assumption $\alpha/2-1/4\le\gamma<\alpha+1/4$,

$$\sup_{t>0}|L^*_{m+2}(w^-_\alpha,\mathcal{F}_m(fw_\alpha);t)|u(t)\le\mathcal{C}\|\mathcal{F}_m(fw_\alpha)u\|_\infty\log m\,. \tag{10.159}$$

Since

$$\begin{aligned}|\mathcal{F}_m(fw_\alpha;t)|u(t)\le\|fu\|_\infty\left\{\sum_{k=1}^{j}\frac{\lambda_{m,k}(w_\alpha)u(t)}{|x_{m+1,k}(w)-t|u(x_{m+1,k}(w))}+\right.\\ \left.+\sum_{k=1}^{j}\frac{\lambda_{m,k}(w_\alpha)}{|x_{m+1,k}(w)-t|}\right\},\end{aligned}$$

taking into account (10.148) and using estimates (10.153) and (10.157),

$$|\mathcal{F}_m(fw_\alpha;t)|u(t) \le \|fu\|_\infty .$$

We omit the proof of (8.115) since it follows by similar arguments.

□

In order to prove Theorem 8.118 we recall some additional results. Let $x_{m,j}(w_{\alpha+1})$ defined as

$$x_{m,j}(w_{\alpha+1}) = x_{m,j(m)}(w_{\alpha+1}) =$$
$$= \min\{x_{m,k}(w_{\alpha+1}) \,:\, x_{m,k}(w_{\alpha+1}) \ge \theta a_m \quad , \quad k = 1,2,\dots,m\},$$

and define the following subspace

$$\mathcal{P}^*_{m+1} = \{q \in \mathbb{P}_{m+1} \,:\, q(x_{m,i}(w_{\alpha+1})) = q(a_m) = 0, \quad x_{m,i}(w_{\alpha+1}) > x_{m,j}(w_{\alpha+1})\}. \tag{10.160}$$

In [20] the authors proved , that for any polynomial $Q_{m+1} \in \mathcal{P}^*_{m+1} \subset \mathbb{P}_{m+1}$, it is $L^*_{m+1}(\bar{w}, Q_{m+1}) = Q_{m+1}$. Moreover, setting

$$\tilde{E}_m(f)_{u,\infty} := \inf_{P_m \in \mathcal{P}^*_m} \|[f - P_m]u\|_\infty$$

they estimate $\tilde{E}_m(f)_{u,\infty}$ by the best approximation error $E_M(f)_{u,\infty}$, where M is a proper fraction of m,

$$\tilde{E}_m(f)_{u,\infty} \le \mathcal{C}\left\{E_M(f)_u + e^{-Am}\|fu\|_\infty\right\}, \quad \forall f \in C_u \tag{10.161}$$

where $M = \left[m(\theta/(1+\theta))^\beta\right]$, the constants $0 < A \ne A(m,f)$, $0 < \mathcal{C} \ne \mathcal{C}(m,f)$ [20].

Proof of Theorem 8.2. We prove (8.118). Let $Q_{m+1} \in \mathcal{P}^*_{m+1}$.

$$|H(fw_\alpha;t) - H^{(A)}_m(fw_\alpha;t)|u(t) \le |\mathcal{F}(fw_\alpha;t) - Q_{m+1}(t)|u(t) +$$
$$+ |L^*_{m+1,1}(w_{\alpha+1}, \mathcal{F}_{m+1}(fw_\alpha) - Q_{m+1};t)|u(t).$$

Since the assumptions of Theorem 7.2 are satisfied with $\eta = \alpha+1, \tau = 1$, for $\alpha < 1$, we have

$$|H(fw_\alpha;t) - H^{(A)}_m(fw_\alpha;t)|u(t) \le \mathcal{C}|\mathcal{F}(fw_\alpha;t) - Q_{m+1}(t)|u(t) +$$
$$+ |\mathcal{F}_{m+1}(fw_\alpha;t) - Q_{m+1}(t)|u(t)\log m$$

Taking the infimum on $Q_{m+1} \in \mathcal{P}^*_{m+1}$ and by (10.161),

$$|H(fw_\alpha;t) - H^{(A)}_m(fw_\alpha;t)|u(t) \le \mathcal{C}\{\|[\mathcal{F}_m(fw_\alpha) - Q_{m+1}]u\|_\infty \log m +$$
$$+ \|[\mathcal{F}(fw_\alpha) - Q_{m+1}]u\|_\infty\}.$$

Taking the infimum on $Q_{m+1} \in \mathcal{P}^*_{m+1}$ and by (10.161),

$$|H(fw_\alpha;t) - H^{(A)}_m(fw_\alpha;t)|u(t) \le \mathcal{C}\{E_M(\mathcal{F}(fw_\alpha))_{u,\infty} +$$
$$+ E_M(\mathcal{F}_{m+1}(fw_\alpha))_{u,\infty}\log m\},$$

where $M = \left[m\left(\theta/(1+\theta)\right)^\beta\right] \sim m$. By Lemmas 10.4 and 10.7, (8.118) follows. We omit the proof of (8.119) since it follows by similar arguments.

□

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