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A set of multi-variable polynomials generalizing the Gegenbauer polynomials

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Abstract[1]. We consider a set of multi-variable polynomials including Chebyshev polynomials of the first and second kind, and the Gegenbauer polynomials too. We derive some properties of this family, including several recurrence relations.

1. Introduction

In [1] a generating function for numerical sequences, including both the Fibonacci and Lucas numbers is presented, and some convolution sums of mixed type, involving these sequences, are derived.

The same procedure can be applied for deriving generating functions [2],[3] of polynomial sets, including the classical Gegenbauer polynomials, and therefore, in particular, the Chebyshev polynomials of the first and second kind, and the Legendre polynomials. The relevant polynomial sequence, depending on two real parameters p and q, is shown to satisfy a 4-term recurrence relation.

In this article some properties of this family are shown, and in particular several recurrence relations are derived.

The same technique is further extended in this article to the case of multi-variable polynomials generalizing the multi-variable Gegenbauer polynomials, including the multi-variable Chebyshev polynomials of the first and second kind [4],[5].

In is worth to note that the Hermite and Gegenbauer polynomials can be considered as "twins", since they - and their generalizations - appear as unique solutions of many characterization problems.

In Sect. 5, the subject is framed in this general context, since in the Rainville book [6] an extension of the Hermite polynomials is given, while the corresponding extension of the Gegenbauer polynomials is missing. This work is intended to complete this gap. In this context, several generalizations of the Hermite and Gegenbauer polynomials existing in literature are also mentioned.

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2. Definition and 4-term recurrence relation

We consider the polynomial set $G_n^{(p,q)}$ where p and q are real parameters (satisfying a consistency condition to be stated later, in equation (3.3)), defined by the generating function

$$\mathcal{G}^{(p,q)}(x,t) := \frac{(1-xt)^p}{(1-2xt+t^2)^{p+q}} = \sum_{n=0}^{\infty} G_n^{(p,q)}(x)\, t^n \;, \tag{2.1}$$

Particular cases are given by (see e.g. [3])

- $p = 1$, $q = 0$
$$G_n^{(1,0)}(x) \equiv T_n(x) \qquad \text{(Chebyshev 1st kind)}$$
- $p = 0$, $q = 1$
$$G_n^{(0,1)}(x) \equiv U_n(x) \qquad \text{(Chebyshev 2nd kind)}$$
- $p = 0$, $q = 1/2$
$$G_n^{(0,1/2)}(x) \equiv P_n(x) \qquad \text{(Legendre)}$$
- $p = 0$, $q = \lambda$
$$G_n^{(0,\lambda)}(x) \equiv C_n^{\lambda}(x) \qquad \text{(Gegenbauer)}$$

Theorem 1. *For every choice of the parameters p and q, verifying* (3.3)*, the polynomial set $G_n^{(p,q)}$ satisfies the* 4*-terms recurrence relation*

$$\left\{\begin{aligned} (n+1)\,G_{n+1}^{(p,q)}(x) &= (3n+2q+p)x\,G_n^{(p,q)}(x) - \\ &\quad - \left[2(n+q-1)x^2 + 2p + 2q + n - 1\right] G_{n-1}^{(p,q)}(x) + \\ &\quad + (2q+p+n-2)x\,G_{n-2}^{(p,q)}(x)\;, \\ G_{-1}^{(p,q)}(x) = 0 \quad , \quad & G_0^{(p,q)}(x) = 1 \quad , \quad G_1^{(p,q)}(x) = (2q+p)x\;. \end{aligned}\right. \tag{2.2}$$

Proof. In fact, differentiating with respect to t both sides of (2.1) we find

$$\begin{aligned} \frac{\partial}{\partial t}(1-xt)^p(1-2xt+t^2)^{-p-q} &= -px(1-xt)^{p-1}(1-2xt+t^2)^{-p-q} + \\ &+ (1-xt)^p\left[2(p+q)(x-t)(1-2xt+t^2)^{-p-q-1}\right] = \\ &= \sum_{n=0}^{\infty}(n+1)\,G_{n+1}^{(p,q)}(x)\,t^n\;. \end{aligned} \tag{2.3}$$

Setting $t = 0$ we find $G_1^{(p,q)}(x) = (2q+p)x$. Furthermore,

$$\left[\frac{-px}{1-xt} + \frac{2(p+q)(x-t)}{1-2xt+t^2}\right]\sum_{n=0}^{\infty} G_n^{(p,q)}(x)\,t^n = \sum_{n=0}^{\infty}(n+1)\,G_{n+1}^{(p,q)}(x)\,t^n\;,$$

and

$$\begin{aligned} &[(2q+p)x - 2(qx^2+p+q)t + (2q+p)xt^2]\sum_{n=0}^{\infty} G_n^{(p,q)}(x)\,t^n = \\ &= \left[1 - 3xt + (1+2x^2)t^2 - xt^3\right]\sum_{n=0}^{\infty}(n+1)\,G_{n+1}^{(p,q)}(x)\,t^n\;. \end{aligned}$$

Therefore, our result follows comparing coefficients of powers of t in the above equation.

Remark 1. The recurrence relation (2.2) may be rewritten under the form:

$$\begin{aligned}&\left[(n+1)G_{n+1}^{(p,q)}(x) - 2x(q+n)G_n^{(p,q)}(x) + (2q+n-1)G_{n-1}^{(p,q)}(x)\right] - \\ &-x\left[nG_n^{(p,q)}(x) - 2x(q+n-1)G_{n-1}^{(p,q)}(x) + (2q+n-2)G_{n-2}^{(p,q)}(x)\right] - \\ &-p\left[xG_n^{(p,q)}(x) - 2G_{n-1}^{(p,q)}(x) + xG_{n-2}^{(p,q)}(x)\right] = 0\ .\end{aligned} \tag{2.4}$$

The recurrence relation (2.4), with $p = 0$, is an iteration of the three term recurrence relation satisfied by the Gegenbauer polynomials C_n^ν [Rainville book [6], equation (13), p. 279]:

$$nC_n^\nu(x) = 2x(\nu+n-1)C_{n-1}^\nu(x) - (2\nu+n-2)C_{n-2}^\nu(x)\ .$$

Remark 2. It is not difficult to find a link between the above introduced polynomials and the generalized Fibonacci-Lucas numbers $\{T^{(\ell,m)}\}$, defined in [1], equation (2):

$$G_{(\ell,m)}(t) = \sum_{n=0}^{\infty} T_n^{(\ell,m)} t^n = \frac{(1+2t)^m}{(1-t-t^2)^{\ell+m}}\ ,$$

where the parameters ℓ and m are non negative integral numbers.
In fact, putting

$$G_{(\ell,m)}(x,y,t) = \sum_{n=0}^{\infty} T_n^{(\ell,m)}(x,y) t^n = \frac{(1+2yt)^m}{(1-xt-t^2)^{\ell+m}}\ ,$$

where ℓ and m are in general real numbers, and comparing these new polynomials with the $G_n^{(p,q)}$ defined in (2.1), we find:

$$G_n^{(\ell,m)}(x) = i^n T_n^{(\ell,m)}(-2ix, ix/2)\ .$$

3. Link with the Gegenbauer polynomials

Theorem 2. *For every choice of the parameters p and q, verifying (3.3), the polynomial $G_n^{(p,q)}$ is represented, in terms of Gegenbauer polynomials as follows*

$$G_n^{(p,q)}(x) = \sum_{k=0}^{n} \binom{p}{n-k} (-1)^{n-k} x^{n-k} C_k^{p+q}(x)\ . \tag{3.1}$$

Proof. Writing equation (2.1) in the form

$$\begin{aligned}\sum_{n=0}^{\infty} G_n^{(p,q)}(x)\, t^n &= (1-xt)^p \frac{1}{(1-2xt+t^2)^{p+q}} = \\ &= \sum_{m=0}^{\infty} \binom{p}{m} (-1)^m x^m t^m \sum_{h=0}^{\infty} C_h^{p+q}(x)\, t^h = \\ &= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{p}{n-k} (-1)^{n-k} x^{n-k} C_k^{p+q}(x)\right] t^n\ ,\end{aligned} \tag{3.2}$$

our result follows.

3.1. **A consistency condition.** The following lemma gives the before announced consistency condition on the parameters p and q in order to work with an effective polynomial set

Lemma 1. *$\deg G_n^{(p,q)} = n$, for all n, if and only if*

$$\frac{(-p)_n}{n!}\,{}_2F_1(-n, p+q; 1+p-n; 2) \neq 0 \quad , \quad \text{for all } n\,. \tag{3.3}$$

Proof. Recall first that the coefficient of highest degree of the Gegenbauer polynomial C_n^ν is $2^n(\nu)_n/n!$. Since the generating function $\mathcal{G}^{(p,q)}(x,t)$ satisfies $\mathcal{G}^{(p,q)}(x,-t) = \mathcal{G}^{(p,q)}(-x,t)$, the polynomial set $\{G_n^{(p,q)}\}_{n\geq 0}$ is symmetric. It follows then from (3.1) that

$$G_n^{(p,q)}(x) = c_n x^n + \pi_{n-2}(x)$$

where π_{n-2} is a polynomial of degree less or equal to $n-2$ and

$$c_n = \sum_{k=0}^{n} \binom{p}{n-k} (-1)^{n-k}(p+q)_k \frac{2^k}{k!} = \frac{(-p)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (p+q)_k}{(1+p-n)_k} \frac{2^k}{k!}\,. \tag{3.4}$$

So

$$G_n^{(p,q)}(x) = \frac{(-p)_n}{n!}\,{}_2F_1\left(-n, p+q; 1+p-n; 2\right) x^n + \pi_{n-2}(x)\,,$$

and the desired result follows.
Particular case: From this lemma, we deduce that $\deg G_1^{(p,q)} = 1$ if and only if $2q+p \neq 0$. This result was also given by the initial conditions of the recurrence relation (2.2).

3.2. **Explicit expression.** It is possible to find the explicit expression of the $G_n^{(p,q)}$ polynomials. In fact, every polynomial like $G_n^{(p,q)}$, belonging to a symmetric polynomial set, i.e. such that: $G_n^{(p,q)}(-x) = (-1)^n G_n^{(p,q)}(x)$, can be written in the form:

$$G_n^{(p,q)}(x) = x^n \sum_{j=0}^{[n/2]} c(n,j)\,\frac{1}{x^{2j}}\,.$$

By using the identity (3.1) and the explicit expression of the Gegenbauer polynomials [Rainville book [6], equation (2) p. 277]:

$$C_n^\nu(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (\nu)_{n-k} (2x)^{n-2k}}{k!(n-2k)!}\,,$$

we find:

$$c(n,j) = \frac{(-1)^{n+j} 2^j}{j!} \sum_{k=2j}^{n} \binom{p}{n-k} \frac{(p+q)_{k-j}}{(k-2j)!} (-2)^k\,.$$

For $j=0$ we recover $c(n,0) = c_n$, already computed in equation (3.4).

Remark 3. It is worth to note that it is not possible to find a simple hypergeometric type representation for the polynomials $G_n^{(p,q)}$. In fact they are represented by a combination of hypergeometric functions. The relevant expression could be derived by using equation (3.1) and one of the hypergeometric representation of the Gegenbauer polynomials.

4. Miscellaneous results

We list in the following some particular relationships linking the $G_n^{(p,q)}$ polynomials shifting the indices p and q:

- Differential recurrence relation

$$\left(G_n^{(p,q)}\right)'(x) = 2(p+q)G_{n-1}^{(p,q+1)}(x) - p\,G_{n-1}^{(p-1,q+1)}(x)\ . \tag{4.1}$$

Proof. It follows from (2.1) differentiating both sides with respect to x.

- Property 1.

$$G_n^{(p,q)}(x) = G_n^{(p-1,q+1)}(x) - x\,G_{n-1}^{(p-1,q+1)}(x)\ , \tag{4.2a}$$

or, in equivalent form

$$G_n^{(p,q)}(x) = x\,G_{n-1}^{(p,q)}(x) + G_n^{(p+1,q-1)}(x)\ . \tag{4.2b}$$

Proof. It follows from

$$\mathcal{G}^{(p,q)}(x) = (1-xt)\mathcal{G}^{(p-1,q+1)}(x)\ .$$

- Property 2.

$$\begin{aligned} &G_n^{(p,q)}(x) - 2x\,G_{n-1}^{(p,q)}(x) + G_{n-2}^{(p,q)}(x) = \\ &= G_n^{(p-1,q)}(x) - x\,G_{n-1}^{(p-1,q)}(x) = G_n^{(p,q-1)}(x)\ . \end{aligned} \tag{4.3}$$

Proof. It follows from

$$(1-2xt+t^2)\mathcal{G}^{(p,q)}(x) = (1-xt)\mathcal{G}^{(p-1,q)}(x)\ ,$$

and from (4.2a).

- Property 3.

$$G_n^{(p,q)}(x) = G_n^{(p,q+1)}(x) - 2x\,G_{n-1}^{(p,q+1)}(x) + G_{n-2}^{(p,q+1)}(x)\ . \tag{4.4}$$

Proof. It follows from

$$(1-2xt+t^2)\mathcal{G}^{(p,q+1)}(x) = \mathcal{G}^{(p,q)}(x)\ .$$

- Property 4.

$$G_n^{(p-1,q+1)}(x) - x\,G_{n-1}^{(p-1,q+1)}(x) = G_n^{(p+1,q-1)}(x) + x\,G_{n-1}^{(p,q)}(x)\ . \tag{4.5}$$

Proof. It follows from (4.2a) and (4.2b).

- Property 5.

$$\begin{aligned} (n+1)G_{n+1}^{(p,q)}(x) = 2(p+q)\left[x\,G_n^{(p,q+1)}(x) - G_{n-1}^{(p,q+1)}(x)\right] - \\ -px\,G_n^{(p-1,q+1)}(x)\ . \end{aligned} \tag{4.6}$$

Proof. Since

$$\frac{\partial}{\partial t}(1-xt)^p(1-2xt+t^2)^{-p-q} =$$

$$= \frac{-px(1-xt)^{p-1}}{(1-2xt+t^2)^{p+q}} + \frac{2(p+q)(x-t)(1-xt)^p}{(1-2xt+t^2)^{p+q+1}} =$$

$$= -px \sum_{n=0}^{\infty} G_n^{(p-1,q+1)}(x)\, t^n + 2(p+q)x \sum_{n=0}^{\infty} G_n^{(p,q+1)}(x)\, t^n -$$

$$-2(p+q) \sum_{n=1}^{\infty} G_{n-1}^{(p,q+1)}(x)\, t^n \ ,$$

recalling equation (2.3) we find the result.

- Property 6.

$$G_n^{(p,q)}(x) = \sum_{k=0}^{n} G_{n-k}^{(0,q)}(x) G_k^{(p,0)}(x) \ , \tag{4.7}$$

Proof. It follows from

$$\mathcal{G}^{(p,q)}(x) = \mathcal{G}^{(0,q)}(x)\, \mathcal{G}^{(p,0)}(x) \ .$$

- Property 7.

$$\begin{aligned} &G_n^{(p,q)}(x) - 2x\, G_{n-1}^{(p,q)}(x) + G_{n-2}^{(p,q)}(x) = \\ &= \sum_{k=0}^{n} \binom{p}{n-k} (-1)^{n-k} x^{n-k} C_k^{p+q-1}(x) \ . \end{aligned} \tag{4.8}$$

Proof. It follows from (4.3) and (3.1).

5. About some generalizations of Hermite and Gegenbauer polynomials in the literature

Classical orthogonal polynomials (the Jacobi, Laguerre and Hermite) form the simplest class of special functions. At the same time, the theory of these polynomials admits wide generalizations. There is an extensive literature about mathematical properties of these polynomials and their applications in various areas. Particular interests are given to Hermite and Gegenbauer polynomials (included in Jacobi polynomials) and their generalizations. Indeed these polynomials appeared as the only solutions for many characterization problems arising in the d-orthogonal polynomial theory. It's well known that Hermite and Gegenbauer polynomials are the only symmetric classical orthogonal polynomials. That means that they satisfy the property: $P_n(-x) = (-1)^n P_n(x)$, and their derivatives are also orthogonal.

Another characterization was given separately by Al-Salam [7] and Von Bachhaus [8] who proved that the only orthogonal polynomials generated by

$$G(2xt - t^2) = \sum_{n=0}^{\infty} c_n P_n(x) t^n \ , \ \ c_n \neq 0 \ , \ \text{ where } \ G(z) = \sum_{n=0}^{\infty} a_n z^n \ , \ \ a_n \neq 0 \ ,$$

are the Hermite and Gegenbauer polynomials. For Hermite polynomials, $G(z) = e^z$ and for Gegenbauer polynomials, $G(z) = (1-z)^{(-\lambda)}$. There are in the literature three other characterization problems for which the only solutions are different generalizations of Hermite and Gegenbauer polynomials:

Generalization 1. The Dunkl operator T_μ is defined by

$$T_\mu f(x) = f'(x) + \mu \, \frac{f(x) - f(-x)}{x} \quad , \quad \mu > -\frac{1}{2} \, ,$$

f being a differentiable function. The operator T_0 is reduced to the derivative operator.

An orthogonal polynomial set $\{P_n\}_{n\geq 0}$ is called T_μ-classical if $\{T_\mu P_{n+1}\}_{n\geq 0}$ is also orthogonal.

Ben Cheikh and Gaied [9] proved that the only symmetric T_μ-classical orthogonal polynomials are the generalized Hermite polynomials $\{H_n^\mu\}_{n\geq 0}$ and the generalized Gegenbauer polynomials $\left\{S_n^{(\alpha,\mu-1/2)}\right\}_{n\geq 0}$ given respectively, in terms of Laguerre polynomials L_n^α, $n \geq 0$, and Jacobi polynomials $P_n^{(\alpha,\beta)}$, $n \geq 0$, by:

$$H_{2m}^\mu(x) = \frac{(-1)^m (2m)!}{\left(\mu + \frac{1}{2}\right)_m} \, L_m^{\mu-1/2}(x^2) \, ,$$

$$H_{2m+1}^\mu(x) = \frac{(-1)^m (2m+1)!}{\left(\mu + \frac{1}{2}\right)_{m+1}} \, x \, L_m^{\mu+1/2}(x^2) \, ,$$

$$S_{2m}^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+1)_m}{(\beta+1)_m} \, P_m^{(\alpha,\beta)}(2x^2 - 1) \, ,$$

$$S_{2m+1}^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+1)_{m+1}}{(\beta+1)_{m+1}} \, x \, P_m^{(\alpha,\beta+1)}(2x^2 - 1) \, ,$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

The polynomial set $\{H_n^\mu\}_{n\geq 0}$ is orthogonal with respect the weight function [10], [11]

$$|x|^{2\mu} e^{-x^2} \quad ; \quad -\infty < x < +\infty \, .$$

The polynomial set $\left\{S_n^{(\alpha,\beta)}\right\}_{n\geq 0}$ is orthogonal with respect the weight function [10]

$$|x|^{2\beta+1}(1 - x^2)^\alpha \quad ; \quad -1 < x < 1 \, .$$

Generalization 2. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the effect of the functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. Let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials in $\mathcal{P}$ such that $\deg P_n(x) = n$ for all n. In this case, we call also $\{P_n\}_{n\geq 0}$ is a polynomial set. The corresponding monic polynomial sequence $\{\hat{P}_n\}_{n\geq 0}$ is given by $P_n = \lambda_n \hat{P}_n$, $n \geq 0$, where λ_n is the normalization coefficient and its dual sequence $\{u_n\}_{n\geq 0}$ is defined by $\langle u_n, \hat{P}_m\rangle = \delta_{n,m}$, $n, m \geq 0$.

Definition 1. Let d be an arbitrary positive integer. The polynomial sequence $\{P_n\}_{n\geq 0}$ is called a *d-orthogonal polynomial sequence* (d-OPS) with respect to the d-dimensional functional $\mathcal{U} = {}^t(u_0, \cdots, u_{d-1})$ if it fulfills [12], [13]

$$\begin{cases} \langle u_k, P_m P_n\rangle = 0 \quad , \quad m > dn + k \quad , \quad n \geq 0 \, , \\ \langle u_k, P_n P_{dn+k}\rangle \neq 0 \quad , \quad n \geq 0 \, , \end{cases}$$

for each integer k belonging to $\{0, 1, \cdots, d-1\}$.

These orthogonality conditions are equivalent to the fact that the sequence $\{P_n\}_{n\geq 0}$ satisfies a $(d+1)$-order recurrence relation [12], which we write in the *monic* form as

$$\hat{P}_{m+d+1}(x) = (x-\beta_{m+d})\hat{P}_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} \hat{P}_{m+d-1-\nu}(x) \quad , \quad m \geq 0 \, ,$$

with the initial conditions

$$\begin{cases} \hat{P}_0(x) = 1 \quad , \quad \hat{P}_1(x) = x - \beta_0 \quad \text{and if} \quad d \geq 2 : \\ \hat{P}_n(x) = (x-\beta_{n-1})\hat{P}_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} \hat{P}_{n-2-\nu}(x) \quad , \quad 2 \leq n \leq d \, , \end{cases}$$

and the regularity conditions

$$\gamma_{n+1}^{0} \neq 0 \quad , \quad n \geq 0 \, .$$

When $d = 1$, the above recurrence relation is the well-known second-order recurrence relation satisfied by orthogonal polynomials:

$$\begin{cases} \hat{P}_{n+2}(x) = (x-\beta_{n+1})\hat{P}_{n+1}(x) - \gamma_{n+1}\hat{P}_n(x) \quad , \quad n \geq 0 \, , \\ \hat{P}_0(x) = 1 \quad , \quad \hat{P}_1(x) = x - \beta_0 \, . \end{cases} \tag{5.1}$$

Definition 2. Let d be a positive integer. A polynomial set $(P_n)_n$ is called *d-symmetric* if $P_n(w_d x) = w_d^n P_n(x)$ for all n, where $w_d = \exp(2i\pi/(d+1))$.

Definition 3. A polynomial set $\{P_n\}_n$ is called *classical d-orthogonal* if and only if both $\{P_n\}_n$ and $\{P'_{n+1}\}_n$ are d-orthogonal.

Definition 4. A polynomial set $\{P_n\}_n$ is called of *Boas-Buck type* if it is generated by

$$A(t)B(xC(t)) = \sum_{n=0}^{\infty} c_n P_n(x) t^n \quad , \quad c_n \neq 0 \, ,$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n t^n \;\; , \;\; B(t) = \sum_{n=0}^{\infty} b_n t^n \;\; , \;\; C(t) = \sum_{n=1}^{\infty} c_n t^n \;\; , \;\; a_n b_n c_1 \neq 0 \;\; , \;\; \forall n \, .$$

Ben Cheikh and Ben Romdhane [14] proved that the only d-symmetric classical OPSs of Boas-Buck type are the Gould-Hopper polynomials $\{G_n\}_{n\geq 0}$ and Humbert polynomials $\{h_n^{(\nu)}\}_{n\geq 0}$ given respectively by their generating functions:

$$\exp\left((d+1)xt - t^{d+1}\right) = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \, ,$$

and

$$\frac{1}{\left(1-(d+1)xt+t^{d+1}\right)^{\nu}} = \sum_{n=0}^{\infty} h_n^{(\nu)}(x) t^n \, .$$

For $d = 1$, the Gould-Hopper polynomials and Humbert polynomials are respectively reduced to Hermite polynomials and Gegenbauer polynomials. Various properties for these d-OPSs related to were given by Gould and Hopper [15], Humbert

[16], Ben Cheikh and Douak [17], Ben Cheikh and Ben Romdhane [14] and Lamiri and Ouni [18],[19].

Generalization 3.

Definition 5. A polynomial set $\{P_n\}_n$ is called T_μ*-classical* d*-orthogonal* if and only if both $\{P_n\}_n$ and $\{T_\mu P_{n+1}\}_n$ are d-orthogonal.

Definition 6. There exists a unique linear isomorphism called *intertwining operator* V_μ such that

$$V_\mu\left(\mathcal{P}_n\right)=\mathcal{P}_n \quad , \quad V_\mu(1)=1 \quad \text{and} \quad T_\mu V_\mu=V_\mu \frac{d}{dx} \ ,$$

where $\mathcal{P}_n$ denotes the set of polynomials of degree less or equal to n.

V_μ is given explicitly by:

$$V_\mu(x^n)=\frac{(1/2)_{[(n+1)/2]}}{(\mu+1/2)_{[(n+1)/2]}}\ x^n \ .$$

Ben Cheikh and Gaied [20] proved that the only d-symmetric T_μ-classical OPSs of Boas-Buck type are the Gould-Hopper type polynomials $\{G_n^{(\mu)}\}_{n\geq 0}$ and Humbert type polynomials $\{h_n^{(\nu,\mu)}\}_{n\geq 0}$ given respectively by:

$$G_n^{(\mu)}=V_\mu(G_n) \quad \text{and} \quad h_n^{(\nu,\mu)}=V_\mu\left(h_n^{(\nu)}\right) \ .$$

Various properties for these d-OPSs were given by Ben Cheikh and Gaied [20].

Now, it's worth to recall that Hermite and Gegenbauer polynomials are symmetric and satisfy a recurrence relation of type (5.1) and to note that all the above generalizations have corresponding ones to these two main properties. On other hand, we give a further generalization of Hermite polynomials introduced by Rainville [6], p. 237, who considered the polynomial set $\{g_n\}_{n\geq 0}$ generated by

$$\exp\left((2xt-t^2\right)\,{}_1F_1\left(1+\alpha;1+\beta;t^2\right)=\sum_{n=0}^{\infty} g_n(x)\frac{t^n}{n!} \ ,$$

and showed that these polynomials satisfy the following four-terms recurrence relation:

$$(2\beta+n)g_n(x)-(2\beta+2n-1)(2x)g_{n-1}(x)+$$
$$+(n-1)[2(2\alpha+n)+(2x)^2]g_{n-2}(x)-2(n-1)(n-2)(2x)g_{n-3}(x)=0 \ .$$

We refer to such recurrence relation as of type $RR_n[4(0,1,2,1)]$ (which means a recurrence relation linking four consecutive polynomials belonging to a polynomial set, the highest index is n, and where the first coefficient is a polynomial in x of degree 0, the second of degree 1, the third of degree 2 and the fourth of degree 1). Similarly, the recurrence relation (1) may be viewed as of type $RR_{n+2}[3(0,1,0)]$.

The polynomial set $\{g_n\}_{n\geq 0}$ is then a generalization of Hermite polynomials which is symmetric and satisfies a recurrence relation of type $RR_n[4(0,1,2,1)]$.

In this paper, we provide a new polynomial set $\{G_n^{p,q}\}_{n\geq 0}$ having the same properties and the question is: "*what corresponds to the orthogonality for polynomials satisfying a recurrence relation of type* $RR_n[4(0,1,2,1)]$?"

Notice by the way that every orthogonal polynomial set satisfies a recurrence relation of type $RR_n[4(0,1,2,1)$ according to the identity:

$$RR_n[3(0,1,0)]+(x+a)RR_{n-1}[3(0,1,0)]=RR_n[4(0,1,2,1)]$$

since every orthogonal polynomial set satisfies a recurrence relation of the type $RR_n[3(0,1,0)]$.

6. The multi-variable case

In [4] a set of multi-variable polynomials extending to the case of $r-1$ variables the classical Chebyshev polynomials of the first and second kind was introduced specializing the corresponding Lucas polynomials in r variables.

In order to generalize this set, introducing the above parameters p, q, and preserving the notation used in [4], we will consider in the following the multi-variable polynomial set defined by the generating function

$$\begin{aligned} &\mathcal{G}^{(p,q)}(u_1, u_2, \cdots, u_{r-1}, z) := \\ &= \frac{[r-(r-1)u_1 z + \cdots + (-1)^{r-1}u_{r-1}z^{r-1}]^p}{[1-u_1 z + u_2 z^2 + \cdots + (-1)^{r-1}u_{r-1}z^{r-1} + (-1)^r z^r]^{p+q}} = \\ &= \sum_{n=0}^{\infty} G^{(p,q)}_{n+r-2}(u_1, u_2, \cdots, u_{r-1}) z^n \ . \end{aligned} \tag{6.1}$$

We consider first the case when $r = 3$. In this case, the following theorem can be proved

Theorem 3. *For every choice of the parameters p and q, such that the consistency condition* $\deg G_n^{(p,q)} = n$ *is verified, the polynomial set* $\left\{G_n^{(p,q)}\right\} \equiv \left\{G_n^{(p,q)}(u_1, u_2)\right\}$ *satisfies the* 6*-terms recurrence relation*

$$\begin{aligned} &3(n+1)G^{(p,q)}_{n+2} = (p+3q+5n)u_1 G^{(p,q)}_{n+1} + \\ &+2[(1-q-n)u_1^2 + (2-2p-3q-2n)u_2]G^{(p,q)}_n + \\ &+[(p+5q-6+3n)u_1 u_2 + 3(3p+3q-2+n)]G^{(p,q)}_{n-1} + \\ &+[2(3-2p-3q-n)u_1 + (3-2q-n)u_2^2]G^{(p,q)}_{n-2} + \\ &+(p+3q-4+n)u_2 G^{(p,q)}_{n-3} \ , \end{aligned} \tag{6.2}$$

and the initial conditions

$$\begin{aligned} &G^{(p,q)}_{-2} = G^{(p,q)}_{-1} = G^{(p,q)}_0 = 0 \ , \\ &G^{(p,q)}_1 = 3^p \quad , \quad G^{(p,q)}_2 = 3^p \left(\frac{p}{3} + q\right) u_1 \ . \end{aligned} \tag{6.3}$$

Proof. Differentiating with respect to z both sides of (6.1), in the particular case when $r = 3$, yields

$$\begin{aligned} &\frac{\partial}{\partial z}(3-2u_1 z + u_2 z^2)^p (1-u_1 z + u_2 z^2 - z^3)^{-p-q} = \\ &= 2p(-u_1 + u_2 z)(3-2u_1 z + u_2 z^2)^{p-1}(1-u_1 z + u_2 z^2 - z^3)^{-p-q} + \\ &+(3-2u_1 z + u_2 z^2)^p \left[(p+q)(u_1 - 2u_2 z + 3z^2)(1-u_1 z + u_2 z^2 - z^3)^{-p-q-1}\right] = \\ &= \sum_{n=0}^{\infty}(n+1)G^{(p,q)}_{n+2}(u_1, u_2) z^n \ . \end{aligned}$$

Putting $z = 0$ in the this equation we find

$$G^{(p,q)}_2 = 3^p \left(\frac{p}{3} + q\right) u_1 \ .$$

Furthermore,

$$\left[\frac{2p(-u_1+u_2z)}{3-2u_1z+u_2z^2}+\frac{(p+q)(u_1-2u_2z+3z^2)}{1-u_1z+u_2z^2-z^3}\right]\sum_{n=0}^{\infty}G_{n+1}^{(p,q)}(u_1,u_2)z^n=$$

$$=\sum_{n=0}^{\infty}(n+1)G_{n+2}^{(p,q)}(u_1,u_2)z^n\ ,$$

so that

$$\begin{gathered}\big[2p(-u_1+u_2z)(1-u_1z+u_2z^2-z^3)+\\ +(p+q)(u_1-2u_2z+3z^2)(3-2u_1z+u_2z^2)\big]\sum_{n=0}^{\infty}G_{n+1}^{(p,q)}(u_1,u_2)z^n=\\ =(3-2u_1z+u_2z^2)(1-u_1z+u_2z^2-z^3)\sum_{n=0}^{\infty}(n+1)G_{n+2}^{(p,q)}(u_1,u_2)z^n\ ,\end{gathered}$$

and therefore the recurrence relation (6.2) follows by comparing the coefficients of same powers of z in the last equation. Other initial conditions in (6.3) easily follow.

The same procedure can be used in the general case, but in order to avoid cumbersome computations, it is convenient to let

$$\begin{gathered}N:=N(u_1,u_2,\cdots,u_{r-1}):=\\ :=r-(r-1)u_1z+\cdots+(-1)^{r-1}u_{r-1}z^{r-1}\ ,\\ D:=D(u_1,u_2,\cdots,u_{r-1}):=\\ :=1-u_1z+u_2z^2+\cdots+(-1)^{r-1}u_{r-1}z^{r-1}+(-1)^rz^r\ ,\end{gathered}\tag{6.4}$$

so that equation (6.1) becomes

$$\frac{N^p}{D^{p+q}}=\sum_{n=0}^{\infty}G_{n+r-2}^{(p,q)}(u_1,u_2,\cdots,u_{r-1})z^n\ ,\tag{6.5}$$

and subsequently (denoting by suffix the derivative with respect to z),

$$\frac{\partial}{\partial z}N^pD^{-p-q}=\frac{pN_zN^{p-1}}{D^{p+q}}-\frac{(p+q)D_zN^p}{D^{p+q+1}}=$$

$$=\sum_{n=0}^{\infty}(n+1)G_{n+r-1}^{(p,q)}(u_1,u_2,\cdots,u_{r-1})z^n\ ,$$

$$\left[p\frac{N_z}{N}-(p+q)\ \frac{D_z}{D}\right]\sum_{n=0}^{\infty}G_{n+r-2}^{(p,q)}(u_1,u_2,\cdots,u_{r-1})z^n=$$

$$=\sum_{n=0}^{\infty}(n+1)G_{n+r-1}^{(p,q)}(u_1,u_2,\cdots,u_{r-1})z^n\ ,$$

$$[pN_zD-(p+q)D_zN]\sum_{n=0}^{\infty}G_{n+r-2}^{(p,q)}z^n=N\,D\sum_{n=0}^{\infty}(n+1)G_{n+r-1}^{(p,q)}z^n\ .\tag{6.6}$$

Therefore, we can derive the following general result

Theorem 4. *For every choice of the parameters p and q, such that the consistency condition* $\deg G_n^{(p,q)} = n$ *is verified, the polynomial set* $\left\{G_n^{(p,q)}\right\} \equiv \left\{G_n^{(p,q)}(u_1, u_2, \cdots, u_{r-1})\right\}$ *satisfies the 2r-terms recurrence relation*

$$
\begin{aligned}
r(n+1)\, G_{n+r-1}^{(p,q)} &= [p+qr+(2r-1)n]\, u_1\, G_{n+r-2}^{(p,q)} - \\
&-\{[q(r-1)+(r-1)(n-1)]u_1^2 + \\
&+[2^2p+2qr+(2r-2)(n-1)]\, u_2\}\, G_{n+r-3}^{(p,q)} + \\
&+\{[p+(3r-4)q+(2r-3)(n-2)]\, u_1u_2 + \\
&+[3^2p+3qr+(2r-3)(n-2)]\, u_3\}\, G_{n+r-4}^{(p,q)} + \\
&+\cdots+ \\
&+\left\{(-1)^k[p(1-k)+q(r-k)+(r-1)(n-k)]\, u_1u_k + \right.\\
&+(-1)^{k-1}[2p(-3+k)-2q(r-k+1)-(r-2)(n-k)]\, u_2u_{k-1} + \\
&+(-1)^{k-2}[3p(5-k)+3q(r-k+2)+(r-3)(n-k)]\, u_3u_{k-2} + \\
&\left.+\cdots+(-1)^k r(n-k)u_{k+1}\right\} G_{n+r-2-k}^{(p,q)} + \\
&+\cdots+ \\
&+\{[p+(3r-4)q+3(-2r+4+n)]\, u_{r-1}u_{r-2} + \\
&+3[3p+qr-2r+4+n]\, u_{r-3}\}\, G_{n-r+2}^{(p,q)} + \\
&+\{-2(2p+qr-2r+3+n)\, u_{r-2} - \\
&-[q(r-1)-2r+3+n]\, u_{r-1}^2\}\, G_{n-r+1}^{(p,q)} + \\
&+[p+qr-2(r-1)+n]\, u_{r-1}\, G_{n-r}^{(p,q)}\ ,
\end{aligned}
\tag{6.7}
$$

and the initial conditions

$$
\begin{aligned}
&G_{-r+1}^{(p,q)} = G_{-r+2}^{(p,q)} = \cdots = G_0^{(p,q)} = G_1^{(p,q)} = \cdots = G_{r-3}^{(p,q)} = 0\ , \\
&G_{r-2}^{(p,q)} = r^p \quad , \quad G_{r-1}^{(p,q)} = r^p\left(\frac{p}{r}+q\right)u_1\ .
\end{aligned}
\tag{6.8}
$$

Note that the general term of equation (6.7)*, is valid provided that* $0 \le k \le r-2$*, since we must assume* $u_r = u_{r+1} = \cdots := 0$.

Proof. By differentiating with respect to z, multiplying factors in the left and right hand sides of equation (6.6), and equating the coefficients of same powers of z in the same equation, we find, without changing signs nor compacting terms, in order to show the construction rule

$$
\begin{aligned}
r(n+1)\, G_{n+r-1}^{(p,q)} &= -\{p[(r-1)u_1]-(p+q)[ru_1]-[ru_1+(r-1)u_1]n\}\, G_{n+r-2}^{(p,q)} + \\
&+\{p[(r-1)u_1u_1+2(r-2)u_2]-(p+q)[(r-1)u_1u_1+2ru_2] - \\
&-[ru_2+(r-1)u_1u_1+(r-2)u_2](n-1)\}\, G_{n+r-3}^{(p,q)} - \\
&-\{p[(r-1)u_1u_2+2(r-2)u_2u_1+3(r-3)u_3] - \\
&(p+q)[(r-2)u_1u_2+2(r-1)u_2u_1+3ru_3] + \\
&+[ru_3+(r-1)u_2u_1+(r-2)u_1u_2+(r-3)u_3](n-2)\}\, G_{n+r-4}^{(p,q)} + \\
&+\cdots+
\end{aligned}
$$

$$+\left\{p\left[-(-1)^k(r-1)u_1u_k+(-1)^{k-1}2(r-2)u_2u_{k-1}-\right.\right.$$
$$-(-1)^{k-2}3(r-3)u_3u_{k-2}+\cdots+(-1)^{k+1}(k+1)(r-k-1)u_{k+1}\right]-$$
$$-(p+q)\left[-(-1)^k((r-k)u_1u_k)+(-1)^{k-1}2(r-k+1)u_2u_{k-1}-\right.$$
$$-(-1)^{k-2}3(r-k+2)u_3u_{k-2}+\cdots+(-1)^{k+1}(k+1)ru_{k+1}\right]-$$
$$-\left[(-1)^{k+1}ru_{k+1}-(-1)^k(r-1)u_1u_k+(-1)^{k-1}(r-2)u_2u_{k-1}-\right.$$
$$\left.\left.-(-1)^{k-2}(r-3)u_3u_{k-2}+\cdots+(-1)^{k+1}(r-k-1)u_{k+1}\right](n-k)\right\}G^{(p,q)}_{n+r-2-k}+$$
$$+\ \cdots\ +$$
$$+(-1)^{2r-3}\left\{p\left[(r-1)u_{r-1}u_{r-2}+2(r-2)u_{r-2}u_{r-1}+3(r-3)u_{r-3}\right]-\right.$$
$$-(p+q)\left[3ru_{r-3}+2(r-1)u_{r-2}u_{r-1}+(r-2)u_{r-1}u_{r-2}\right]-$$
$$\left.-\left[3u_{r-3}+2u_{r-2}u_{r-1}+u_{r-1}u_{r-2}\right]\left[n-(2r-4)\right]\right\}G^{(p,q)}_{n-r+2}+$$
$$+(-1)^{2r-2}\left\{p\left[(r-1)u_{r-1}u_{r-1}+2(r-2)u_{r-2}\right]-\right.$$
$$-(p+q)\left[2ru_{r-2}+(r-1)u_{r-1}u_{r-1}\right]-$$
$$\left.-\left[2u_{r-2}+u_{r-1}u_{r-1}\right]\left[n-(2r-3)\right]\right\}G^{(p,q)}_{n-r+1}+$$
$$+(-1)^{2r-1}\left\{p(r-1)u_{r-1}-(p+q)ru_{r-1}-u_{r-1}[n-(2r-2)]\right\}G^{(p,q)}_{n-r}\ .$$

Rearranging this equation, the recurrence (6.7) follows.

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