Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 9(2010), pp. 41–56.

On obtaining dual sequences via inversion coefficients

Youssef Ben Cheikh and Imed Lamiri

Dedicated to Professor P.E. Ricci on the occasion of his 65th birthday

Abstract[1]. In this paper, we express explicitly the dual sequence of Boas-Buck polynomials and we derive integral representations of the dual sequence of two classes of polynomial sets given by their inversion formulas. That can be useful in studying the d-orthogonality of some polynomials, and allows us to unify the treatment of the dual sequence problem for many well-known polynomials in the literature.

1. Introduction

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the effect of the functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. A polynomial sequence $\{P_n\}_{n\geq 0}$ is called a *polynomial set* (PS, for shorter) if and only if $\deg P_n = n$ for all non-negative integer n. The *dual sequence* (DS, for shorter) $\{u_r\}_{r\geq 0}$ associated with the PS $\{P_n\}_{n\geq 0}$ is defined by $\langle u_r, P_n\rangle = \delta_{r,n}$; $r, n \in \mathbb{N} = \{0, 1, 2, \cdots\}$; where $\delta_{r,n}$ denotes the Kronecker symbol. Among the problems related to this notion, we quote the following.

Problem P: For a given PS $\{P_n\}_{n\geq 0}$, find an integral representation of its DS $\{u_r\}_{r\geq 0}$ of the type:

$$\langle u_r, x^n\rangle = \int_0^\infty t^n \psi_r(t)\, dt \quad , \quad \text{if } n \geq r\ . \tag{1.1}$$

Note that if $n < r$, $\langle u_r, x^n\rangle = 0$.
Such a problem arises in various fields of mathematics. In particular, in studying d-orthogonal polynomials. That was introduced by Maroni [15] and Van Iseghem [19] as follows.

Let d be a positive integer. A PS $\{P_n\}_{n\geq 0}$ is called *d-orthogonal* (d-OPS, for shorter) with respect to the d-dimensional functional vector $\Gamma =^t (\Gamma_0, \Gamma_1, \cdots, \Gamma_{d-1})$ if it

---

[1]Authors' address: Y. Ben Cheikh, Université de Monastir, Département de Mathématiques, 5019 Monastir, Tunisia; e-mail: : `youssef.bencheikh@planet.tn`.

I. Lamiri, École Supérieure des Sciences et Technologies de Hammam Sousse, Département de Mathématiques, Rue Lamine Abassi, 4011 Hammam Sousse, Tunisia;
e-mail: `Imed.Lamiri@infcom.rnu.tn`.

Keywords. Dual sequence, Boas-Buck polynomials, inversion formula, Meijer G-function, d-orthogonality, linear functionals.

AMS Subject Classification. 44A60, 33C47, 42C05.

satisfies the following orthogonality relations:

$$(1.2)\qquad \begin{cases} \langle \Gamma_k, P_r P_n \rangle = 0 \quad , & r > nd + k,\ n \in \mathbb{N}\ , \\ \langle \Gamma_k, P_n P_{nd+k} \rangle \neq 0 \quad , & n \in \mathbb{N}\ , \end{cases}$$

for each integer k belonging to $\mathbb{N}_d = \{0, 1, \cdots, d-1\}$.
For $d = 1$, this notion is reduced to the known notion of orthogonality.
As proved by Douak and Maroni [7], a PS $\{P_n\}_{n\geq 0}$ is d-orthogonal with respect to a d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \cdots, \Gamma_{d-1})$ if and only if it is also d-orthogonal with respect to the vector $\mathcal{U} = {}^t(u_0, u_1, \cdots, u_{d-1})$, where the functionals $u_0, u_1, \cdots, u_{d-1}$ are the d first elements of the dual sequence $\{u_n\}_{n\geq 0}$ associated with the PS $\{P_n\}_{n\geq 0}$. Consequently, for a d-OPS, we consider the d first elements of the corresponding dual sequence as the d-dimensional functional vector ensuring the d-orthogonality of these polynomials.

Problem P contains in fact two questions:
Question 1: Find explicit expression of the sequences

$$(\alpha_n(r))_{n\geq 0} = (\langle u_r, x^n \rangle)_{n\geq 0} \quad , \quad r \in \mathbb{N}\ .$$

Question 2 (*Moment problem*): Find a suitable function $\psi_r(t)$ such that (1.1).
Among the methods developed to solve Question 1, we could mention the one which requires that the polynomials P_n be orthogonal. Then $\langle u_r, x^n \rangle$, $n \geq 0$ are the corresponding Fourier coefficients.
A second one, introduced by the first author [1] is based on the lowering operator and a formal power series associated with the involved PS. More precisely, it was shown that if σ is the lowering operator of the PS $\{P_n\}_{n\geq 0}$, the DS of $\{P_n\}_{n\geq 0}$ is given by

$$\langle u_r, x^n \rangle = \frac{1}{r!}[\sigma^r \phi(\sigma)(x^n)]_{x=0}\ ,$$

where $\phi(t) = \sum_{n=0}^{\infty} \alpha_n t^n, \alpha_0 \neq 0$, is a suitable power series.
A third one (see, for instance [20]) which was deduced from the inversion formula:

$$x^n = \sum_{j=0}^{n} I_j(n) P_j(x) \quad , \quad n \in \mathbb{N}\ ,$$

provides

$$(1.3)\qquad \langle u_r, x^n \rangle = I_r(n) \quad , \quad n \geq r\ .$$

In this work, we treat the two previous questions, using the inversion formulas, for some general classes of PSs. More precisely, we give a solution to Question 1 for Boas-Buck polynomials and a solution to Question 2 for two classes of PSs having specific inversion formulas and containing many known PSs.

2. Dual sequence of a Boas-Buck polynomial set

Let us, firstly, introduce the following definitions and notations. We will denote by:
- $\mathcal{S}_0$ the set of all formal power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_0 \neq 0$;
- $\widehat{\mathcal{S}}_0$ the set of all formal power series $B(t) = \sum_{n=0}^{\infty} b_n t^n$, $b_n \neq 0$ for all n;
- $\mathcal{S}_1$ the set of all formal power series $C(t) = \sum_{n=1}^{\infty} c_n t^n$, $c_1 \neq 0$.

Let $C \in \mathcal{S}_1$. By C^* we mean the composition inverse of C given by the relations: $C^*(C(t)) = t$ and $C(C^*(s)) = s$.

Definition 2.1. A PS $\{P_n\}_{n\geq 0}$ is said to be of *Boas-Buck type* (or a *Boas-Buck PS*) if it is generated by:

$$A(t)B(xC(t)) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n , \tag{2.1}$$

where $(A, B, C) \in \mathcal{S}_0 \times \widehat{\mathcal{S}}_0 \times \mathcal{S}_1$.

This class of PSs contains the most known PSs.
Two particular subclasses are worth to note. The first one is the Brenke polynomials where $C(t) = t$. The second one corresponds to Sheffer polynomials, where $B(t) = e^t$.
We state the following.

Theorem 2.2. *Let $\{P_n\}_{n\geq 0}$ be a Boas-Buck PS generated by the identity* (2.1).
Then the dual sequence $\{u_r\}_{r\geq 0}$ associated with $\{P_n\}_{n\geq 0}$ is given by

$$\langle u_r, x^n \rangle = \frac{\alpha_r(n)}{b_n} \quad , \quad n \geq r ,$$

where

$$\frac{C^{*k}(t)}{k!A(C^*(t))} = \sum_{n=k}^{\infty} \alpha_k(n) t^n . \tag{2.2}$$

Proof. We recall that, a Boas-Buck PS generated by the identity (2.1) have the following inversion formula [2]:

$$x^n = \sum_{j=0}^{n} \frac{\alpha_j(n)}{b_n} P_j(x) ,$$

where the coefficients $\alpha_j(n)$ are given by the identity (2.2).
That, by virtue of (1.3), leads to the desired result.

□

Next, we illustrate an application of Theorem 2.2.
From Theorem 2.2, one can deduce an explicit expression of the linear functional u_0 ensuring the orthogonality of an orthogonal PS of Boas-Buck type. In fact we have the following.

Corollary 2.3. *Let $\{P_n\}_{n\geq 0}$ be an orthogonal PS generated by* (2.1). *Then the linear functional u_0 for which the orthogonality holds is given by*

$$\langle u_0, x^n \rangle = \frac{\alpha_n}{b_n}$$

where $(1/A)(C^*(t)) = \sum_{n=0}^{\infty} \alpha_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$.

Notice by the way that the characterization of all orthogonal PSs of Boas-Buck type is an open question as mentioned in Ismail's book [10]. In the meanwhile, Meixner [16] gave the only five orthogonal PSs of Sheffer type and Chihara [6] gave the only eight orthogonal PSs of Brenke type. Next, we show how one can deduce known results from Corollary 2.3 for some PSs belonging to theses two subclasses of Boas-Buck polynomials by considering two examples.

Example 2.4 (Hermite polynomials)**.** The Hermite polynomials are generated by [11]:

$$G(x,t) = \underbrace{\exp(-t^2)}_{A(t)} \exp(x \underbrace{2t}_{C(t)}) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n .$$

Observing that

$$C^*(t) = \frac{t}{2} , \quad \frac{1}{A(C^*(t))} = \exp\left(\frac{t^2}{4}\right) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}k!} \frac{t^{2k}}{(2k)!} = \sum_{n=0}^{\infty} \langle u_0, x^n \rangle \frac{t^n}{n!} , \; b_n = \frac{1}{n!} .$$

Then the linear functional u_0 for which the orthogonality holds is given by:

$$\langle u_0, x^n \rangle = \begin{cases} 0 , & \text{if } n \text{ is odd} , \\ \dfrac{n!}{2^n (n/2)!} = \dfrac{1}{\sqrt{\pi}} \Gamma\left(\dfrac{n+1}{2}\right) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^{+\infty} x^n e^{-x^2} \, dx , & \text{if } n \text{ is even} , \end{cases}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^n e^{-x^2} \, dx .$$

Example 2.5 (Laguerre polynomials)**.** The Laguerre polynomials are generated by [11]:

$$\underbrace{e^t}_{A(t)} \underbrace{{}_0F_1 \left(\begin{matrix} - \\ \alpha+1 \end{matrix} ; -xt \right)}_{B(xC(t))} = \sum_{n=0}^{\infty} \underbrace{\frac{n! L_n^{(\alpha)}(x)}{(\alpha+1)_n}}_{P_n(x)} \frac{t^n}{n!} ,$$

where ${}_pF_q(z)$ designates the generalized hypergeometric function given by [14]:

$$(2.3) \qquad {}_pF_q \left(\begin{matrix} (a_p) \\ (b_q) \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{[a_p]_k}{[b_q]_k} \frac{z^k}{k!} ,$$

with $[a_r]_p = \prod_{i=1}^{r} (a_i)_p$, $(a_i)_p = \Gamma(a_i + p)/\Gamma(a_i)$ the Pochhammer symbol, and (a_r) the set $\{a_1, \cdots, a_r\}$.
In this case, we have:

$$C(t) = C^*(t) = -t \quad ; \quad \frac{1}{A(C^*(t))} = \exp(t) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{n!}}_{\alpha_n} t^n ,$$

$$B(t) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{n!(\alpha+1)_n}}_{b_n} t^n .$$

It follows that:

$$\langle u_0, x^n \rangle = (\alpha+1)_n = \frac{1}{\Gamma(\alpha+1)} \int_0^{+\infty} x^n x^{\alpha} e^{-x} \, dx \quad , \quad (\Re(\alpha) > -1) .$$

3. Integral representations

In this section, we derive integral representations of the dual sequences of two classes of PSs given by their inversion formulas. Then, as applications, we consider some special cases. Some d-orthogonal PSs will be discussed.

3.1. **Theorems.**

Theorem 3.1. *Let* $\{P_n\}_{n\geq 0}$ *be a PS satisfying the following inversion formula of the type:*

$$x^n = \sum_{j=0}^{n} \xi_j [h(j)]^{l(j)n} \, \frac{\prod_{i=1}^{m} \Gamma(\lambda_i(j)+n)}{\prod_{i=1}^{s} \Gamma(\rho_i(j)+n)} \, P_j(x) \ , \tag{3.1}$$

where $m \geq s$ *and* $h(j), l(j), \lambda_i(j), \rho_i(j)$ *are real numbers. Then the dual sequence* $\{u_r\}_{r\geq 0}$ *associated with* $\{P_n\}_{n\geq 0}$ *is given by*

$$\langle u_r, x^n \rangle = \int_0^\infty t^n \psi_r(t)\, dt \quad , \quad n \geq r \ ,$$

where

$$\psi_r(t) = \frac{\xi_r}{t} G_{s,\ m}^{m,\ 0}\left(\frac{t}{(h(r))^{l(r)}} \,\middle|\, \begin{matrix} (\rho_s) \\ (\lambda_m) \end{matrix} \right), \qquad \text{if } m > s \ ,$$

$$\psi_r(t) = \chi_{[0,(h(r))^{l(r)}]} \frac{\xi_r}{t} \, G_{m,\ m}^{m,\ 0}\left(\frac{t}{(h(r))^{l(r)}} \,\middle|\, \begin{matrix} (\rho_m) \\ (\lambda_m) \end{matrix} \right), \ \text{if } m = s \text{ and } \sum \rho_i - \lambda_i > 0 \ .$$

$G_{p,\ q}^{m,\ n}$ *designates the Meijer's G-function defined by* [14, p. 143]*:*

$$G_{p,\ q}^{m,\ n}\left(z \,\middle|\, \begin{matrix} (a_p) \\ (b_q) \end{matrix} \right) = (2\pi i)^{-1} \int_L z^\tau \, \frac{\prod_{j=1}^{m} \Gamma(b_j - \tau) \prod_{j=1}^{n} \Gamma(1 - a_j + \tau)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \tau) \prod_{j=n+1}^{p} \Gamma(a_j - \tau)} \, d\tau \ .$$

We refer to [14, p. 144] *for the details regarding the type of the contour* L *and for the conditions on the involved parameters.*

Proof. According to the identities (1.3) and (3.1), we get:

$$\langle u_r, x^n \rangle = \xi_r [h(r)]^{l(r)n} \, \frac{\prod_{i=1}^{m} \Gamma(\lambda_i(r)+n)}{\prod_{i=1}^{s} \Gamma(\rho_i(r)+n)} \quad , \quad n \geq r \ . \tag{3.2}$$

We consider the following two cases.

Case 1: $m > s$. Using the identity (3.2) and the Mellin transform of the Meijer's G-function given by [14, p. 157]:

$$\begin{aligned} &\int_0^\infty G_{p,\ q}^{m,\ l}\left(\mu t \,\middle|\, \begin{matrix} (a_p) \\ (b_q) \end{matrix} \right) t^{s-1}\, dt = \\ &= \frac{\mu^{-s} \prod_{j=1}^{m} \Gamma(b_j + s) \prod_{j=1}^{l} \Gamma(1 - a_j - s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - s) \prod_{j=l+1}^{p} \Gamma(a_j + s)} \ , \end{aligned} \tag{3.3}$$

we obtain

$$\langle u_r, x^n \rangle = \xi_r [h(r)]^{l(r)n} \int_0^\infty G_{s,\ m}^{m,\ 0}\left(t \,\middle|\, \begin{matrix} (\rho_s) \\ (\lambda_m) \end{matrix} \right) t^{n-1}\, dt \ .$$

That, upon the change of variable $t = t'/[h(r)]^{l(r)}$, leads to the desired result.

Case 2: $m = s$. Recall that the first author and Douak [4] showed that

$$(3.4)\qquad \begin{aligned} {}_pF_q\left(\begin{matrix}(a_p)\\ (\alpha_q+\beta_q+1)\end{matrix};x\right) &= \prod_{i=1}^{q}\left(\frac{\Gamma(\alpha_i+1+\beta_i)}{\Gamma(\alpha_i+1)}\right)\times \\ &\times\int_0^1 G^{q,\ 0}_{q,\ q}\left(t\,\middle|\,\begin{matrix}(\alpha_q+\beta_q)\\ (\alpha_q)\end{matrix}\right)\,{}_pF_q\left(\begin{matrix}(a_p)\\ (\alpha_q+1)\end{matrix};xt\right)dt\ ,\end{aligned}$$

where $\sum_{j=1}^{d+1}\beta_j > 0$.
This identity, for $x = 0$ and $q = m$, is reduced to

$$(3.5)\qquad \prod_{j=1}^{m}\left[\frac{\Gamma(\alpha_j+1)}{\Gamma(\alpha_j+1+\beta_j)}\right] = \int_0^1 G^{m,\ 0}_{m,\ m}\left(t\,\middle|\,\begin{matrix}(\alpha_m+\beta_m)\\ (\alpha_m)\end{matrix}\right)dt\ .$$

Thus, for $\sum_{i=1}^{m}\rho_i - \lambda_i > 0$, the identity (3.2) can be rewritten under the form

$$\langle u_r, x^n\rangle = \xi_r[h(r)]^{l(r)n}\int_0^1 G^{m,\ 0}_{m,\ m}\left(t\,\middle|\,\begin{matrix}(\rho_m(r)+n-1)\\ (\lambda_m(r)+n-1)\end{matrix}\right)dt\ .$$

That, by virtue of the transformation [17, p. 46]:

$$(3.6)\qquad z^{\sigma}G^{m,\ n}_{p,\ q}\left(z\,\middle|\,\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_q\end{matrix}\right) = G^{m,\ n}_{p,\ q}\left(z\,\middle|\,\begin{matrix}\alpha_1+\sigma,\cdots,\alpha_p+\sigma\\ \beta_1+\sigma,\cdots,\beta_q+\sigma\end{matrix}\right)\ ,$$

leads to the following identity

$$\langle u_r, x^n\rangle = \xi_r[h(r)]^{l(r)n}\int_0^1 t^{n-1}G^{m,\ 0}_{m,\ m}\left(t\,\middle|\,\begin{matrix}(\rho_m(r))\\ (\lambda_m(r))\end{matrix}\right)dt\ .$$

Using the change of the variable $X = t[h(r)]^{l(r)}$, we get:

$$\langle u_r, x^n\rangle = \xi_r\int_0^{[h(r)]^{l(r)}} X^{n-1}G^{m,\ 0}_{m,\ m}\left(\frac{X}{[h(r)]^{l(r)}}\,\middle|\,\begin{matrix}(\rho_m(r))\\ (\lambda_m(r))\end{matrix}\right)dt\ .$$

□

Theorem 3.2. *Let $\{P_n\}_{n\geq 0}$ be a PS satisfying the following inversion formula of the type:*

$$(3.7)\qquad x^n = \sum_{j=0}^{[n/(d+1)]}\frac{\xi^j}{j!}\,\mu_{n-(d+1)j}\,\frac{\prod_{l=1}^{m}\Gamma(\lambda_l+pj+n)}{\prod_{l=1}^{s}\Gamma(\rho_l+qj+n)}\,P_{n-(d+1)j}(x)\ ,$$

where $m \geq s$, $p, q \in \mathbb{Z}$ and ξ, μ_j, λ_j and ρ_j are real numbers. Then the dual sequence $\{u_r\}_{r\geq 0}$ associated with $\{P_n\}_{n\geq 0}$ is given by

$$\langle u_r, x^n\rangle = \delta_{i,i'}\int_0^{\infty} t^n\psi_r(t)\,dt\quad ,\quad n \geq r\ ,$$

with

$$\begin{cases} n = i+(d+1)k\quad , & k\in\mathbb{N}\ ,\ i\in\mathbb{N}_{d+1}\ ,\\ r = i'+(d+1)k'\quad , & k'\in\mathbb{N}\ ,\ i'\in\mathbb{N}_{d+1}\ ,\end{cases}$$

and ψ_r the function defined by

- *for* $m(d+p+1) > s(d+q+1)+1$,

(3.8)
$$\psi_r(t) = (d+1)T_r t^{-(r+1)} \times$$
$$\times G^{m(d+p+1),\ 0}_{s(d+q+1)+1,\ m(d+p+1)}\left(\frac{t^{d+1}}{\xi_1}\,\middle|\,\begin{matrix}1, \Delta(d+1+q, \rho_1+r), \cdots, \Delta(d+1+q, \rho_s+r)\\ \Delta(d+1+p, \lambda_1+r), \cdots, \Delta(d+1+p, \lambda_m+r)\end{matrix}\right)$$

- *for* $m(d+p+1) = s(d+q+1)+1$,

(3.9)
$$\psi_r(t) = \chi_{[0,\xi_1^{1/(d+1)}]}(d+1)T_r t^{-(r+1)} \times$$
$$\times G^{m(d+p+1),\ 0}_{m(d+p+1),\ m(d+p+1)}\left(\frac{t^{d+1}}{\xi_1}\,\middle|\,\begin{matrix}0, \Delta(d+1+q, \rho_1+r), \cdots, \Delta(d+1+q, \rho_s+r)\\ \Delta(d+1+p, \lambda_1+r), \cdots, \Delta(d+1+p, \lambda_m+r)\end{matrix}\right)$$

where $\Delta(r, \lambda)$ *designates the array of* r *parameters* $(\lambda + j - 1)/r$, $j = 1, \cdots, r$,

(3.10)
$$T_r = \mu_r \, \frac{\prod_{l=1}^{m}\left[\dfrac{\Gamma(\lambda_l + r)}{\prod_{j=0}^{d+p}\Gamma((\lambda_l + r + j)/(d+1+p))}\right]}{\prod_{l=1}^{s}\left[\dfrac{\Gamma(\rho_l + r)}{\prod_{j=0}^{d+q}\Gamma((\rho_l + r + j)/(d+1+q))}\right]} \qquad \textit{and}$$

$$\xi_1 = \xi \, \frac{(d+1+p)^{(d+1+p)m}}{(d+1+q)^{(d+1+q)s}} \, .$$

Proof. Let n and r be two integers. Put

(3.11)
$$\begin{cases} n = i + (d+1)k \quad , \quad k \in \mathbb{N} \quad , \quad i \in \mathbb{N}_{d+1} \, , \\ r = i' + (d+1)k' \quad , \quad k' \in \mathbb{N} \quad , \quad i' \in \mathbb{N}_{d+1} \, , \\ k'' = k - k' \, . \end{cases}$$

According to (1.3) and (3.7), we get

(3.12)
$$\langle u_r, x^n \rangle = \delta_{i,i'} \mu_r \frac{\xi^{k''}}{k''!} \, \frac{\prod_{l=1}^{m}[\lambda_l + r + (d+1+p)k'']}{\prod_{l=1}^{s}[\rho_l + r + (d+1+q)k'']} \, .$$

The use of the Gauss's multiplication theorem for the Gamma function [17]:

(3.13)
$$\frac{\Gamma(a+mk)}{\Gamma(a)} = m^{m\,k} \prod_{j=0}^{m-1} \frac{\Gamma\left((a+j)/m + k\right)}{\Gamma\left((a+j)/m\right)} \quad , \quad k = 0, 1, 2, \cdots \, ,$$

leads to

(3.14)
$$\langle u_r, x^n \rangle = \delta_{i,i'} T_r \xi_1^{k''} \times$$
$$\times \frac{\prod_{l=1}^{m}\left[\prod_{j=0}^{d+p}\Gamma\left((\lambda_l + r + j)/(d+1+p) + k''\right)\right]}{\Gamma(1+k'')\prod_{l=1}^{s}\left[\prod_{j=0}^{d+q}\Gamma\left((\rho_l + r + j)/(d+1+q) + k''\right)\right]} \, ,$$

where T_r and ξ_1 are given by (3.10).
We consider the following two cases.

Case 1. $m(d+p+1) > s(d+q+1)+1$. Using the identity (3.14) and the Mellin transform of the Meijer's G-function given by (3.3), we obtain

$$\langle u_r, x^n\rangle = \delta_{i,i'} T_r \int_0^\infty t^{k''-1} \times$$
$$\times G^{m(d+p+1),\ 0}_{s(d+q+1)+1,\ m(d+p+1)}\left(\frac{t}{\xi_1}\,\middle|\,\begin{matrix}1, \Delta(d+1+q, \rho_1+r), \cdots, \Delta(d+1+q, \rho_s+r)\\ \Delta(d+1+p, \lambda_1+r), \cdots, \Delta(d+1+p, \lambda_m+r)\end{matrix}\right) dt\ .$$

Applying the change of variable $t = X^{d+1}$ to obtain (3.8).

Case 2. $m(d+p+1) = s(d+q+1)+1$. The identity (3.14) under the transformation (3.5), gives

$$\langle u_r, x^n\rangle = \delta_{i,i'} T_r \int_0^1 \frac{(\xi_1 t)^{k''}}{t} \times$$
$$\times G^{m(d+p+1),\ 0}_{m(d+p+1),\ m(d+p+1)}\left(t\,\middle|\,\begin{matrix}0, \Delta(d+1+q, \rho_1+r), \cdots, \Delta(d+1+q, \rho_s+r)\\ \Delta(d+1+p, \lambda_1+r), \cdots, \Delta(d+1+p, \lambda_m+r)\end{matrix}\right) dt\ .$$

That, upon the change of variable $\xi_1 t = X^{d+1}$, leads to the desired result.

□

3.2. **Special cases.**

3.2.1. *Brafman polynomials.* The Brafman polynomials $B^1_n((a_p), (b_q); x)$ are generated by [17]:

$$e^t\ {}_pF_q\left(\begin{matrix}(a_p)\\ (b_q)\end{matrix}\ ; -xt\right) = \sum_{n=0}^{\infty} B^1_n(x; (a_p), (b_q))\, \frac{t^n}{n!}\ .$$

The corresponding inversion formula is given by [2]:

$$\frac{[a_p]_n}{[b_q]_n}\, x^n = \sum_{m=0}^{n} \binom{n}{m} (-1)^m B^1_m((a_p), (b_q); x)\ . \tag{3.15}$$

Using the definition of the Pochhammer symbol, the identity (3.15) can be rewritten in the form:

$$\begin{aligned} x^n &= \sum_{m=0}^{n} \frac{(-1)^m \prod_{i=1}^p \Gamma(a_i)}{m! \prod_{i=1}^q \Gamma(b_i)} \times \\ &\times \frac{\Gamma(n+1) \prod_{i=1}^q \Gamma(b_i+n)}{\Gamma(n-m+1) \prod_{i=1}^p \Gamma(a_i+n)}\ B^1_m((a_p), (b_q); x)\ . \end{aligned} \tag{3.16}$$

Hence, for $p \le q+1$, the application of the Theorem 3.1, leads to the dual sequence $\{u_r\}_{r\ge 0}$ associated with the Brafman PS $\{B^1_m((a_p), (b_q); x)\}_{n\ge 0}$ given by:

$$\langle u_r, x^n\rangle = \int_0^\infty t^n \psi_r(t)\, dt \quad , \quad n \ge r\ , \tag{3.17}$$

where

$$\psi_r(t) = \xi G^{q+1,\ 0}_{p+1,\ q+1}\left(t\,\middle|\,\begin{matrix}(a_p-1), -r\\ (b_q-1), 0\end{matrix}\right) \quad \text{with } \xi = \frac{(-1)^r \prod_{j=1}^p \Gamma(a_j)}{r! \prod_{j=1}^q \Gamma(b_j)}\ . \tag{3.18}$$

Two particular cases are worthy to note:

Example 3.3 (Laguerre polynomials)**.** In the case $p = 0$ and $q = 1$ the Brafman polynomials are reduced to the Laguerre ones [11] $\{n!/(\alpha+1)_n\, L_n^{\alpha}(x)\}_{n\geq 0}$, which is orthogonal for $\alpha > -1$ with respect to the well-known weight function $\psi_0(x) = t^{\alpha}e^{-t}/\Gamma(\alpha+1)$ on the interval $0 \leq x < +\infty$. This weight function is also given by (3.18) for $r = 0$. In fact, from (3.18) we have

$$\psi_0(t) = \frac{1}{\Gamma(\alpha+1)}\, G^{2,\ 0}_{1,\ 2}\left(t \,\middle|\, \begin{matrix} 0 \\ \alpha, 0 \end{matrix} \right) . \tag{3.19}$$

In view of the transformations (3.6) and [17, p. 46]:

$$\begin{aligned} &G^{m,\ n}_{p,\ q}\left(z \,\middle|\, \begin{matrix} \alpha_1, \cdots, \alpha_{p-1}, \beta_1 \\ \beta_1, \cdots, \beta_q \end{matrix} \right) = \\ &= G^{m-1,\ n}_{p-1,\ q-1}\left(z \,\middle|\, \begin{matrix} \alpha_1, \cdots, \alpha_{p-1} \\ \beta_2, \cdots, \beta_q \end{matrix} \right) \quad ; \quad m, p, q \geq 1 \ ; \end{aligned} \tag{3.20}$$

the identity (3.19) can be written in the form

$$\psi_0(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\, G^{1,\ 0}_{0,\ 1}\left(t \,\middle|\, \begin{matrix} - \\ 0 \end{matrix} \right) . \tag{3.21}$$

Then, according to the formula [17, p. 46]:

$$G^{1,\ p}_{p,\ q+1}\left(-z \,\middle|\, \begin{matrix} 1-\alpha_1, \cdots, 1-\alpha_p \\ 0, 1-\beta_1, \cdots, 1-\beta_q \end{matrix} \right) = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\ {}_pF_q\left(\begin{matrix} (\alpha_p) \\ (\beta_q) \end{matrix} ; z \right) , \tag{3.22}$$

we obtain $\psi_0(t) = t^{\alpha}e^{-t}/\Gamma(\alpha+1)$.

Example 3.4 (d-Laguerre polynomials)**.** For $p = 0$ and $q = d$ the Brafman polynomials are reduced to the d-Laguerre ones $\{n!l_n^{(\alpha_d)}(x)\}_{n\geq 0}$ defined by

$$\ell_n^{(\alpha_d)}(x) = \ell_n^{(\alpha_1,\cdots,\alpha_d)}(x) = {}_1F_d\left(\begin{matrix} -n \\ (\alpha_d+1) \end{matrix} ; x \right) . \tag{3.23}$$

These polynomials were deeply investigated by the first author and Douak [4]. They showed that these polynomials are d-orthogonal. Hence, the dual sequence given by (3.17) leads to the d-dimensionnal functional vector $\mathcal{U} ={}^t (u_0, u_1, \cdots, u_{d-1})$ ensuring the d-orthogonality of these polynomials given by:

$$\langle u_r, x^n \rangle = \int_0^{\infty} t^n \psi_r(t)\, dt \quad , \quad n \geq r,\ r \in \mathbb{N}_d \ . \tag{3.24}$$

where

$$\psi_r(t) = \frac{(-1)^r}{r! \prod_{j=0}^{d}\Gamma(\alpha_j+1)}\, G^{d+1,\ 0}_{1,\ d+1}\left(t \,\middle|\, \begin{matrix} -r \\ 0, (\alpha_d) \end{matrix} \right) .$$

For $d = 2$, the PS $\{\ell_n^{(\alpha_d)}\}_{n\geq 0}$ are reduced to the PS $\{B_n^{\alpha_1,\alpha_2}(x)\}_{n\geq 0}$, which was studied separately by the first author and Douak [3], and Van Assche and Yakubovich [18]. In [3] the 2-dimensional functional vector was derived via a solution of a Pearson type equation since these polynomial set is classical.

3.2.2. *Chaunday polynomials.* The Chaunday polynomials $P_n(x;(a_p),(b_q))$ are generated by [17]:

$$(1-t)^{-\lambda}\,{}_pF_q\left(\begin{array}{c}(a_p)\\ (b_q)\end{array};xt\right)=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,P_n(x;(a_p),(b_q))\,t^n\,.$$

The corresponding inversion formula is given by [2]:

$$\frac{[a_p]_n}{[b_q]_n}\,x^n=\sum_{m=0}^{n}\binom{n}{m}(-\lambda)_{n-m}(\lambda)_nP_m(x;(a_p),(b_q))\,. \tag{3.25}$$

Applying Theorem 3.1, we deduce that the dual sequence $\{u_r\}_{r\geq 0}$ associated with the Chaunday polynomials is given by:
• For $q+1=p$:

$$\langle u_r,x^n\rangle=\int_0^1 t^n\psi_r(t)\,dt\quad,\quad n\geq r\,, \tag{3.26}$$

where

$$\psi_r(t)=\xi\,G^{p+1,\;0}_{p+1,\;p+1}\left(t\,\middle|\,\begin{array}{c}1-r,(a_p)\\ 1,-\lambda-r,\;(b_q)\end{array}\right)\qquad\text{with}\;\;\xi=\frac{(-1)^r(\lambda)_r\prod_{j=1}^{p}\Gamma(a_j)}{\Gamma(-\lambda)\prod_{j=1}^{q}\Gamma(b_j)}\,.$$

• For $q+1>p$:

$$\langle u_r,x^n\rangle=\int_0^1 t^n\psi_r(t)\,dt\quad,\quad n\geq r\,, \tag{3.27}$$

where

$$\psi_r(t)=\frac{\xi}{t}\,G^{q+2,\;0}_{p+1,\;q+2}\left(t\,\middle|\,\begin{array}{c}1-r,(a_p)\\ 1,-\lambda-r,(b_q)\end{array}\right)\qquad\text{with}\;\;\xi=\frac{(-1)^r(\lambda)_r\prod_{j=1}^{p}\Gamma(a_j)}{\Gamma(-\lambda)\prod_{j=1}^{q}\Gamma(b_j)}\,.$$

3.2.3. *Jain polynomials.* The Jain polynomials $R_n^{(s)}(\lambda,(a_p),(b_q);x)$ are generated by [17]:

$$(1-t)^{-\lambda}\,{}_pF_q\left(\begin{array}{c}(a_p)\\ (b_q)\end{array};-\frac{s^sxt}{(1-t)^s}\right)=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,R_n^{(s)}(\lambda,(a_p),(b_q);x)t^n\,. \tag{3.28}$$

The corresponding inversion formula is given by [2]:

$$\begin{aligned}&\frac{[a_p]_n}{[b_q]_n}\,\frac{s^{sn}}{(\lambda)_{sn}}\,x^n=\\ &=\sum_{m=0}^{n}(-1)^n\binom{n}{m}\frac{ms+\lambda}{(\lambda+m)_{n(s-1)+1}}\,R_n^{(s)}(\lambda,(a_p),(b_q);x)\,.\end{aligned} \tag{3.29}$$

Applying Theorem 3.1, we obtain the dual sequence $\{u_r\}_{r\geq 0}$ associated with the Jain polynomials given by:

$$\langle u_r,x^n\rangle=\begin{cases}\displaystyle\int_0^{\infty}t^n\psi_r(t)\,dt\quad, & \text{if }\;q+1-p>0\,,\\ \displaystyle\int_0^{(s-1)^{1-s}}t^n\psi_r(t)\,dt\quad, & \text{if }\;q+1-p=0\,,\end{cases} \tag{3.30}$$

where

$$\psi_r(t) = \frac{\xi}{t}\, G^{q+s+1,\ 0}_{p+s,\ q+s+1}\left((s+1)^{s-1}t \,\middle|\, \begin{array}{l} 1-r, (a_p), \Delta(s-1, \lambda+r) \\ 1,\ (b_q), \Delta(s,\lambda) \end{array} \right) , \tag{3.31}$$

$$\xi = \frac{(-1)^r(\lambda+rs)_r \prod_{j=1}^{p}\Gamma(a_j)}{r!\Gamma(\lambda+r)\prod_{j=1}^{q}\Gamma(b_j)}\ \frac{\prod_{j=1}^{p}\Gamma((\lambda+j+r-1)/(s-1))}{\prod_{j=1}^{q}\Gamma((\lambda+j-1)/s)} .$$

From the identity (3.28), it is easy to show that the Jain polynomials have the following hypergeometric representation:

$$R_n^{(s)}(\lambda, (a_p), (b_q); x) =$$

$$= {}_{p+s}F_{q+s}\left(\begin{array}{l} -n, \Delta(s-1, \lambda+n), (a_p) \\ \Delta(s,\lambda), (b_q) \end{array} ; (s-1)^{(s-1)}x \right) . \tag{3.32}$$

Four particular cases are worthy to note:

Example 3.5 (d-Laguerre polynomials)**.** In the case $p = s = 1$, $q = d$, the Jain polynomials are reduced to the d-Laguerre ones defined by (3.23). Then, we rediscovery the d-dimensionnal functional vector given by (3.24) ensuring the d-orthogonality of the d-Laguerre polynomials from the dual sequence defined by (3.30) corresponding to the Jain polynomials .

Example 3.6 (d-Jacobi polynomials)**.** In the case $p = s = d+1$, $q = d$, $(a_p) = \Delta(s,\lambda)$, the identity (3.32) is reduced to the hypergeometric representation of the d-Jacobi polynomials given by:

$$P^{(\lambda,(b_d))}(x) = R_n^{(d+1)}(\lambda, \Delta(d+1,\lambda), (b_q); x) =$$

$$= {}_{d+1}F_d\left(\begin{array}{l} -n, \Delta(d, \lambda+n) \\ (b_d) \end{array} ; d^d x \right) \tag{3.33}$$

This PS was introduced by the authors and Ouni [5] as a d-orthogonal PS of Jacobi type. Hence, the dual sequence given by (3.30) leads to the d-dimensional functional vector $\mathcal{U} = {}^t(u_0, u_1, \cdots, u_{d-1})$ ensuring the d-orthogonality of the d-Jacobi polynomials, which is given by the following integral representation:

$$\langle u_r, x^n \rangle = \int_0^{d^d} t^n \psi_r(t)\, dt \quad , \quad n \geq r , \tag{3.34}$$

where

$$\psi_r(t) = \frac{(-1)^r(\lambda+r(d+1))}{r!(\lambda+r)}\ \frac{\prod_{j=1}^{d}\Gamma((\lambda+j+r-1)/d)}{\prod_{j=1}^{d}\Gamma(b_j)}\ \frac{1}{t}\times$$

$$\times G^{d+1,\ 0}_{d+1,\ d+1}\left(d^d t \,\middle|\, \begin{array}{l} 1-r, \Delta(d,\lambda+r) \\ 1, (b_d) \end{array} \right) .$$

The functional vector defined by (3.34) was given by the second author and Ouni [13] for the particular case $(\alpha_d) = (\alpha_{d,\mu})$ where $(\alpha_{d,\mu}) = \{(\mu+1+j)/(d+1)\ ;\ j \in \mathbb{N}_{d+1}$ and $j \neq d-\mu\}$, $\mu \in \mathbb{N}_{d+1}$. The general case given here appears to be new.

For $d = 1$, 1-Jacobi polynomials are the classical Jacobi polynomials $\Big\{n!/(\alpha+1)_n$ $P_n^{(\alpha,\beta)}(1-2x)\Big\}$ with $\lambda = \alpha+1+\beta$ and $b_1 = \alpha+1$. Moreover, for this case, the identity (3.34) becomes

$$\langle u_0, x^n\rangle = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} \int_0^1 t^n G_{1,\ 1}^{1,\ 0}\left(t \;\middle|\; \begin{matrix} \alpha+\beta \\ \alpha \end{matrix}\right) dt\ .$$

Taking into account the following identity [4]:

$$G_{1,\ 1}^{1,\ 0}\left(x \;\middle|\; \begin{matrix} \alpha+\beta \\ \alpha \end{matrix}\right) = \frac{1}{\Gamma(\beta)}(1-x)^{\beta-1}x^{\alpha}\ , \tag{3.35}$$

we obtain

$$\langle u_0, x^n\rangle = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta)} \int_0^1 t^{n+\alpha}(1-t)^{\beta}\,dt\ .$$

That, upon the change of variable $t = (1-\xi)/2$, leads to

$$\begin{aligned}\langle u_0, (1-2x)^n\rangle &= \sum_{j=0}^{n}\binom{n}{j}\langle u_0, (-2x)^j\rangle = \\ &= \frac{1}{2^{\alpha+\beta+1}}\,\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} \int_{-1}^{1} \xi^n(1-\xi)^{\alpha}(1+\xi)^{\beta}\,d\xi\ .\end{aligned} \tag{3.36}$$

From (3.36), we deduce the well-known weight functions associated to the Jacobi polynomials

$$\left\{\frac{n!}{(\alpha+1)_n}\ P_n^{(\alpha,\beta)}(1-2x)\right\}_{n\geq 0}$$

given by [17]:

$$\int_{-1}^{1}(1-\xi)^{\alpha}(1+\xi)^{\beta}P_m^{(\alpha,\beta)}(\xi)P_n^{(\alpha,\ \beta)}(\xi)\,d\xi = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta)n!\Gamma(n+\alpha+\beta+1)}\ .$$

Example 3.7 (Srivastava-Pathan polynomials)**.** For the case $s = 2$ and fixed two numerator parameters, the Jain polynomials are reduced to the Srivastava-Pathan polynomials generated by:

$$\begin{aligned}(1-t)^{-\lambda}{}_{p+2}F_q\left(\begin{matrix}\frac{\lambda}{2}, \frac{\lambda+1}{2}, (a_p) \\ (b_q)\end{matrix}\ ; \frac{-4xt}{(1-t)^2}\right) &= \\ = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ {}_{p+2}F_q\left(\begin{matrix}-n, \lambda+n, (a_p) \\ (b_q)\end{matrix}\ ; x\right) t^n\ .\end{aligned}$$

From the identity (3.30), we deduce that the dual sequence corresponding with the Srivastava-Pathan polynomials is given by:

$$\langle u_r, x^n\rangle = \begin{cases} \displaystyle\int_0^{\infty} t^n\psi_r(t)1\,dt \quad , & \text{if } q-1-p>0\ , \\ \displaystyle\int_0^{1} t^n\psi_r(t)\,dt \quad , & \text{if } q-1-p=0\ , \end{cases} \tag{3.37}$$

where

$$\psi_r(t) = \frac{\xi}{t}\, G^{q+1,\ 0}_{p+2,\ q+1}\left(t \;\middle|\; \begin{matrix} 1-r, (a_p), \lambda + r \\ 1, (b_q) \end{matrix} \right)$$

with

$$\xi = \frac{(-1)^r(\lambda+2r)\prod_{j=1}^{p}\Gamma(a_j)}{r!\prod_{j=1}^{q}\Gamma(b_j)} .$$

Example 3.8 (Rice polynomials)**.** The Srivastava-Pathan polynomials, for $p = 1$ and $q = 2$, are reduced to the Rice polynomials defined by [17]:

$$H_n^{(\alpha,\beta)}(\gamma,\sigma;x)\frac{(\alpha+1)_n}{n!}\, {}_3F_2\left(\begin{matrix} -n, \lambda+n, \gamma \\ \alpha+1, \sigma \end{matrix} ; x \right) \quad \text{with } \lambda = \alpha+\beta+1 .$$

From the identity (3.37), we deduce that the dual sequence corresponding with the Rice polynomials is given by:

$$\langle u_r, x^n \rangle = \frac{(-1)^r(2r+\lambda)}{r!}\, \frac{\Gamma(\gamma)}{\Gamma(\alpha+1)\Gamma(\sigma)} \times \\ \times \int_0^1 t^n G^{3,\ 0}_{3,\ 3}\left(t \;\middle|\; \begin{matrix} -r, \alpha-1, \alpha+\beta+r \\ 0, \alpha, \sigma-1 \end{matrix} \right) dt . \tag{3.38}$$

3.2.4. *Humbert polynomials.* The Humbert polynomials are defined by the generating relation [8]:

$$(1-(d+1)xt+t^{d+1})^{-\nu} = \sum_{n\geq 0} h^{\nu}_{n,d+1}(x)t^n , \tag{3.39}$$

where $\nu > -1/2$, $\nu \neq 0$.
The corresponding inversion formula is given by [2]:

$$x^n = \sum_{j=0}^{[n/(d+1)]} \frac{(\nu+n-(d+1)j)}{(\nu)_{n+1-j}}\, \frac{n!}{(d+1)^n j!}\, h^{\nu}_{n-(d+1)j,d+1}(x) . \tag{3.40}$$

The d-dimensional functional vector ensuring the d-orthogonality was derived in [13] using the inversion formula. That is, according to Theorem 3.2:

$$\langle u_r, x^n \rangle = \delta_{r,i} \int_0^{d^{-d/(d+1)}} \xi^n \varphi_{r,d}(\xi)\, d\xi , \tag{3.41}$$

where $n = i + (d+1)k$, $k \in \mathbb{N}$, $i \in \mathbb{N}_{d+1}$, $r \in \mathbb{N}_d$ and

$$\varphi_{r,d}(\xi) = \frac{r!}{(d+1)^{r-1}(\nu)_r}\, \frac{\prod_{j=1}^{d}\Gamma((\nu+r+j)/d)}{\prod_{j=1}^{d+1}\Gamma((r+j)/(d+1))} \times \\ \times \xi^{-(r+1)} G^{d+1,\ 0}_{d+1,\ d+1}\left(d^d\xi^{d+1} \;\middle|\; \begin{matrix} \frac{\nu+r+1}{d}, \cdots, \frac{\nu+r+d}{d}, 1 \\ \frac{r+1}{d+1}, \cdots, \frac{r+(d+1)}{d+1} \end{matrix} \right) . \tag{3.42}$$

The Humbert PS $\{h^{\nu}_{n,d+1}\}_{n\geq 0}$ contains as particular cases the Legendre, Tchebychev, Pincherle, Kinney and Gegenbauer ones [9]. Then, by using Theorem 3.2, we can rediscovery the weight functions ensuring the orthogonality of Legendre, Tchebychev and Gegenbauer polynomials, the functional ensuring the 2-orthogonality of Pincherle polynomials, and the functional ensuring the d-orthogonality of Kinney polynomials.

Example 3.9 (Gegenbauer polynomials)**.** By letting $d = 1$ in (3.39), we meet the generating function of Gegenbauer polynomials $\{C_n^\nu(x)\}_{n\geq 0}$. These polynomials are orthogonal with respect to the well-known weight function [11]:

$$\varphi_{0,1}(\xi) = \frac{\nu(\Gamma(\nu))^2}{\pi\Gamma(2\nu)2^{1-2\nu}}(1-\xi^2)^{\nu-1/2} \quad , \quad -1 \leq \xi \leq 1 \,. \tag{3.43}$$

From (3.41) with $d = 1$ and the transformation (3.20), we have

$$\varphi_{0,1}(\xi) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}}\, \xi G_{1,\ 1}^{1,\ 0}\left(\xi^2 \,\middle|\, \begin{matrix} \nu \\ -\frac{1}{2} \end{matrix}\right) .$$

Taking into account the identity (3.35) we obtain

$$\varphi_{0,1}(\xi) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+1/2)}(1-\xi^2)^{\nu-1/2} \,, \tag{3.44}$$

which in view of Gauss's multiplication theorem [17, p. 23]

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad , \quad z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \cdots \,, \tag{3.45}$$

becomes

$$\varphi_{0,1}(\xi) = \frac{\nu(\Gamma(\nu))^2}{\pi\Gamma(2\nu)2^{-2\nu}}(1-\xi^2)^{\nu-1/2} \,.$$

Consequently, the Gegenbauer functional is given by its moments

$$\langle u_0, x^n\rangle = \delta_{0,i}\int_0^1 \xi^{2k}(1-\xi^2)^{\nu-1/2}\frac{\nu\Gamma(\nu)^2}{\pi\Gamma(2\nu)2^{-2\nu}}\, d\xi \,,$$

where $n = i + 2k$, $k \in \mathbb{N}$ and $i = 0, 1$.
Or equivalently

$$\langle u_0, x^n\rangle = \delta_{0,i}\int_{-1}^1 \xi^{n}(1-\xi^2)^{\nu-1/2}\frac{\nu\Gamma(\nu)^2}{\pi\Gamma(2\nu)2^{1-2\nu}}\, d\xi \,. \tag{3.46}$$

3.2.5. *Gould-Hopper polynomials.* The Gould-Hopper polynomials are generated by [17]:

$$\exp[xt - ht^{d+1}] = \sum_{n=0}^{\infty} g_n^{d+1}(x;h)\,\frac{t^n}{n!} \,. \tag{3.47}$$

The corresponding inversion formula is given by:

$$x^n = \sum_{j=0}^{[n/(d+1)]}\frac{(-h)^j}{j!}\,\frac{n!}{(n-(d+1)j)!}\, g_{n-(d+1)j}^{d+1}(x;h) \,. \tag{3.48}$$

The d-dimensional functional vector ensuring the d-orthogonality was derived in [12] using the inversion formula. That is, according to Theorem 3.2:

$$\langle u_r, x^n\rangle = \delta_{r,i}\int_0^\infty \xi^n\psi_{r,d}(\xi)\, d\xi \,, \tag{3.49}$$

where $n = i + (d+1)k$, $k \in \mathbb{N}$, $i = 0, 1, \cdots, d$, $r = 0, 1, \cdots, d-1$, and

$$\psi_{r,d}(\xi) = \frac{(d+1)\xi^{-(r+1)}}{\prod_{j=1}^{d+1}\Gamma((r+j)/(d+1))}\times$$

$$\times G^{d+1,\ 0}_{1,\ d+1}\left(\frac{-\xi^{d+1}}{h(d+1)^{d+1}}\ \middle|\ \begin{array}{l} 1 \\ \frac{r+1}{d+1}, \frac{r+2}{d+1}, \cdots, \frac{r+(d+1)}{d+1} \end{array}\right). \tag{3.50}$$

Example 3.10 (Hermite polynomials)**.** For $d = 1$ and $h = -1$, the Gould-Hopper polynomials are reduced to the Hermite ones $\{H_n(x/2)\}_{n\geq 0}$. The weight function given by (3.50) is reduced to the well-known weight function

$$\psi_{0,1}(\xi) = e^{-\xi^2/\sqrt{\pi}} \quad , \quad \infty < \xi < +\infty \ ,$$

ensuring the orthogonality of the Hermite polynomials $\{H_n(x/2)\}_{n\geq 0}$ (see, for instance, [11, p. 50]).

Acknowledgments. The first author thanks Professor P.E. Ricci and organizers of the 4th workshop on "Advanced Special Functions and Solution of PDES" for the kind invitation. The research was supported by the Ministry of Hight Education and Technology, Tunisia, (02/UR/1501) and King Saud University, Riadh through Grant DSFP/ Math 01.

References

[1] Y. Ben Cheikh, *On obtaining dual sequences via quasi-monomiality*, Georgian Math. J., 9(2002), 413–422.

[2] Y. Ben Cheikh & H. Chaggara, *Connection coefficients between Boas-Buck polynomial sets*, J. Math. Anal. Appl., 319(2006), 665–689.

[3] Y. Ben Cheikh & K. Douak, *On two-orthogonal polynomials related to Bateman's $J_n^{u,\ v}$-function*, Meth. Appl. Anal., 7(2000), 641–662.

[4] Y. Ben Cheikh & K. Douak, *On the classical d-orthogonal polynomials defined by certain generating functions II*, Bull. Belg. Math. Soc., 8(2001), 591–605.

[5] Y. Ben Cheikh, I. Lamiri & A. Ouni, *On Askey-scheme and d-orthogonality, I: a characterization theorem*, J. Comput. Appl. Math., 233(2009), 621–629.

[6] T.S. Chihara, *Orthogonal polynomials with Brenke type generating function*, Duke Math. J., 35(1968), 505–518.

[7] K. Douak & P. Maroni, *Une caractérisation des polynômes classiques de dimension d*, J. Approx. Theory, 82(1995), 177–204.

[8] P. Humbert, *Some extentions of Pincherle's polynomials*, Proc. Edin. Math. Soc., 29(1920), 21–24.

[9] A.F. Horadam, *Gegenbauer polynomials revisted*, Fibonacci Quart., 23(4)(1985), 294–299.

[10] M.E.H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Cambridge University Press, New York, 2005.

[11] R. Koekoek & R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Technical Report 98-17, Faculty of the Technical Mathematics and Informatics, Delft University of Technology. Delft, 1998.

[12] I. Lamiri & A. Ouni, *d-orthogonality of Hermite type polynomials*, Appl. Math. Comput., 202(2008), 24–43.

[13] I. Lamiri & A. Ouni, *d-orthogonality of Humbert and Jacobi type polynomials*, J. Math. Anal. Appl., 341(2008), 24–51.

[14] Y.L. Luke, *The special functions and their approximations*, vol.I, Academic Press, New York, San Francisco, London, 1969.

[15] P. Maroni, *L'orthogonalité et les récurrences de polynômes d'ordre supérieur à deux*, Ann. Fac. Sci. Toulouse, 10(1989), 105–139.

[16] J. Meixner, *Orthogonal polynomsysteme mit einer besonderen gestalt der erzeugenden funktion*, J. London Math. Soc., 9(1934), 6–13.

[17] H.M. Srivastava & H.L. Manocha, *A treatise on generating functions*, Halsted Press (Ellis Harwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1984.

[18] W. Van Assche & S.B. Yakubovich, *Multiple orthogonal polynomials associated with MacDonald functions*, Integral Transform. Spec. Funct., 9(2000), 229–244.

[19] J. Van Iseghem, *Vector orthogonal relations. Vector* QD-*algorithm*, J. Comput. Appl. Math., 19(1987), 141–150.

[20] J.M. Wittaker, *Interpolatory functions theory*, *Treater in Mathematics and Mathemathecal Physics*, vol. 33, Combridge University Press, London 1935.