Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 9(2010), pp. 129–151.

Special polynomials and polynomial bases in hypercomplex function theory

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Dedicated to Professor P.E. Ricci on the occasion of his 65th birthday

Abstract[1]. Hypercomplex function theory generalizes the theory of holomorphic functions of one complex variable by using Clifford algebras and provides the fundamentals of Clifford Analysis as a refinement of Harmonic Analysis in higher dimensions. After a brief introduction to basic concepts like hypercomplex differentiability and a corresponding Taylor series expansion we discuss approaches to some classes of special polynomials and polynomial bases in Clifford Analysis.

1. Introduction

1.1. **Brief historical remarks.** The Swiss mathematician Rudolf Fueter (1880-1950), disciple of David Hilbert, systematically started around 1930 with the foundation of a theory of quaternion valued functions $w = f(q)$ of a quaternion variable q ([20], [21], [22]). Being that time a famous number theorist, he was in the beginning mainly interested in its applications as a tool for the development of new analytical methods in Number Theory (cf. [35]), but soon he became aware that he had found an approach to generalize the classical complex function theory in a way different from that of holomorphic functions of several complex variables. He coined the designation *Hypercomplex Function Theory* for such a theory of functions with values in a *hypercomplex system* (nowadays called *algebras*). Analogous attempts had been made before Fueter and - very curiously - even after him until recent times (a detailed description is given in [15]). But in that time only the work of the Romanian mathematicians G. Moisil and N. Théodoresco (cf. [41], [42]), developed mainly for boundary value problems in $\mathbb{R}^3$ and using a matrix approach without

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Financial support from the research unit "Matemática e Aplicações" of the University of Aveiro, through the Portuguese Foundation for Science and Technology (FCT), gratefully acknowledged.

Keywords. Hypercomplex function theory, generalized Bernoulli and Euler polynomials, monogenic polynomial bases.

AMS Subject Classification. 30G35, 05A19, 11B68.

mentioning quaternions, came close to Fueter's ideas. As Fueter explains in [23], only in 1936 he had knowledge about those papers.

Although until the 50-ties several students of Fueter contributed together with him to the further development of Hypercomplex Function Theory, the breakthrough came only after a systematical work of the Belgian mathematician R. Delanghe and his school in the end of the 60-ties (cf. [14]). The publication of their book [7], which title *Clifford Analysis* soon was used as designation for the field of higher dimensional Analysis involving *Clifford algebras*, contributed very much to an increasing interest in this field. Indeed, we would like to stress, that nowadays Clifford Analysis should not be considered as a straightforward attempt to generalize Complex Analysis (even if this is suggested, due to the obvious association with Ahlfors' classical book *Complex Analysis* and the fact that $\mathbb{C}$ is the simplest and unique commutative Clifford algebra). Due to our objectives, we prefer here to follow the classical designation of Fueter, keeping in mind that almost all basic ideas of Clifford Analysis have their origin in Hypercomplex Function Theory (cf. [16]).

Another important step with important impact was the application of Clifford analytic tools to elliptic boundary value problems in higher dimensions and properly developed numerical tools (cf. [24]). The first text book with a parallel treatment of classical and hypercomplex function theory is the recently published [25], which contains also a CD-ROM with corresponding software.

It is also necessary to notice the relationship between Clifford Analysis and Harmonic Analysis. The possibility of factorizing the second order Laplace operator (in several real variables) by a product of two generalized hypercomplex first order Cauchy-Riemann operators by the use of a corresponding Clifford algebra, suggests to consider Clifford Analysis as some kind of refinement of Harmonic Analysis.

Finally we would like to call attention to the fact, that in present time the research on function theoretic methods and its applications in all fields of Clifford Analysis is subject to an intensive study in Austria, Australia, Belgium, China, Egypt, Finland, France, Germany, Israel, Italy, Japan, Mexico, Poland, Portugal, Russia, UK, USA and other countries.

Potential theory, differential geometry, operator theory, BVP of partial differential equations, analytic number theory, discrete and computational mathematics and their corresponding applications in Sciences and Engineering are some examples of other mathematical fields with relationship to Clifford Analysis. It seems to us also remarkable that a Fields Medallist of 2006, namely Terence Tao, started his career in Clifford Analysis.

1.2. **Clifford algebras.** A finite-dimensional algebra with a unit element over the field of real or complex numbers was formerly known as a *hypercomplex system*. Examples of hypercomplex systems are: the real numbers, the complex numbers, the *quaternions*, and the Cayley numbers or *octonions* (in this list each successive system is obtained by doubling the preceding one). Other examples include *double* and *dual numbers*, and the *Clifford numbers* as elements of a *Clifford algebra* of rank 2^n. They are, for $n = 3$, also known as *Clifford-Lipschitz numbers*. Another very important example of hypercomplex systems are *complete matrix algebras over* $\mathbb{R}$. Clifford numbers possess an isomorphic representation as elements of a $(2^n \times 2^n)$ matrix algebra. The matrix representations of complex numbers and quaternions are well known special cases.

Clifford algebras as associative non-commutative algebras over the field of real or complex numbers can be defined in several ways (cf. [13]).

The easiest way is the introduction of the basis of the algebra through the multiplication rules of an orthonormal basis of the underlying vector space of dimension n. This vector space, in general, has not to be Euclidean, but could be also a Minkowski or, more general, a pseudo-Euclidean vector space.

In a more general way a Clifford algebra is constructed from a finite dimensional space with (not necessarily positive definite) inner product, introducing an algebra multiplication which both reflects the properties of this inner product and a corresponding outer product following the concept of W. K. Clifford (1878).

Let $\mathcal{V}$ be a finite dimensional vector space. A function $\mathcal{B} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is a non-degenerate symmetric bilinear form (*inner product*) if for all $\lambda \in \mathbb{R}$ and x, y and z in $\mathcal{V}$,

$$\mathcal{B}(x, y) = \mathcal{B}(y, x),$$

$$\mathcal{B}(\lambda x + y, z) = \lambda\mathcal{B}(x, z) + \mathcal{B}(y, z) \text{ and}$$

$$\mathcal{B}(x, u) = 0, \ \forall u \text{ implies } x = 0.$$

Based on this general background about vector spaces a Clifford algebra over $\mathbb{R}^{p,q}$ with signature (p, q), $p+q = n$, is denoted by $\mathcal{C}\ell_{p,q}$ and it is generated by $\mathbb{R}^{p,q}$ modulo the relation

$$x \cdot x = x^2 = -\mathcal{B}(x, x) \ ,$$

where

$$\mathcal{B}(e_i, e_i) = -1 \quad , \quad i = 1, \cdots, p \ ,$$

$$\mathcal{B}(e_i, e_i) = 1 \quad , \quad i = p + 1, \cdots, n \ ,$$

assuming an orthonormal basis $(e_1, e_2, \cdots, e_n)$ of $\mathbb{R}^{p,q}$. Applying

$$\mathcal{B}(e_i, e_j) = \delta_{ij}$$

to the square of the vector $e_i + e_j$, where $i \neq j$, leads to

$$e_i^2 + e_i e_j + e_j e_i + e_j^2 = -\mathcal{B}(e_i + e_j, e_i + e_j) =$$

$$= -\mathcal{B}(e_i, e_i) - \mathcal{B}(e_i, e_j) - \mathcal{B}(e_j, e_i) - \mathcal{B}(e_j, e_j) = e_i^2 + e_j^2$$

from which the non-commutativity property

$$e_i e_j + e_j e_i = -2\mathcal{B}(e_i, e_j) = -2\delta_{ij}$$

follows.

Let us now consider the universal Clifford algebra $\mathcal{C}\ell_{0,n}$, formed by elements of the form

$$\alpha = \sum_A \alpha_A e_A \quad , \quad \alpha_A \in \mathbb{R} \ ,$$

with $A \subseteq \{1, \cdots, n\}$, $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \leq l_1 < \cdots < l_r \leq n$ and $e_\emptyset =: e_0 =: 1$.

The conjugate of $\alpha \in \mathcal{C}\ell_{0,n}$ is defined by

$$\overline{\alpha} = \sum_A \alpha_A \overline{e}_A$$

where $\overline{e}_A = \overline{e}_{l_r} \overline{e}_{l_{r-1}} \cdots \overline{e}_{l_1}$, being $\overline{e}_j = -e_j$ $(j = 1, \cdots, n)$, $\overline{e}_0 = e_0 = 1$.

The Euclidean space $\mathbb{R}^{n+1}$ is embedded in the Clifford algebra $\mathcal{C}\ell_{0,n}$ by the identification of each element $x = (x_0, x_1, \cdots, x_n)$ of $\mathbb{R}^{n+1}$ with

$$z = x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathcal{A} := \mathrm{span}_{\mathbb{R}}\{1, e_1, \cdots, e_n\} \subset \mathcal{C}\ell_{0,n} .$$

The elements of $\mathcal{A}$ are called paravectors.
The conjugate to z is then given by

$$\bar{z} = x_0 - x_1 e_1 - \cdots - x_n e_n .$$

Like in the complex case the norm of $z \in \mathcal{A}$ is defined by

$$|z| := \sqrt{z\bar{z}} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} .$$

It immediately follows that each non-zero $z \in \mathcal{A}$ is invertible and its inverse is $z^{-1} = \bar{z}/|z|^2$.

The extension of this norm to the norm of $\alpha \in \mathcal{C}\ell_{0,n}$ is straightforward and leads to

$$|\alpha| = \left(\sum_A \alpha\overline{\alpha} \right)^{1/2} = \left(\sum_A \alpha_A^2 \right)^{1/2} .$$

1.3. **Two approaches to hypercomplex function theory.** The usual approach to hypercomplex function theory considers $\mathcal{C}\ell_{0,n}$-valued functions of the form $f(z) = \sum_A f_A(z) e_A, \;\; f_A(z) \in \mathbb{R}$, as mappings

$$f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{A} \longmapsto \mathcal{C}\ell_{0,n} .$$

Of course, the big advantage of this approach is to deal with *only one* hypercomplex variable z. Compared with the ordinary case of two real and one complex variables ($\mathbb{R}^2 \cong \mathbb{C}$) it reflects the reduction of the real dimension $(n+1)$ *to dimension 1*, i.e. *by* n, compared to $\mathbb{R}^{n+1}$.

A second hypercomplex structure of $\mathbb{R}^{n+1}$ different from that given by $\mathcal{A}$ consists in the following isomorphism:

$$\mathbb{R}^{n+1} \cong \mathcal{H}^n = \{\vec{z} = (z_1, \cdots, z_k) : z_k = x_k - x_0 e_k; \; x_0, x_k \in \mathbb{R}\} ,$$

which reduces the real dimension only *by* 1. More detailed, this means to take n copies $\mathbb{C}_k$ of $\mathbb{C}$ identifying $i \cong e_k, \;\; (k = 1, \cdots, n)$; $x_0 \cong \Re z$; $x_k \cong \Im z$; where $z \in \mathbb{C}$, and $\mathbb{C}_k := -e_k\mathbb{C}$. Then $\mathcal{H}^n$ is the cartesian product $\mathcal{H}^n := \mathbb{C}_1 \times \cdots \times \mathbb{C}_n$.

Hence, $\mathcal{C}\ell_{0,n}$-valued functions $f(z) = \sum_A f_A(z) e_A$ are considered as mappings

$$f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{H}^n \longmapsto \mathcal{C}\ell_{0,n} .$$

Naturally the question arises, if in the same way also the generalization of the one complex variable theory to the theory of holomorphic functions in several complex variables could be explained? The answer is affirmative and throws by a simple detail light on the duality of both higher dimensional theories - the several complex variable and the hypercomplex one. Notice that in the case of functions of several complex variables the real vector space $\mathbb{R}^{2n}$ equipped with the complex multiplication leads to $\mathbb{C}^n$ by reducing the real dimension *by the half.* Indeed, this is the third way of generalizing the ordinary case of two real and one complex variables ($\mathbb{R}^2 \cong \mathbb{C}^1$) - an amazing play with three different interpretations of the change from 2 to 1. As we will see, the use of $\mathcal{H}^n$ instead of $\mathcal{A}$ for the pre-image set is just the clue to the natural hypercomplex generalization of the classical Cauchy resp. Weierstrass approaches to holomorphic functions.

2. Hypercomplex differentiability

2.1. **Definition of a hypercomplex differentiable function.** The adequate definition of hypercomplex differentiability as generalization of complex differentiability remained an unsolved problem until the end of the 80-ties of the last century (cf. [32]). Based on the use of the hypercomplex structure expressed by $\mathbb{R}^{n+1} \cong \mathcal{H}^n$ and the fact that differentiability is equivalent with the possibility of local linearization of the increment of the considered function, we have the following

Definition 2.1. Let f be a continuous mapping from a neighborhood of $\vec{z_*} \in \mathcal{H}^n$ into $\mathcal{C}\ell_{0,n}$. Then f is called *left hypercomplex differentiable* (resp. *right hypercomplex differentiable*) at $\vec{z_*}$ if there exists a left (resp. right) $\mathcal{C}\ell_{0,n}$-linear mapping ℓ such that

$$\lim_{\Delta\vec{z}\to 0} \frac{|f(\vec{z_*}+\Delta\vec{z}) - f(\vec{z_*}) - \ell(\Delta\vec{z})|}{\|\Delta\vec{z}\|} = 0 \, .$$

We say that a function f is *hypercomplex differentiable* in $\Omega \subset \mathbb{R}^{n+1} \cong \mathcal{H}^n$ if it is hypercomplex differentiable at all points of Ω (cf. [33]).

Through this definition the well known Cauchy approach to holomorphic functions is generalized. In [47], which discussed the adequate generalization of Cauchy's as well as Weierstrass' approach (by the concept of convergent power series) to holomorphic functions and gave a negative answer, this possibility has been overseen (cf. also [15]).

Following [33], the equivalence of the Cauchy approach and the so called Riemann approach by a generalized system of Cauchy-Riemann equations, is stated in the

Theorem 2.1. *Let $f = f(\vec{z})$ be continuously real differentiable in an open set $\Omega \subset \mathcal{H}^n$. Then f is hypercomplex L- (R-) differentiable in Ω, if and only if $Df = 0$ ($fD = 0$) in Ω, where D is the hypercomplex differential operator* (*generalized Cauchy-Riemann or Dirac operator*) *defined by*

$$D = \partial_{x_0} + e_1\partial_{x_1} + \cdots + e_n\partial_{x_n}$$

which acts on the function f from the left (*resp. right*) *like indicated by its position on the left* (*right*) *side of f.*

Real differentiable solutions of $Df = 0$ (resp. $fD = 0$) are called left (resp. right) *monogenic functions.* This designation has historically been used in [7] and later on as long as the appropriate definition of hypercomplex differentiability was not clarified. For stressing more the complete coincidence with the situation in the complex case, recently in [25] and other papers f is called (Clifford) holomorphic.

2.2. **Examples and remarks.** We illustrate now the aforementioned facts in relation to the classical complex case.

1. A system of partial differential equations which describes the flow of a non-compressible fluid without sources nor sinks is given in the following way. Consider an open set $\Omega \subset \mathbb{R}^3$ and a continuously differentiable vector field (the velocity field of the flow) $\vec{g}$ on Ω. Then $\vec{g}$ satisfies the *Riesz system*

$$\begin{cases} div\, \vec{g} = 0 \\ curl\, \vec{g} = 0 \, . \end{cases}$$

In the equivalent hypercomplex setting we consider

$$f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3 \simeq \mathcal{A} = \mathrm{span}_{\mathbb{R}}\{1, e_1, e_2\}$$

which leads to the hypercomplex function $f = f_0 + f_1\, e_1 + f_2\, e_2$. Identifying $\overrightarrow{g} = (g_0, g_1, g_2) = (f_0, -f_1, -f_2)$, then the Riesz system can be written with respect to f in a compact form as

$$Df = 0 \qquad \text{or} \qquad fD = 0$$

and represents obviously a generalized Cauchy-Riemann system in $\mathbb{R}^3$.

2. The hypercomplex variables

$$f_k(z) = z_k := x_k - x_0 e_k = -\frac{1}{2}[ze_k + ze_k] \quad , \quad (k = 1, \cdots, n)$$

are left- and right-monogenic functions, also called *totally regular variables* (cf. [14]).

3. The identity function $f(z) = z \in \mathcal{A}$ is not monogenic unless $n = 1$ (the classical complex case), since $Df = fD = 1 - n$.
4. Powers of z, i.e. $f(z) = z^n$ and simple products of the totally regular variables like $z_j \cdot z_k,\ \ j \neq k$, are *not monogenic.*
5. Symmetric products of the totally regular variables in the form

$$z_1 \times z_2 := \frac{1}{2}[z_j \cdot z_k + z_k \cdot z_j] = x_j x_k - x_0 x_k e_j - x_0 x_j e_k$$

are left- and right-monogenic. More general, it has been proven in [34], that if $\nu = (\nu_1, \cdots, \nu_n)$ is a multi-index, all homogeneous monogenic polynomials of degree $|\nu| = k$ can be obtained as linear combinations (from the left or from the right) of generalized powers given in the form

$$\vec{z}^{\,\nu} := z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} \quad = \underbrace{z_1 \times \cdots \times z_1}_{\nu_1} \times \cdots \times \underbrace{z_n \times \cdots \times z_n}_{\nu_n} = \\ = \frac{1}{k!} \sum_{\pi(i_1, \cdots, i_k)} z_{i_1} \cdots z_{i_k} \ ,$$

where the sum is taken over *all* permutations of $\{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\}$ and $z_j = x_j - x_0 e_j$, $j = 1, \cdots, n$.

Moreover, all functions of the form $f(z) = \vec{z}^{\,\nu}$, are left and right monogenic and $Cl_{0,n}$ - linear independent. Therefore they can be used as basis for generalized power series. Following [34] it has been shown, that the generalized power series of the form

$$P\left(\vec{z}\right) = \sum_{k=0}^{\infty} \left(\sum_{|\nu|=k} \vec{z}^{\,\nu} c_\nu \right), c_\nu \in Cl_{0,n}$$

generates in the neighborhood of the origin a monogenic function $f(\vec{z})$ and coincides in the interior of its domain of convergence with the Taylor series of $f(\vec{z})$, i.e, in a neighborhood of the origin we have

$$f(\vec{z}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{|\nu|=k} \vec{z}^{\,\nu} \binom{k}{\nu} \frac{\partial^{|\nu|} f(\vec{0})}{\partial \vec{x}^{\,\nu}} \right) \ ,$$

where $\vec{x} = (x_1, \cdots, x_n)$.

6. Let
$$\overline{D} = \partial_{x_0} - e_1\partial_{x_1} - \cdots - e_n\partial_{x_n}$$
be the conjugate operator to D. Like in the complex case we have
$$D\overline{D} = \overline{D}D = \Delta_{n+1} ,$$
and therefore all real components $f_A(x)$ of a monogenic function f are also harmonic functions of $(n+1)$ real variables ("refinement" of harmonic function theory).
7. The analogy or similarity of the fundamental concepts in the hypercomplex case with the complex case, particularly the well defined concept of hypercomplex differentiability, suggest to ask for the role of the conjugate differential operator in the whole theory. Does $\overline{D}$ play the same role as $\partial_z = (1/2)(\partial_{x_0} - i\partial_{x_1})$ in $\mathbb{C}$, where in the case of a holomorphic function f we have the coincidence with its ordinary complex derivative, i.e.
$$(\partial_{x_0} - i\partial_{x_1})f = 2\frac{df}{dz} = 2\partial_{x_0} = 2f' \ ?$$

The answer to the last question will be given in the next section. For more details we refer to [15], [36] or [25].

3. The hypercomplex derivative as areolar derivative in the sense of Pompeiu

In the following we are using different hypercomplex differential forms without going into the details of their definition (cf. [36]). We mention only that we define the outer products of the dz_k $(k = 0, 1, \cdots, n)$, $z_0 := x_0$, in the same way as defining the complex differential forms by their corresponding real components. For example, $dz := dx + idy$ and $d\bar{z} := dx - idy$, which leads to their outer product $dz \wedge d\bar{z} = -2idx \wedge dy = 2idy \wedge dx$ by applying the multiplication rules for the outer product of the real differential forms dx and dy. Remarkable, that the form of the hypercomplex variables z_k leads again to a system of alternating differential forms, i.e. we have $dz_k \wedge dz_l = -dz_l \wedge dz_k;\ \ k \neq l$. This can be seen immediately and implies extremely easy calculations if applied to corresponding integral representations, for example in connection with the hypercomplex version of Stokes' or Cauchy's integral formulae (cf. [36]). The following theorem shows that the generalized Cauchy-Riemann operator D can be characterized as a areolar derivative in the sense of Pompeiu. The concept of *areolar derivative in the sense of Pompeiu* originated from [44] has been discussed in the hypercomplex context in [40], [36] and [25].

Theorem 3.1. *Let $z_\star$ be a fixed point in a positively oriented differentiable and contractible domain $\Omega \subset \mathcal{H}^n$ with smooth boundary. Consider a regular sequence of subdomains $\{\Omega_n\}$ which is shrinking to $z_\star$ if n tends to infinity and whereby $z_\star$ belongs to all Ω_n. For a real differentiable function f defined in Ω holds*

$$(fD)(z_\star) = \lim_{n\to\infty} \frac{1}{\text{mes } \Omega_n} \int_{\partial\Omega_n} f(z)\, dz_1 \wedge \cdots \wedge dz_n \tag{3.1}$$

i.e. the (right) generalized Cauchy-Riemann operator D is a (right) generalized areolar derivative in the sense of Pompeiu of $f = f(z)$ over Ω at $z_\star$.

The proof of this theorem relies on the application of the hypercomplex form of Stokes' theorem and the mean value property (cf. [25]), but it is out of the scope of this survey. Our main concern is the corresponding interpretation of the hypercomplex derivative of a monogenic function, which plays an essential role in the forthcoming subjects and will be given after some

Remarks.

1. $$\text{mes}\,\Omega_n = \int_{\Omega_n} dz_0 \wedge dz_1 \wedge \cdots \wedge dz_n$$
   and therefore $dV = dz_0 \wedge dz_1 \wedge \cdots \wedge dz_n$ is the corresponding (real valued) volume element.
2. In the corresponding expression of the left generalized areolar derivative the integrand in (3.1) has the form
   $$dz_1 \wedge \cdots \wedge dz_n f(z) =: d\sigma_{(n)} f(z)$$
3. In terms of dx_k the hypercomplex surface element $d\sigma_{(n)}$ is expressed by
   $$d\sigma_{(n)} = d\hat{x}_0 - e_1 d\hat{x}_1 + \cdots + (-1)^n e_n d\hat{x}_n$$
   where, like usual, $d\hat{x}_k$ $(k = 0, \cdots, n)$ means that in the ordered outer product of the 1-forms dx_k the factor dx_k is absent.

Now we will answer the question about the role of the conjugate operator $\overline{D}$ applied to a monogenic function. For the recognition of $(1/2)\overline{D}f$ as an areolar derivative of a monogenic function f we need also the hypercomplex $(n-1)$ - differential form

$$d\sigma_{(n-1)} := -e_1 d\hat{x}_{0,1} + e_2 d\hat{x}_{0,2} + \cdots + (-1)^n e_n d\hat{x}_{0,n} \,.$$

where $d\hat{x}_{0,m}$ $(m = 1, \cdots, n)$ means that in the ordered outer product of the 1-forms dx_k $(k = 0, \ldots, n)$ the factors dx_0 and dx_m are absent.

Theorem 3.2. *Let $\mathcal{S} \subset \Omega$ be an oriented differentiable n-dimensional hypersurface with boundary $\partial\mathcal{S}_m$ and z^* be a fixed point in $\mathcal{S}$. Consider a sequence of subdomains $\{\mathcal{S}_m\}$ which is shrinking to z^* if $m \to \infty$. Suppose now that the function f is left monogenic in Ω, i.e. $Df = 0$. Then the left hypercomplex derivative* $(1/2)\overline{D}$ *is a generalized areolar derivative in the sense of Pompeiu of the form*

$$\frac{1}{2}\,\overline{D}f = \lim_{m\to\infty} \left[\int_{\mathcal{S}_m} d\sigma_{(n)}\right]^{-1} \int_{\partial\mathcal{S}_m} (d\sigma_{(n-1)} f) \,.$$

Remarks.

1. It thus follows that in case of monogenic functions the nature of the derivative $(1/2)\overline{D}$ is also that of an areolar derivative, but related to an oriented differentiable hypersurface of dimension n (not of full dimension $(n+1)$). This fact coincides exactly with the situation in the plane and the existence of the ordinary complex derivative df/dz equal to the (plane-)areolar derivative $\partial f/\partial z$.
2. Moreover, a function f is monogenic iff f has a uniquely defined *hypercomplex areolar derivative f' in the sense of Pompeiu* (cf. [26]). It holds, that $f' := (1/2)\overline{D}f = (1/2)(\partial_{x_0} f - e_1\partial_{x_1} f - \cdots - e_n\partial_{x_n} f) = \partial_{x_0} f$ due to the fact that $Df = \partial_{x_0} f + e_1\partial_{x_1} f + \cdots + e_n\partial_{x_n} f = 0$.

After this short survey about some fundamental concepts of hypercomplex function theory, the following sections are dedicated to the construction of special sets of polynomials in $(n+1)$ real variables that have been considered very recently. One of the main reasons was the fact that classical special sets of polynomials did almost not have their counterpart in the context of Clifford Analysis, but seem to have rather high significance for the development of new hypercomplex numerical methods related to approximation problems in higher dimensions. Another reason was how the systematical use of the hypercomplex derivative could give new insights in the structure of special monogenic functions and their relationship with classical special functions of one or several complex variables. Therefore the following sections are dedicated to Appell sets of special monogenic polynomials, a generating function approach to generalized Bernoulli and Euler polynomials as well as to the important question of complete sets of polynomials in certain function spaces of monogenic functions.

4. Appell sequences of special monogenic polynomials in $\mathbb{R}^{n+1}$

Such an elementary problem as the construction of a suitable generalization of the complex exponential function to the hypercomplex case has been considered for a long time, for instance in [6] and [46], using the method of Cauchy-Kovalevskaya extension and corresponding series representations. The problem is not only the non-commutativity of the used algebra, but also the fact that it is not possible to deal in the same way as usual with the non-monogenic ordinary power of the underlying variable $z \in \mathcal{A}$ as mentioned in example 4 of subsection 2.2. A more recent approach can be find in [25], using as building blocks solutions of ordinary differential equations which the real harmonic components have to satisfy. Analogously to the complex case, this approach started from the differential equation $f' = f$ with the initial condition $f(0) = 1$ taking into account that $Df = 0$. The derivative has to be understood as the hypercomplex derivative $f' = (1/2\ \overline{D})f$ of f.

Here we use another method for the construction that relies on the idea to find first an Appell sequence of homogeneous monogenic polynomials and after, as a direct consequence, to obtain the exponential function as its elementary generating function.

4.1. Appell sequence in terms of a paravector variable and its conjugate.

In this section, we start by considering special homogeneous monogenic polynomials of degree k of the form

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n)\, x^{k-s}\, \bar{x}^s\ , \tag{4.2}$$

where $x = x_0 + \underline{x} \in \mathcal{A}$, with $\underline{x} = x_1e_1 + \cdots + x_ne_n$, and $T_s^k(n)$ are suitable defined real numbers, such that for a given $n \geq 1$ the sequence $\{\mathcal{P}_k^n(x)\}_{k\in\mathbb{N}_0}$ forms an Appell sequence with respect to the hypercomplex derivative[2].

[2] In the article [2] and some subsequent papers of the same authors, also special monogenic polynomials in terms of x and $\bar{x}$ similar to (4.2) have been considered. But the authors have been concerned with the extension of the theory of basic sets of polynomials in one complex variable, as introduced by J. M. Whittaker and B. Cannon, to the setting of Clifford analysis. At the time of publication of [2] the concept of hypercomplex differentiability ([33]) or the corresponding use of the hypercomplex derivative, first published in [26], have not been at disposal for the investigation of Appell sequences of monogenic polynomials.

We recall that a sequence of polynomials $P_0(x),\ P_1(x), \cdots$ is said to form an Appell sequence if

i. $P_k(x)$ is of exact degree k, for each $k = 0, 1, \cdots$;
ii. $P'_k(x) = kP_{k-1}(x)$, for each $k = 1, 2, \cdots$.

The basic idea is that the polynomials of an Appell sequence behave like power-law functions under the differention (see e.g. [4, 5]). As usually, the sequence is normalized by making $P_0(x) = 1$.

Due to the fact that it is not an easy task to obtain by induction the general expression of T_s^k $(s = 0, \cdots, k, k = 0, 1, \cdots)$, we developed in [37] a recursive matrix method with the additional (but natural assumption from the complex point of view) condition $\mathcal{P}_k^n(1) = 1$ and obtained

$$T_s^k(n) = \frac{k!}{(n)_k} \frac{((n+1)/2)_{(k-s)}\,((n-1)/2)_{(s)}}{(k-s)!s!}$$

as the general expression for the unknown coefficients.

In this context an essential role play the coefficients $c_k(n)$ in the representation of the $\mathcal{P}_k^n(x)$ in terms of the generalized powers $z_1^{\nu_1} \times \cdots \times z_n^{\nu_n}$, which is given by

$$\mathcal{P}_k^n(x) = \mathbf{P}_k(z_1, \cdots, z_n) = c_k(n) \sum_{|\nu|=k} z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} \binom{k}{\nu} e_1^{\nu_1} \times \cdots \times e_n^{\nu_n} \ ,$$

where $\binom{k}{\nu} = k!/\nu_1! \cdots \nu_n!$.
Using the initial value, it is straightforward to show that at the same time the $c_k(n)$ are equal to the alternating sum of $T_s^k(n)$, i.e.

$$c_k(n) = \sum_0^k (-1)^s T_s^k(n) \ .$$

As an example, the first $T_s^k(n)$ and $c_k(n)$, $n \geq 1$, are given by

$c_k(n) := \sum_0^k (-1)^s T_s^k(n)$	$T_s^k(n)$				
1	1				
$\frac{1}{n}$	$\frac{n+1}{2n}$	$\frac{n-1}{2n}$			
$\frac{1}{n}$	$\frac{n+3}{4n}$	$\frac{n-1}{2n}$	$\frac{n-1}{4n}$		
$\frac{3}{n(n+2)}$	$\frac{(n+5)(n+3)}{8n(n+2)}$	$\frac{3(n-1)(n+3)}{8n(n+2)}$	$\frac{3(n^2-1)}{8n(n+2)}$	$\frac{(n-1)(n+3)}{8n(n+2)}$	
$\frac{3}{n(n+2)}$	$\frac{(n+5)(n+7)}{16n(n+2)}$	$\frac{(n+5)(n-1)}{4n(n+2)}$	$\frac{3(n^2-1)}{8n(n+2)}$	$\frac{n^2-1}{4n(n+2)}$	$\frac{(n+5)(n-1)}{16n(n+2)}$

Remarks. It holds:

1. $\sum_0^k T_s^k(n) \equiv 1$ as a consequence of the condition $\mathcal{P}k^n(1) = 1$.
2.

$$c_k(n) = \begin{cases} \dfrac{k!!(n-2)!!}{(n+k-1)!!} \ , & \text{if } k \text{ is odd} \\ c_{k-1} \ , & \text{if } k \text{ is even} \end{cases}$$

The following table for the case $n = 2$ of the first hypercomplex Appell polynomials in terms of z_1 and z_2 reveals clearly that the use of those monogenic hypercomplex variables leads to symmetric expressions total different from the representation in terms of x and $\bar{x}$:

k	$\mathcal{P}_k(x)$	$\mathbf{P_k}(z_1, z_2)$
0	1	1
1	$\frac{3}{4}x + \frac{1}{4}\bar{x}$	$\frac{1}{2}(z_1e_1 + z_2e_2)$
2	$\frac{5}{8}x^2 + \frac{1}{4}x\bar{x} + \frac{1}{8}\bar{x}^{\ 2}$	$-\frac{1}{2}(z_1^2 + z_2^2)$
3	$\frac{35}{64}x^3 + \frac{15}{64}x^2\bar{x} + \frac{9}{64}x\bar{x}^{\ 2} + \frac{5}{64}\bar{x}^{\ 3}$	$-\frac{3}{8}(z_1^3e_1 + z_1^2 \times z_2e_2 + z_1 \times z_2^2e_1 + z_2^3e_2)$
4	$\frac{63}{128}x^4 + \frac{28}{128}x^3\bar{x} + \frac{18}{128}x^2\bar{x}^{\ 2} + \frac{12}{128}x\bar{x}^{\ 3} + \bar{x}^{\ 4}$	$\frac{3}{8}(z_1^4 + 2z_1^2 \times z_2^2 + z_2^4)$

Further analysis showed for all polynomials and independent from n the validity of a binomial type theorem of the form

$$\mathcal{P}_k^n(x_0 + \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} \mathcal{P}_{k-s}^n(x_0)\mathcal{P}_s^n(\underline{x}) = \sum_{s=0}^{k} \binom{k}{s} c_s x_0^{k-s}\underline{x}^s \tag{4.3}$$

or, equivalently,

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} c_s \binom{k}{s} \left(\frac{x+\bar{x}}{2}\right)^{k-s} \left(\frac{x-\bar{x}}{2}\right)^{s} .$$

We can also notice that for Legendre polynomials $L_k(x)$ holds a similar formula with $\binom{k}{s}^2$ instead of $c_s\binom{k}{s}$, namely

$$L_k(x) = \sum_{s=0}^{k} \binom{k}{s}^2 \left(\frac{x-1}{2}\right)^{k-s} \left(\frac{x+1}{2}\right)^{s}$$

as a consequence of Rodrigues' formula

$$L_k(x) = \frac{1}{2^k k!}\,\frac{d}{dx^k}(x^2-1)^k .$$

From (4.3) and $c_0(n) \equiv 1$ follows in the case of $\underline{x} \equiv 0$ that

$$\mathcal{P}_k^n(x_0) = x_0^k ,$$

as consequence of the construction of $\{\mathcal{P}_k^n(x)\}_{k\in\mathbb{N}_0}$ as Appell sequence with $\mathcal{P}_0^n(n) = c_0(n) \equiv 1$.

In the case of $x_0 \equiv 0$ we obtain the essential property:

$$\mathcal{P}_k^n(\underline{x}) = c_k(n)\underline{x}^k .$$

Finally we mention some relationships of monogenic functions generated by $\{\mathcal{P}_k^n(x)\}_{k\in\mathbb{N}_0}$ with Special Functions (cf. [19]).

If we define now (as noticed in the beginning of this section) the corresponding hypercomplex exponential function by

$$\mathrm{Exp}_n(x) := exp(\mathcal{P}_k^n(x)) = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)}{k!} ,$$

then $\mathrm{Exp}_n(xt)$ is an *exponential generating function* of the sequence $\{\mathcal{P}_k^n(x)\}_{k\in\mathbb{N}_0}$.

With $\omega(x) := \underline{x}/|\underline{x}|$ and $\omega^2 = -1$ as the equivalent for the imaginary unit i we can prove the following result (cf.[19]):

Theorem 4.1. *The* Exp_n*-function can be written in terms of Bessel functions of the first kind,* $J_a(x)$*, for orders* $a = n/2-1, n/2$ *as*

$$\mathrm{Exp}_n(x_0+\underline{x}) = e^{x_0}\Gamma\left(\frac{n}{2}\right)\left(\frac{2}{|\underline{x}|}\right)^{(n/2)-1}\left(J_{n/2-1}(|\underline{x}|)+\omega(x)J_{n/2}(|\underline{x}|)\right) .$$

Finally we mention some further relationships of monogenic functions generated by $\{\mathcal{P}_k^n(x)\}_{k\in\mathbb{N}_0}$ to other Special Functions:

1. The generalized exponential function Exp_n suggests, for example, to consider the series of $\cosh(\mathcal{P}_k^n(\underline{x}))$ or $\sinh(\mathcal{P}_k^n(\underline{x}))$. This leads to expressions with *modified Bessel functions of first kind.*
2. The analogous study of the series $\arccos(\mathcal{P}_k^n(\underline{x}))$ and $\arcsin(\mathcal{P}_k^n(\underline{x}))$ leads to expressions with *elliptic integrals of type K and E.*
3. The series $\arctan(\mathcal{P}_k^n(\underline{x}))$ leads to expressions with the LerchPhi-function etc.

4.2. **Appell sequences generated by n-tuples of real parameters.** In [10] a new family of monogenic Appell polynomials was defined. More precisely, we consider a fixed n-tuple of real parameters $\mu = (\mu_1,\cdots,\mu_n)$, such that $|\mu|^2 = \mu_1^2+\cdots+\mu_n^2 \neq 0$. For each degree of homogeneity $k \geq 0$, we define the homogeneous monogenic polynomials

$$\mathcal{M}_k^{n,\mu}(z_1,\cdots,z_n) := (z_1\,\mu_1+\cdots+z_n\,\mu_n)^k\left(\frac{\hat{\mu}}{|\mu|}\right)^k , \tag{4.4}$$

where

$$\hat{\mu} = \frac{\mu_1 e_1+\cdots+\mu_n e_n}{|\mu|} , \tag{4.5}$$

with the property that $\hat{\mu}^2 = -1$.

We remark that in the expression (4.4) the power should be understood in the permutational product sense (see [34] for details about this product in $C\ell_{0,n}$).

Theorem 4.2. *For each fixed n-tuple $\mu = (\mu_1,\cdots,\mu_n)$ of real parameters such that $|\mu| \neq 0$, the sequence $(\mathcal{M}_k^{n,\mu})_{k\geq 0}$ is an Appell sequence.*

Proof. It is clear that, for each k, $\mathcal{M}_k^{n,\mu}$ has exactly degree of homogeneity k. The hypercomplex derivative of $\mathcal{M}_k^{n,\mu}$ is given by

$$\begin{aligned}(\mathcal{M}_k^{n,\mu}(z_1,\cdots,z_n))' &= k\,(z_1\,\mu_1+\cdots+z_n\,\mu_n)^{k-1}\,(-\mu_1 e_1-\cdots-\mu_n e_n)\times\\ &\times\frac{(\mu_1 e_1+\cdots+\mu_n e_n)^k}{|\mu|^{2k}} =\\ &= k\,(z_1\,\mu_1+\cdots+z_n\,\mu_n)^{k-1}\,\frac{(\mu_1 e_1+\cdots+\mu_n e_n)^{k-1}}{|\mu|^{2k-2}} = k\,\mathcal{M}_{k-1}^{n,\mu}(z_1,\cdots,z_n) .\end{aligned}$$

We stress the fact that the new family of monogenic Appell polynomials defined by (4.4) is not a generalization of the Appell sequence $\{\mathcal{P}_k^n\}_{k\in\mathbb{N}_0}$. Indeed, there are no particular choices of the parameter $\mu = (\mu_1,\cdots,\mu_n)$ such that $\mathcal{P}_k^n \equiv \mathcal{M}_k^{n,\mu}$, as we can see in the following example:

Example. For $n = 2$ and $k = 2$, we have

$$\mathbf{P}_2(z_1, z_2) = -\frac{1}{2}(z_1^2 + z_2^2)$$

and

$$\begin{aligned} \mathcal{M}_2^{2,\mu}(z_1, z_2) &= (z_1\,\mu_1 + z_n\,\mu_2)^2 \left(\frac{\hat{\mu}}{|\mu|}\right)^2 = \\ &= -\frac{1}{|\mu|^2}\left(z_1^2\,\mu_1^2 + 2z_1 \times z_2\,\mu_1\,\mu_2 + z_2^2\,\mu_2^2\right) . \end{aligned}$$

The identity of $\mathbf{P}_2$ and $\mathcal{M}_2^{2,\mu}$ would imply that $\mu_1 = \mu_2 = 0$, which contradicts the definition of $\mathcal{M}_2^{2,\mu}$.

Naturally, the Appell sequence defined by (4.4) lead to a new method of generating exponential monogenic functions in $\mathbb{R}^{n+1}$, with respect to a fixed n-tuple of real parameters $\mu = (\mu_1, \cdots, \mu_n)$, such that $|\mu| \neq 0$:

$$E(z_1, \cdots, z_n; \mu_1, \cdots, \mu_n, t) := \sum_{k=0}^{\infty} \frac{1}{k!}\,\mathcal{M}_k^{n,\mu}(z_1, \cdots, z_n)\,t^k \quad , \quad t \in \mathbb{R} .$$

The particular choice of the parameter $t = 1$ defines the exponential function

$$Exp_n^{\mu}(z_1, \cdots, z_n) := \sum_{k=0}^{\infty} \frac{1}{k!}\,\mathcal{M}_k^{n,\mu}(z_1, \cdots, z_n) \quad , \quad |\mu| \neq 0 .$$

Theorem 4.3. *For each fixed n-tuple of real parameters $\mu = (\mu_1, \cdots, \mu_n)$ such that $|\mu| \neq 0$, the exponential function Exp_n^{μ} has the following properties:*

1. $Exp_n^{\mu}(z_1, \cdots, z_n)\big|_{\underline{x}=0} = e^{x_0}$, $x_0 \in \mathbb{R}$
2. $(Exp_n^{\mu}(\lambda z_1, \cdots, \lambda z_n))' = \lambda\, Exp_n^{\mu}(\lambda z_1, \cdots, \lambda z_n)$, $\lambda \in \mathbb{R}$.

Proof. We begin by proving property 1. We have

$$Exp_n^{\mu}(z_1, \cdots, z_n)\big|_{\underline{x}=0} = \sum_{k=0}^{\infty} \frac{1}{k!}\,(-e_1\,x_0\mu_1 - \cdots - e_n\,x_0\mu_n)^k \left(\frac{\hat{\mu}}{|\mu|}\right)^k .$$

Taking into account the definition of $\hat{\mu}$ given by (4.5), we obtain

$$\begin{aligned} Exp_n^{\mu}(z_1, \cdots, z_n)\big|_{\underline{x}=0} &= \sum_{k=0}^{\infty} \frac{1}{k!}\,(-1)^k\,x_0^k\,(e_1\mu_1 + \cdots + e_n\mu_n)^{2k}\,\frac{1}{|\mu|^{2k}} = \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\,(-1)^k x_0^k \frac{\left(-|\mu|^2\right)^k}{|\mu|^{2k}} = \sum_{k=0}^{\infty} \frac{1}{k!}\,x_0^k . \end{aligned}$$

Property 2 is a consequence of the homogeneity of the polynomials $\mathcal{M}_k^{n,\mu}$ and the fact that they form Appell sequences.

Remark. For $n = 2$ the Appell sequences (4.4) can easily be written in terms of the generalized powers as

$$\mathcal{M}_k^{2,\mu}(z_1, z_2) = \begin{cases} \dfrac{(-1)^l}{|\mu|^{2l}} \displaystyle\sum_{s=0}^{2l} \binom{2l}{s} \mu_1^{2l-s}\mu_2^s z_1^{2l-s} \times z_2^s \quad , \quad k = 2l \\[2ex] \dfrac{(-1)^l}{|\mu|^{2l}} \displaystyle\sum_{s=0}^{2l+1} \binom{2l+1}{s} \mu_1^{2l+1-s}\mu_2^s z_1^{2l+1-s} \times z_2^s \dfrac{\hat{\mu}}{|\mu|} \quad , \quad k = 2l+1 \end{cases}$$

and for the particular case of $|\mu| = 1$, the polynomials $\mathcal{M}_k^{2,\mu}$ are generated by the complex powers $z^k,\ z = x + iy \in \mathbb{C}$, in the sense described in [18].

5. Hypercomplex Bernoulli and Euler polynomials

Bernoulli polynomials and Euler polynomials in one real or complex variable have been studied over the last two centuries. Their application in several branches of mathematics justifies the amount of literature about them, mainly about the first one. Applications in Combinatorics, Number Theory and Numerical Analysis are well known.

There are various ways to define those polynomials but one of the most common is using the exponential generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} \quad , \quad |t| < 2\pi \tag{5.6}$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{\infty} E_k(x)\frac{t^k}{k!} \quad , \quad |t| < \pi\ , \tag{5.7}$$

where $B_k(x)$ and $E_k(x)$ denote the k-th Bernoulli and Euler polynomials, respectively. Bernoulli numbers, B_k, and Euler numbers, E_k, are given by $B_k := B_k(0)$ and $E_k := 2^k E_k(1/2)$ (cf. [1, 12]).

A considerable part of literature about those polynomials and numbers handles with generalizations and applications. In [39] and [38] we have studied the possibility to extend the referred polynomials to higher dimensions using concepts of Clifford Analysis. New polynomials were obtained as well as their properties were studied.

The introduction of the hypercomplex exponential function by the formal series

$$\mathbf{Exp}(\vec{t}, \vec{z}) := \exp\left(t_1 z_1 + \cdots + t_n z_n\right) = \sum_{k=0}^{\infty} \frac{1}{k!}(t_1 z_1 + \cdots + t_n z_n)^k\ , \tag{5.8}$$

with $\vec{t} = (t_1, \cdots, t_n) \in \mathbb{R}^n,\ \vec{z} = (z_1, \cdots, z_n) \in \mathcal{H}^n$, permits to define Bernoulli polynomials, $B_{j_1,\cdots,j_n}(z_1, \cdots, z_n)$, $j_k \in \mathbb{N}_0$, $(k = 1, \cdots, n)$, and Euler polynomials, $E_{j_1,\cdots,j_n}(z_1, \cdots, z_n)$, $j_k \in \mathbb{N}_0$ $(k = 1, \cdots, n)$, in several hypercomplex variables, as coefficients of the multiple power series ordered with respect to the degree of homogeneity by

$$\begin{aligned} \mathbf{Exp}(\vec{t}, \vec{z}) &= \left(\sum_{r=0}^{\infty} \frac{1}{(r+1)!}(t_1 + \cdots + t_n)^r\right) \times \\ &\times \left(\sum_{|j|=0}^{\infty} \frac{1}{j!}\, B_{j_1,\cdots,j_n}(z_1, \cdots, z_n) t_1^{j_1} \cdots t_n^{j_n}\right)\ , \end{aligned} \tag{5.9}$$

and

$$(5.10)\quad \begin{aligned} 2\,\mathbf{Exp}(\vec{t},\vec{z}) &= \left(1+\sum_{r=0}^{\infty}\frac{1}{r!}(t_1+\cdots+t_n)^r\right)\times \\ &\times\left(\sum_{|j|=0}^{\infty}\frac{1}{j!}\,E_{j_1,\cdots,j_n}(z_1,\cdots,z_n)t_1^{j_1}\cdots t_n^{j_n}\right), \end{aligned}$$

respectively.

The expressions (5.9) and (5.10) can be considered as generalizations of

$$\sum_{s=0}^{\infty}\frac{(xt)^s}{s!}=\left(\sum_{r=0}^{\infty}\frac{t^r}{(r+1)!}\right)\left(\sum_{k=0}^{\infty}B_k(x)\,\frac{t^k}{k!}\right),$$

and

$$2\sum_{s=0}^{\infty}\frac{(xt)^s}{s!}=\left(1+\sum_{r=0}^{\infty}\frac{t^r}{r!}\right)\left(\sum_{k=0}^{\infty}E_k(x)\,\frac{t^k}{k!}\right),$$

for several variables, obtained by using (5.6) and(5.7) in their equivalent form

$$e^{xt}=\frac{e^t-1}{t}\sum_{k=0}^{\infty}B_k(x)\,\frac{t^k}{k!}$$

and

$$2e^{xt}=(1+e^t)\sum_{k=0}^{\infty}E_k(x)\,\frac{t^k}{k!}\,.$$

From (5.9) and (5.10), using (5.8) and comparing both sides, an expression can be derived to calculate the referred polynomials:

$$\sum_{s+j=\sigma}\frac{1}{(|s|+1)s!j!}\,B_j(\vec{z})=\frac{\vec{z}^{\sigma}}{\sigma!}$$

and

$$E_\sigma(\vec{z})+\sum_{s+j=\sigma}\frac{1}{s!j!}\,E_j(\vec{z})=2\frac{\vec{z}^{\sigma}}{\sigma!}\,,$$

where $s=(s_1,\cdots,s_n)$, $j=(j_1,\cdots,j_n)$, and $\sigma=(\sigma_1,\cdots,\sigma_n)$.

Remark 5.1. The generalization of Genocchi polynomials (cf.[31]), $G_n(x)$, is entirely similar to the above polynomials, since their generating function is analogous to those of the Bernoulli polynomials and Euler polynomials. Extensions of other special polynomials to several hypercomplex variables will appear in a forthcoming paper.

6. Hypercomplex polynomials and special matrices

The interest has recently emerged around the matrix representation of polynomials (see, for instance, [3, 29]). This representation allows to deal with them in an easier way and also to derive interesting combinatorial identities. Bernoulli and Euler polynomials are, evidently, object of special attention. The representation of polynomials is connected with the well known Pascal matrix, P, and consequently with the "creation matrix", H, since $P=e^H$ (cf. [3, 11, 29, 48]).

In the case of polynomials in several variables the problem of the matrix representation is more complicated. We need to choose a suitable monomial order to write the polynomials, since for each homogeneous degree one has to deal with products of several variables.

As a general tool of dealing with the matrix representation of multivariate polynomials, we introduce a block Pascal matrix, $\mathcal{P}$, and a block creation matrix, $\mathbb{H}$, which simulate, in some sense, the classical ones. In fact, we consider $\mathcal{P} = [\mathcal{P}_{sr}]$ such that

$$\mathcal{P}_{sr} = \begin{cases} \binom{s}{r} P & , \quad s \geq r \\ \\ O & , \qquad \text{otherwise} \;\; s, r = 0, \cdots, n \end{cases}$$

(O is the null matrix of order $n+1$), and $\mathbb{H} = [\mathbb{H}_{sr}]$, where

$$\mathbb{H}_{sr} = \begin{cases} H & , \quad s = r \\ sI & , \quad s = r+1 \\ O & , \quad \text{otherwise} \;\; s, r = 0, \cdots, n \end{cases}$$

(I is the identity matrix of order $n+1$), both block matrices of order $(n+1)(n+1)$.

Most properties of P and H (cf. [3, 11]) are fulfilled for their block versions, $\mathcal{P}$ and $\mathbb{H}$. In particular,

- The inverse of the block Pascal matrix, $\mathcal{P}^{-1} = [(P^{-1})_{sr}]$, is

$$(\mathcal{P}^{-1})_{sr} = \begin{cases} \binom{s}{r} (-1)^{s-r} P^{-1} & , \quad s \geq r \\ \\ 0 & , \quad \text{otherwise} \;\; s, r = 0, \cdots, n \; ; \end{cases}$$

- The symmetric block Pascal matrix, $\mathcal{S} = [\mathcal{S}_{sr}]$ with $\mathcal{S}_{sr} = \binom{s+r}{r} PP^T$, $s, r = 0, \cdots, n$, is $\mathcal{P}\mathcal{P}^T$;
- $\mathbb{H}$ is nilpotent of order $2n$;
- $\mathcal{P}$ and $\mathbb{H}$ are related by

$$\mathcal{P} = e^{\mathbb{H}} = \sum_{k=0}^{2n} \frac{\mathbb{H}^k}{k!} \, . \tag{6.11}$$

In order to pursuit the matrix representation of hypercomplex polynomials, in particular Bernoulli and Euler polynomials, we have defined in [39] the hypercomplex Pascal matrix of order $(n+1)(n+1)$, $\mathcal{P}(z_1, z_2) = [\mathcal{P}_{sr}(z_1, z_2)]$, where

$$\mathcal{P}_{sr}(z_1, z_2) = \begin{cases} \binom{s}{r} P(z_1) \times z_2^{s-r} & , \quad s \geq r \\ \\ 0 & , \quad \text{otherwise} \; s, r = 0, \cdots, n \; , \end{cases}$$

and $P(z_1)$ is the generalized Pascal matrix considered in [3, 48]

$$P(z_1) = e^{z_1 H} = \sum_{k=0}^{2n} \frac{(z_1 H)^k}{k!} \; .$$

Introducing the block matrix $F(z_1, z_2) = [F_{sr}(z_1, z_2)]$:

$$F_{sr}(z_1, z_2) = \begin{cases} z_1 H & , \quad s = r \\ s z_2 I & , \quad s = r+1 \\ O & , \quad \text{otherwise} \;\; s, r = 0, \cdots, n \; , \end{cases}$$

it is possible to conclude that

$$\mathcal{P}(z_1, z_2) = e^{F(z_1,z_2)} = \sum_{k=0}^{2n} \frac{F(z_1, z_2)^k}{k!} \ . \tag{6.12}$$

It is worth noting that, setting $z_1 = z_2 = 1$ in (6.12), formula (6.11) is obtained.

Recall that the hypercomplex polynomial Bernoulli matrix and the hypercomplex polynomial Euler matrix, introduced in [39] and [38], are block matrices of order $(n+1)(n+1)$, $\mathcal{B}(z_1, z_2) = [\mathcal{B}_{ij}^{sr}(z_1, z_2)]$ and $\mathcal{E}(z_1, z_2) = [\mathcal{E}_{ij}^{sr}(z_1, z_2)]$ such that

$$\mathcal{B}_{ij}^{sr}(z_1, z_2) = \begin{cases} \binom{i}{j}\binom{s}{r} B_{i-j,s-r}(z_1, z_2) & , \quad i \geq j \wedge s \geq r \\ 0 \quad , \quad \text{otherwise} \ \ i, j, s, r = 0, \cdots, n \end{cases} \tag{6.13}$$

and

$$\mathcal{E}_{ij}^{sr}(z_1, z_2) = \begin{cases} \binom{i}{j}\binom{s}{r} E_{i-j,s-r}(z_1, z_2) & , \quad i \geq j \wedge s \geq r \\ 0 \quad , \quad \text{otherwise} \ \ i, j, s, r = 0, \cdots, n \ . \end{cases} \tag{6.14}$$

$\mathcal{B}_{ij}^{sr}(z_1, z_2)$ and $\mathcal{E}_{ij}^{sr}(z_1, z_2)$ denote the entry in the position ij of the block in the position sr. The matrices $\mathcal{B} := \mathcal{B}(0, 0)$ and $\mathcal{E} := \mathcal{E}(0, 0)$ are called, respectively, Bernoulli matrix and Euler matrix.

Relations between the hypercomplex Pascal matrix and (6.13) as well as (6.14),

$$\begin{aligned} \mathcal{B}(z_1, z_2) &= \mathcal{P}(z_1, z_2)\mathcal{B} \ , \\ \mathcal{E}(z_1, z_2) &= \mathcal{P}(z_1, z_2)\mathcal{E} \ , \end{aligned} \tag{6.15}$$

were studied in [39] and [38], thereby extending to higher dimensions results obtained by Zhang and Wang in [49].

As a consequence of the matrix representation of polynomials, in [3, 29] the authors obtained the transformation matrix between the Taylor expansion of a real function and an expansion in terms of Bernoulli polynomials. They achieved to the matrix

$$L = \sum_{k=0}^{n} \frac{H^k}{(k+1)!}$$

that satisfies $Lb(x) = \xi(x), \quad x \in \mathbb{R}$, where $b(x) = [B_0(x) \cdots B_n(x)]^T$ is the vector of univariate Bernoulli polynomials and $\xi(x) = [1 \ \ x \ \cdots \ x^n]^T$.

In our case, due to (6.15) and noting that $\mathcal{P}(z_1, z_2)\mathcal{B} = \mathcal{B}\mathcal{P}(z_1, z_2)$, the matrix $\mathcal{B}^{-1}$ has a similar action on $\mathcal{B}(z_1, z_2)$ as L in the univariate case.

Representing by

$$\mathbf{b}(z_1, z_2) := [B_{0,0}(z_1, z_2) \ \cdots \ B_{n,0}(z_1, z_2)| \cdots |B_{0,n}(z_1, z_2) \ \cdots \ B_{n,n}(z_1, z_2)]^T$$

and

$$\mathbf{p}(z_1, z_2) := [1 \ \ z_1 \ \cdots \ z_1^n | z_2 \ \ z_1 \times z_2 \ \cdots \ z_1^n \times z_2 | \cdots | z_2^n \ \ z_1 \times z_2^n \ \cdots \ z_1^n \times z_2^n]^T$$

the first column of $\mathcal{B}(z_1, z_2)$ and $\mathcal{P}(z_1, z_2)$, respectively, we have

$$\mathcal{B}^{-1}\mathbf{b}(z_1, z_2) = \mathbf{p}(z_1, z_2) \ . \tag{6.16}$$

Notice that $\mathbf{p}(z_1, z_2) = vec(\xi(z_1) \circledast \xi(z_2)^T)$, with $\circledast$ the Kronecker product (cf. [28]) in the Clifford Analysis setting.

Let $F = [f_{ij}]$ be the matrix whose entries $f_{ij}, (i, j = 0, \cdots, n)$ are the coefficients of $z_1^i \times z_2^j$ in the Taylor expansion of a function of hypercomplex variables, $f(z_1, z_2)$. Then it is possible to write the Taylor expansion in the form

$$f(z_1, z_2) = (vec(F))^T vec(\xi(z_1) \circledast \xi(z_2)^T) + \cdots = (vec(F))^T \mathbf{p}(z_1, z_2) + \cdots \ , \tag{6.17}$$

where *vec* stands for the usual vectorization of matrices [28]. Moreover, since (6.16), (6.17) becomes

$$f(z_1, z_2) = (vec(F))^T \mathcal{B}^{-1} \mathbf{b}(z_1, z_2) + \cdots \ .$$

In other words, we achieve the expansion of $f(z_1, z_2)$ in terms of hypercomplex Bernoulli polynomials.

In a completely analogous way, it is possible to obtain an expansion in terms of hypercomplex Euler polynomials:

$$f(z_1, z_2) = (vec(F))^T \mathcal{E}^{-1} \mathbf{e}(z_1, z_2) + \cdots \ ,$$

where $e(z_1, z_2)$ is the first column of $\mathcal{E}(z_1, z_2)$.

7. Monogenic polynomial bases

Following the notations of Delanghe (cf. [17]), we denote by $M^+(n;k)$ the $C\ell_{0,n}$-linear space of homogeneous monogenic polynomials of degree k with values in $C\ell_{0,n}$ and by $\mathcal{P}(n;k)$ the space of $C\ell_{0,n}$-valued homogeneous polynomials of degree k in $\mathbb{R}^n$. An element $P \in \mathcal{P}(n;k)$ can be expressed as

$$P(x) = \sum_{|\underline{\alpha}|=k} \underline{x}^{\underline{\alpha}} \, a_{\underline{\alpha}} \quad , \quad a_{\underline{\alpha}} \in C\ell_{0,n} \ .$$

The dimension of $M^+(n;k)$ can be obtained from the already mentioned Cauchy-Kovalevskaya extension (CK) applied to the space $\mathcal{P}(n;k)$. As a consequence of $M^+(n;k) = CK(\mathcal{P}(n;k))$ (see [7]), we get that

$$\dim M^+(n;k) = \dim \mathcal{P}(n;k) = \binom{n+k-1}{n-1} \ .$$

On the other hand, we can consider the $\mathbb{R}$-linear space of $\mathcal{A}$-valued homogeneous monogenic polynomials of degree k in $\mathbb{R}^{n+1}$, that we denote by $M(\mathcal{A};n;k)$. The operator $\overline{D}$ determines an isomorphism between the real vector spaces $\mathcal{H}(n;k+1)$ of homogeneous harmonic polynomials of degree $k+1$ and $M(\mathcal{A};n;k)$ (cf. [17]). As a consequence,

$$\dim M(\mathcal{A};n;k) = \binom{n+k}{n-1} + \binom{n+k-1}{n-1} \ .$$

From now on, we restrict ourselves to the case $n = 2$, the most relevant case in pratical applications. Consequently, the dimension of the spaces $M^+(k) := M^+(2;k)$ and $M(\mathcal{A};k) := M(\mathcal{A};2;k)$ are equal to $k+1$ and $2k+3$, respectively.

We remark that the elements of $M(\mathcal{A};k)$ are polynomial solutions of the Riesz system described in subsection 2.2.

7.1. $C\ell_{0,2}$**-linear space of homogeneous monogenic polynomials of degree** $\boldsymbol{k}$**.** A canonical basis of $M^+(k)$ is formed by the set of generalized powers $\{z_1^{k-j} \times z_z^j : j = 0, \cdots, k\}$. The proof of this fact has been done in different ways by several authors (e.g., Fueter [22], Sudbery [47], Delanghe [14], Malonek [34]). Another basis for the space $M^+(k)$, generated by a pair of real numbers was built in [8]. More precisely, given $k+1$ pairs of real numbers (a_i, b_i) $(i, j = 0, \cdots, k)$, such that $a_i^2 + b_i^2 \neq 0$, and $a_i\, b_j - a_j b_i \neq 0, i \neq j$, the set of the corresponding homogeneous monogenic polynomials defined by $H_{k,(a_i,b_i)} := (a_i\, z_1 + b_i\, z_2)^k$, form a basis of $M^+(k)$.

Motivated by this result we can built bases of $M^+(k)$ with elements $\mathcal{M}_k^{2,\mu}$ of the Appell sequence defined in subsection 4.2 (briefly called Appell bases).

Theorem 7.1. *For each* $k \in \mathbb{N}_0$*, the monogenic polynomials*

$$\left\{\mathcal{M}_k^{2,\mu^{(i)}},\ \mu^{(i)} \in \mathbb{R}^2 : |\mu^{(i)}| \neq 0,\ \mu_1^{(i)}\mu_2^{(j)} - \mu_1^{(j)}\mu_2^{(i)} \neq 0,\ i \neq j,\ i, j = 0, \cdots, k\right\}$$

form a family of Appell bases of the $C\ell_{0,2}$*-linear space* $M^+(k)$*.*

Proof. Clearly it is enough to prove the $C\ell_{0,2}$-linear independence of such a set for a given pair of real parameters $\mu^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)})$. By using the binomial formula (see [34])

$$\left(z_1\, \mu_1^{(i)} + z_2\, \mu_2^{(i)}\right)^k = \sum_{l=0}^{k} \binom{k}{l} \left(\mu_1^{(i)}\right)^{k-l} \left(\mu_2^{(i)}\right)^l z_1^{k-l} \times z_2^l$$

and the fact that the generalized powers form a basis of $M^+(k)$, we obtain homogeneous linear systems of equations whose determinants are Vandermonde determinants that are different from zero if $\mu_1^{(i)}\mu_2^{(j)} - \mu_1^{(j)}\mu_2^{(i)} \neq 0,\ i \neq j\ (i, j = 0, \cdots, k)$ holds.

Example. Choosing the parameters

$$\mu^{(i)} = \left(\cos i \frac{\pi}{k+1}, \sin i \frac{\pi}{k+1}\right) \quad , \quad i = 0, \cdots, k\ ,$$

the required conditions,

$$\begin{aligned} \mu_1^{(i)}\mu_2^{(j)} - \mu_1^{(j)}\mu_2^{(i)} &= \cos i \frac{\pi}{k+1}\ \sin j \frac{\pi}{k+1} - \cos j \frac{\pi}{k+1}\ \sin i \frac{\pi}{k+1} = \\ &= \sin (j-i)\, \frac{\pi}{k+1} = \left|\mu^{(i)} \times \mu^{(j)}\right| \neq 0 \quad , \quad i \neq j\ , \end{aligned}$$

are satisfied, due to the choice of $(k+1)$ pairwise non collinear directions $\mu^{(i)}$ $(i = 0, \cdots, k)$.

7.2. $\mathbb{R}$**-linear space of homogeneous monogenic polynomials of degree** $\boldsymbol{k}$**.** An explicit construction of a basis for the space $M(\mathcal{A}; k)$ was made in [30] by combining in an appropriate way the known generalized powers. Indeed, the author found that a basis of $M(\mathcal{A};k)$ is formed by the homogeneous monogenic polynomials

$$\begin{aligned} P_k^j &:= \binom{k}{j} {z_1}^{k-j} \times {z_2}^{j} \quad , \quad j=0,\cdots,k \,, \\ \widetilde{P}_k^l &:= P_k^l e_1 + P_k^{l-1} e_2 = \\ &= \binom{k}{l} {z_1}^{k-l} \times {z_2}^{l} e_1 + \binom{k}{l-1} {z_1}^{k-l+1} \times {z_2}^{l-1} e_2 \quad , \quad l=0,\cdots,k+1 \end{aligned}$$

(with the convention $P_n^{-1} = P_n^{n+1} \equiv 0$).

These results were later generalized to arbitrary dimensions in [17].

A detailed study about orthonormal bases of $M(\mathcal{A};k)$ was done in [9], with respect to the inner product defined by

$$\langle f, g\rangle_{0,L_2(B)} = \int_B \mathbf{Sc}(\overline{f}\, g)\, dV \,, \tag{7.18}$$

where $B := B(0,1)$ denotes the unit ball in $\mathbb{R}^3$ and $\mathbf{Sc}(.)$ represents the scalar part of its argument.

The aforementioned polynomial basis $\{P_k^j,\ \widetilde{P}_k^l : j=0,\cdots,k,\ l=0,\cdots,k+1\}$ is not orthogonal with respect to this inner product (see [8]). The use of the Gram-Schmidt orthonormalization procedure is analytically very difficult to apply, numerically requires a high storage, is very time consuming and unstable. For that reason, another approach was followed in [9] constructing an orthonormal basis of the space $M(\mathcal{A};k)$, by using the factorization of the Laplace operator, $\Delta_3 = \overline{D}\, D = D\, \overline{D}$. It is clear that this factorization permits to obtain homogeneous monogenic polynomials by applying the operator $\overline{D}$ to real-valued homogeneous harmonic polynomials.

We start by considering spherical coordinates $x = r\,\omega$, $\omega = \omega(\theta,\varphi) \in S := \partial B$. Taking the standard basis of spherical harmonics of degree $k+1$ in $\mathbb{R}^3$ (see, e.g., [45] or [43]) and applying the operator $\overline{D}$ to their extensions into the ball, we get the following $2k+3$ homogeneous monogenic polynomials of degree k,

$$\begin{aligned} X_k^m(x) &:= r^k \big[A^{m,k} \cos m\varphi + \big(B^{m,k} \cos\varphi \cos m\varphi - C^{m,k} \sin\varphi \sin m\varphi\big)\, e_1 + \\ &\quad + \big(B^{m,k} \sin\varphi \cos m\varphi + C^{m,k} \cos\varphi \sin m\varphi\big)\, e_2\big] \quad , \quad m=0,\cdots,k+1 \\ Y_k^l(x) &:= r^k \big[A^{l,k} \sin l\varphi + \big(B^{l,k} \cos\varphi \sin l\varphi + C^{l,k} \sin\varphi \cos l\varphi\big)\, e_1 + \\ &\quad + \big(B^{l,k} \sin\varphi \sin l\varphi - C^{l,k} \cos\varphi \cos l\varphi\big)\, e_2\big] \quad , \quad l=1,\cdots,k+1 \,, \end{aligned}$$

where $A^{m,k}, B^{m,k}, C^{m,k}$ $(m=0,\cdots,k+1)$ are functions that depend only on θ. They are defined by using the associated Legendre functions and their derivatives. For details about the construction, see [9].

Taking, for each $k \in \mathbb{N}$,

$$\widetilde{X}_k^m := \frac{1}{\binom{k+m+1}{k}}\, X_k^m \quad , \quad m=0,\cdots,k+1$$

$$\widetilde{Y}_k^l := \frac{1}{\binom{k+l+1}{k}}\, Y_k^l \quad , \quad l=1,\cdots,k+1$$

we have that

$$\left(\widetilde{X}_k^m\right)' = k\widetilde{X}_{k-1}^m \quad , \quad m = 0, \cdots, k+1$$
$$\left(\widetilde{Y}_k^l\right)' = k\widetilde{Y}_{k-1}^l \quad , \quad l = 1, \cdots, k+1 \; ,$$

i.e.,

$$\left\{\widetilde{X}_k^m,\ \widetilde{Y}_k^l : m = 0, \cdots, k+1,\ l = 1, \cdots, k+1\right\}_{k\in\mathbb{N}_0} \tag{7.19}$$

form an Appell sequence of homogeneous monogenic polynomials. We notice that the special case $m = 0$ coincides with the Appell sequence of homogeneous monogenic polynomials $\{\mathcal{P}_k^n(x)\}_{k\in\mathbb{N}_0}$, constructed in section 4 by other methods.

The main properties of the Appell sequence (7.19) are stated in the following theorems proved in [9]:

Theorem 7.2. *For each k, the set*

$$\left\{\widetilde{X}_k^m,\ \widetilde{Y}_k^l,\ m = 0, \cdots, k+1,\ l = 1, \cdots, k+1\right\}$$

is orthogonal with respect to the inner product (7.18)*.*

From this result, it follows

Theorem 7.3. *For each k, the set*

$$\left\{\widetilde{X}_k^m,\ \widetilde{Y}_k^l,\ m = 0, \cdots, k+1,\ l = 1, \cdots, k+1\right\} \; ,$$

form a orthogonal Appell basis of the space $M(\mathcal{A}; k)$.

We remark that the constructed basis contains two monogenic "constants", i.e, two functions that depend only on the variables x_1 and x_2 and that behave like constants with respect to the hypercomplex derivation. In fact, it holds $\left(\widetilde{X}_n^{n+1}\right)' = \left(\widetilde{Y}_n^{n+1}\right)' \equiv 0$ (see [9]).

The norms of the constructed homogeneous monogenic polynomials were explicitly calculated by using the recurrence formulae of the associated Legendre functions (see [9]). Hence, we can easily normalize the constructed orthogonal polynomial basis in order to get a complete orthonormal system of $\mathcal{A}$-valued functions of the $\mathbb{R}$-linear Hilbert space $L_2(B)$ of $C\ell_{0,2}$-valued functions. Recently, this system was used to describe the pointwise and local behavior of general $\mathcal{A}$-valued functions in $\mathbb{R}^3$ (see [27]).

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