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Fourier solution of the 2D Neumann problem for the Helmholtz equation

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Abstract[1]. The interior and exterior Neumann problems for the Helmholtz equation in starlike planar domains are addressed by using a suitable Fourier-like technique. Attention is in particular focused on normal-polar domains whose boundaries are defined by the so called "superformula" introduced by J. Gielis. A dedicated numerical procedure based on Mathematica© computer algebra system is developed in order to validate the proposed approach. In this way, highly accurate approximations of the solution, featuring properties similar to classical ones, are obtained. Computed results are found to be in good agreement with theoretical findings on Fourier series expansion presented by L. Carleson.

1. Introduction

Many applications of mathematical physics and electromagnetics are connected with the Laplacian:

- The wave equation $v_{tt} = a^2 \Delta v$;
- The heat propagation $v_t = \kappa \Delta v$;
- The Laplace equation $\Delta v = 0$;
- The Helmholtz equation $\Delta v + k^2 v = 0$;
- The Schrödinger equation $-\dfrac{h^2}{2m}\,\Delta\psi + V\psi = E\psi$.

In recent papers [1]-[4], the classical Fourier method [5], [6] for solving the Dirichlet problem for the Laplace equation in canonical domains has been extended in

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order to address the same differential problem in a starlike domain, i.e. a domain $\mathcal{D}$ which is normal with respect to a suitable polar co-ordinate system, and whose boundary $\partial\mathcal{D}$ can be interpreted as an *anisotropically stretched unit circle*, centered at the origin.

In this contribution, a suitable technique useful to compute the coefficients of Fourier-like expansions representing solutions of Neumann boundary-value problems (*BVP*s) for the classical Helmholtz equation in complex planar domains is presented. In particular, the boundary of the considered domains is defined by the so called "superformula" introduced by J. Gielis [7]. Regular functions are assumed to describe the boundary data, but the proposed approach can be easily generalized considering weakened hypotheses.

In order to verify and validate the relevant technique, a suitable numerical procedure based on the computer algebra system Mathematica© has been developed. By using such procedure, a point-wise convergence of the Fourier-series representation of the solution in regular points of the boundary, with Gibbs-like phenomena potentially occurring in singular points, has been observed. The obtained numerical results are in good agreement with theoretical findings by L. Carleson [8].

2. THE LAPLACIAN IN STRETCHED POLAR CO-ORDINATES

We introduce in the real plane the usual polar co-ordinate system

$$x = \rho \cos\vartheta \quad , \quad y = \rho \sin\vartheta \ , \tag{2.1}$$

and the polar equation of $\partial\mathcal{D}$

$$\rho = R(\vartheta) \quad , \quad 0 \le \vartheta \le 2\pi \ , \tag{2.2}$$

where $R(\vartheta)$ is a piece-wise C^2 function in $[0, 2\pi]$. We suppose that the domain $\mathcal{D}$ satisfies

$$0 \le \rho \le R(\vartheta) \ , \tag{2.3}$$

and therefore $\min_{\vartheta\in[0,2\pi]} R(\vartheta) > 0$.

We introduce the stretched radius ϱ^* such that

$$\rho = \varrho^* R(\vartheta) \ , \tag{2.4}$$

and the curvilinear (i.e. stretched) co-ordinates ϱ^*, ϑ in the x, y plane

$$x = \varrho^* R(\vartheta) \cos\vartheta \quad , \quad y = \varrho^* R(\vartheta) \sin\vartheta \ . \tag{2.5}$$

Therefore, $\mathcal{D}$ is obtained by assuming $0 \le \vartheta \le 2\pi$, $0 \le \varrho^* \le 1$.

Remark 1. Note that, in the stretched co-ordinate system the original domain $\mathcal{D}$ is transformed into the unit circle. Hence, in such co-ordinate system we can use all the classical techniques, including the separation of variables, to solve the transformed Helmholtz equation.

We consider a $L^2(\mathcal{D})$ function $v(x, y) = v(\rho\cos\vartheta, \rho\sin\vartheta) = u(\rho, \vartheta)$ and the Laplace operator in polar co-ordinates

$$\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \vartheta^2} \ . \tag{2.6}$$

We can represent this operator in the new stretched co-ordinate system ϱ^*, ϑ by setting

$$U(\varrho^*, \vartheta) = u(\varrho^* R(\vartheta), \vartheta) \ . \tag{2.7}$$

In this way we readily find (denoting for shortness $R(\vartheta) := R$)

$$\frac{\partial u}{\partial \rho} = \frac{1}{R}\,\frac{\partial U}{\partial \varrho^*}\,, \tag{2.8}$$

$$\frac{\partial^2 u}{\partial \rho^2} = \frac{1}{R^2}\,\frac{\partial^2 U}{\partial \varrho^{*2}}\,, \tag{2.9}$$

$$\frac{\partial u}{\partial \vartheta} = -\varrho^*\frac{R'}{R}\,\frac{\partial U}{\partial \varrho^*} + \frac{\partial U}{\partial \vartheta}\,, \tag{2.10}$$

$$\frac{\partial^2 u}{\partial \vartheta^2} = \varrho^*\frac{2R'^2 - RR''}{R^2}\,\frac{\partial U}{\partial \varrho^*} + \varrho^{*2}\frac{R'^2}{R^2}\,\frac{\partial^2 U}{\partial \varrho^{*2}} - 2\varrho^*\frac{R'}{R}\,\frac{\partial^2 U}{\partial \varrho^*\partial\vartheta} + \frac{\partial^2 U}{\partial \vartheta^2}\,. \tag{2.11}$$

Substituting equations (2.8)-(2.11 into equation (2.6) finally yields

$$\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\,\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\,\frac{\partial^2 u}{\partial \vartheta^2} =$$

$$= \frac{1}{R^2}\left(1 + \frac{R'^2}{R^2}\right)\frac{\partial^2 U}{\partial \varrho^{*2}} + \frac{1}{\varrho^* R^2}\left(1 + \frac{2R'^2 - RR''}{R^2}\right)\frac{\partial U}{\partial \varrho^*} -$$

$$-\frac{2R'}{\varrho^* R^3}\,\frac{\partial^2 U}{\partial \varrho^*\partial\vartheta} + \frac{1}{\varrho^{*2}R^2}\,\frac{\partial^2 U}{\partial \vartheta^2}\,. \tag{2.12}$$

As it can be easily noticed, for $\varrho^* = \rho$, $R(\vartheta) \equiv 1$, the expression of the Laplacian in polar co-ordinates is recovered.

3. AN EQUIVALENT FORMULATION

For further computations, it is convenient to introduce the function

$$\Upsilon := \Upsilon(\vartheta) := \frac{1}{R(\vartheta)} \quad , \quad 0 \le \vartheta \le 2\pi\,, \tag{3.1}$$

in order to describe the boundary $\partial\mathcal{D}$. In this way, the unit circle is obtained by assuming $\Upsilon(\vartheta) \equiv 1$.

Under the mentioned assumption, the stretched co-ordinates ϱ^*, ϑ in the x, y plane can be expressed as follows

$$x = \frac{\varrho^*}{\Upsilon(\vartheta)}\cos\vartheta \quad , \quad y = \frac{\varrho^*}{\Upsilon(\vartheta)}\sin\vartheta\,. \tag{3.2}$$

Once again, after setting for shortness

$$U(\varrho^*, \vartheta) = u\left(\frac{\varrho^*}{\Upsilon(\vartheta)}, \vartheta\right)\,, \tag{3.3}$$

the Laplacian is found to be

$$\Delta u = \left(\Upsilon^2 + \Upsilon'^2\right)\frac{\partial^2 U}{\partial \varrho^{*2}} + \frac{\Upsilon^2 + \Upsilon\Upsilon''}{\varrho^*}\,\frac{\partial U}{\partial \varrho^*} + \frac{2\Upsilon\Upsilon'}{\varrho^*}\,\frac{\partial^2 U}{\partial \varrho^*\partial\vartheta} + \frac{\Upsilon^2}{\varrho^{*2}}\,\frac{\partial^2 U}{\partial \vartheta^2}\,. \tag{3.4}$$

For $\varrho^* = \rho$, $\Upsilon(\vartheta) \equiv 1$, equation (2.6) is recovered.

4. THE NEUMANN PROBLEM FOR THE HELMHOLTZ EQUATION

Let us consider the interior Neumann problem for the Helmholtz equation in a starlike domain $\mathcal{D}$, whose boundary is described by the polar equation $\rho = R(\vartheta)$,

$$\left\{ \begin{array}{lll} \Delta v(x,y) + k^2 v(x,y) = 0 & , & (x,y) \in \overset{\circ}{\mathcal{D}} \, , \\ \dfrac{\partial v}{\partial \nu}(x,y) = f(x,y) & , & (x,y) \in \partial\mathcal{D} \, . \end{array} \right. \tag{4.1}$$

where $k > 0$ denotes the propagation constant, and $\hat{\nu} = \hat{\nu}(\vartheta)$ is the outward-pointing normal to the boundary $\partial\mathcal{D}$. We prove the following theorem.

Theorem 4.1. *Let*

$$\psi(\vartheta) = \frac{d}{d\vartheta} \ln R(\vartheta) = \frac{R'(\vartheta)}{R(\vartheta)} \, , \tag{4.2}$$

and

$$\begin{gathered} f(R(\vartheta)\cos\vartheta, R(\vartheta)\sin\vartheta) = F(\vartheta) = \\ = \frac{1}{\sqrt{1+\psi(\vartheta)^2}} \sum_{m=0}^{+\infty} (\alpha_m \cos m\vartheta + \beta_m \sin m\vartheta) \, , \end{gathered} \tag{4.3}$$

where

$$\left\{ \begin{array}{c} \alpha_m \\ \beta_m \end{array} \right\} = \frac{\epsilon_m}{2\pi} \int_0^{2\pi} F(\vartheta) \sqrt{1+\psi(\vartheta)^2} \left\{ \begin{array}{c} \cos m\vartheta \\ \sin m\vartheta \end{array} \right\} d\vartheta \, , \tag{4.4}$$

$$\epsilon_m = \left\{ \begin{array}{lll} 1 & , & m = 0 \\ 2 & , & m \neq 0 \end{array} \right.$$

being the Neumann's symbol [9]*. Then, the interior boundary-value problem for the Helmholtz equation* (4.1) *admits a classical solution*

$$v(x,y) \in L^2(\mathcal{D}) \tag{4.5}$$

such that the following Fourier-Bessel series expansion holds

$$\begin{gathered} v(\varrho^* R(\vartheta)\cos\vartheta, \varrho^* R(\vartheta)\sin\vartheta) = U(\varrho^*,\vartheta) = \\ = \sum_{m=0}^{+\infty} J_m(k\varrho^* R(\vartheta))(A_m \cos m\vartheta + B_m \sin m\vartheta) \, . \end{gathered} \tag{4.6}$$

For each index m, define

$$\begin{gathered} \left[\begin{array}{c} \xi_m(\vartheta) \\ \eta_m(\vartheta) \end{array} \right] = \left[\begin{array}{cc} \cos m\vartheta & -\sin m\vartheta \\ \sin m\vartheta & \cos m\vartheta \end{array} \right] \cdot \\ \cdot \left[\begin{array}{l} -kJ_{m+1}(kR(\vartheta)) + \dfrac{m}{R(\vartheta)} J_m(kR(\vartheta)) \\ -\dfrac{m}{R(\vartheta)} J_m(kR(\vartheta))\psi(\vartheta) \end{array} \right] . \end{gathered} \tag{4.7}$$

Hence, the coefficients A_m*,* B_m *in* (4.6) *can be determined by solving the infinite linear system*

$$\sum_{m=0}^{+\infty} \begin{bmatrix} \mathrm{X}^+_{n,m} & \mathrm{Y}^+_{n,m} \\ \mathrm{X}^-_{n,m} & \mathrm{Y}^-_{n,m} \end{bmatrix} \cdot \begin{bmatrix} A_m \\ B_m \end{bmatrix} = \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} , \tag{4.8}$$

where

$$\mathrm{X}^\pm_{n,m} = \frac{\epsilon_n}{2\pi} \int_0^{2\pi} \xi_m(\vartheta) \left\{ \begin{array}{c} \cos n\vartheta \\ \sin n\vartheta \end{array} \right\} d\vartheta , \tag{4.9}$$

$$\mathrm{Y}^\pm_{n,m} = \frac{\epsilon_n}{2\pi} \int_0^{2\pi} \eta_m(\vartheta) \left\{ \begin{array}{c} \cos n\vartheta \\ \sin n\vartheta \end{array} \right\} d\vartheta , \tag{4.10}$$

with m*,* $n \in \mathbb{N}_0$*.*

Proof. After noting that in the stretched co-ordinates system for the x, y plane the domain $\mathcal{D}$ becomes the unit circle, we can use the usual eigenfunction method [5] and separation of variables (with respect to the variables ρ, ϑ). As a consequence, elementary solutions of the problem can be searched in the form

$$u(\rho, \vartheta) = U\left(\frac{\rho}{R(\vartheta)}, \vartheta\right) = \mathrm{P}(\rho)\Theta(\vartheta) . \tag{4.11}$$

Substituting into the Helmholtz equation we easily find that the functions $\mathrm{P}(\cdot)$, $\Theta(\cdot)$ must satisfy the ordinary differential equations

$$\frac{d^2\Theta(\vartheta)}{d\vartheta^2} + \mu^2\Theta(\vartheta) = 0 , \tag{4.12}$$

$$\rho^2 \frac{d^2\mathrm{P}(\rho)}{d\rho^2} + \rho\frac{d\mathrm{P}(\rho)}{d\rho} + (k^2\rho^2 - \mu^2)\mathrm{P}(\rho) = 0 , \tag{4.13}$$

respectively.

The parameter μ is a separation constant whose choice is governed by the physical requirement that at any point in the real plane the scalar field $u(\rho, \vartheta)$ must be single-valued. So, by setting

$$\mu = m \in \mathbb{N}_0 , \tag{4.14}$$

we find

$$\Theta(\vartheta) = a_m \cos m\vartheta + b_m \sin m\vartheta , \tag{4.15}$$

where a_m, $b_m \in \mathbb{R}$ denote arbitrary constants. The radial function $\mathrm{P}(\cdot)$ satisfying (4.13) can be readily expressed as follows

$$\mathrm{P}(\rho) = c_m J_m(k\rho) + d_m Y_m(k\rho) \quad , \quad (c_m, d_m \in \mathbb{R}) . \tag{4.16}$$

As usual we have to assume $d_m = 0$ for the boundedness of the solution. Therefore, the general solution of the interior Neumann problem (4.1) can be searched in the form

$$u(\rho, \vartheta) = \sum_{m=0}^{+\infty} J_m(k\rho)(A_m \cos m\vartheta + B_m \sin m\vartheta) . \tag{4.17}$$

Imposing the Neumann boundary condition yields

$$F(\vartheta) = \frac{\partial u}{\partial \nu}(R(\vartheta), \vartheta) = \hat{\nu}(\vartheta) \cdot \nabla u(R(\vartheta), \vartheta) , \tag{4.18}$$

where

$$\nabla u(\rho,\vartheta) = \hat{\rho}\frac{\partial u(\rho,\vartheta)}{\partial\rho} + \hat{\vartheta}\frac{1}{\rho}\,\frac{\partial u(\rho,\vartheta)}{\partial\vartheta}\ , \tag{4.19}$$

and

$$\hat{\nu}(\vartheta) = \frac{\hat{\rho} - \psi(\vartheta)\hat{\vartheta}}{\sqrt{1+\psi(\vartheta)^2}}\ . \tag{4.20}$$

Hence, combining equations above and using the Fourier's projection method, formulas (4.8)–(4.10) easily follow.

Remark 2. It is worth noting that an approximate solution of the infinite linear system (4.8) can be obtained by solving the corresponding finite system with m, $n = 0, 1, \cdots, N$. Such solution is convergent as N approaches infinity since it may be regarded as the solution of a vectorial integral equation with L^2 kernel [10]. In fact, substituting in (4.8) the discrete index m with a continuous parametex μ and, consequently, putting $A_m = A(\mu)$, $B_m = B(\mu)$, $\underline{\zeta}(\mu) = [A(\mu), B(\mu)]^T$, $\xi_m(\vartheta) = \xi(\mu,\vartheta)$, $\eta_m(\vartheta) = \eta(\mu,\vartheta)$, $\underline{\Xi}(\mu,\vartheta) = [\xi(\mu,\vartheta), \eta(\mu,\vartheta)]^T$ yields

$$\frac{1}{\sqrt{1+\psi(\vartheta)^2}}\int_0^{+\infty}\underline{\Xi}(\mu,\vartheta)\cdot\underline{\zeta}(\mu)\,d\mu = F(\vartheta)\ . \tag{4.21}$$

Let $m_R := \min_{\vartheta\in[0,2\pi]} R(\vartheta)$, $M_R := \max_{\vartheta\in[0,2\pi]} R(\vartheta)$, and $M_\psi := \max_{\vartheta\in[0,2\pi]} \psi(\vartheta)$. Then, we can estimate [11]

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{2\pi}\left\|\frac{\underline{\Xi}(\mu,\vartheta)}{\sqrt{1+\psi(\vartheta)^2}}\right\|^2 d\vartheta d\mu \le \int_0^{+\infty}\int_0^{2\pi}\left[\xi(\mu,\vartheta)^2 + \eta(\mu,\vartheta)^2\right] d\vartheta d\mu \le \\
&\le \int_0^{+\infty}\int_0^{2\pi}\left[k^2 J_{\mu+1}(kR(\vartheta))^2 + \mu^2\frac{\psi(\vartheta)^2+1}{R(\vartheta)^2}\,J_\mu(kR(\vartheta))^2 + \right. \\
&\left. + \frac{2k\mu}{R(\vartheta)}\left|J_\mu(kR(\vartheta))J_{\mu+1}(kR(\vartheta))\right|\right] d\vartheta d\mu \le 2\pi\int_0^{+\infty}\left[k^2\left(\frac{ekM_R}{2\mu+2}\right)^{2\mu+2} + \right. \\
&\left. + \frac{\mu}{m_R}\left(\frac{M_\psi^2+1}{m_R}\,\mu + k^2 M_R\right)\left(\frac{ekM_R}{2\mu}\right)^{2\mu}\right] d\mu < +\infty\ ,
\end{aligned} \tag{4.22}$$

for all m_R, M_R, $M_\psi \in \mathbb{R}^+$. This inequality confirms that the kernel of the integral equation (4.21) is L^2.

In a similar way, the exterior Neumann problem

$$\begin{cases} \Delta v(x,y) + k^2 v(x,y) = 0 & , \quad (x,y) \in \mathbb{R}^2 \setminus \mathcal{D}\ , \\ \dfrac{\partial v}{\partial\nu}(x,y) = f(x,y) & , \quad (x,y) \in \partial\mathcal{D}\ , \end{cases} \tag{4.23}$$

subject to the Sommerfeld radiation condition [12]

$$\lim_{\rho\to+\infty}\sqrt{\rho}\left(\frac{\partial}{\partial\rho} - ik\right)v(x,y) = 0\ , \tag{4.24}$$

may be addressed. In particular, the following theorem can be easily proved.

Theorem 4.2. *Under the hypotheses of the previous theorem, the exterior boundary-value problem for the Helmholtz equation* (4.23)–(4.24) *admits a classical solution*

$$v(x,y) \in L^2(\mathbb{R}^2 \setminus \mathcal{D}) \tag{4.25}$$

such that the following Fourier-Hankel series expansion holds

$$v(\varrho^* R(\vartheta)\cos\vartheta, \varrho^* R(\vartheta)\sin\vartheta) = U(\varrho^*,\vartheta) =$$

$$= \sum_{m=0}^{+\infty} H_m^{(1)}(k\varrho^* R(\vartheta))(A_m \cos m\vartheta + B_m \sin m\vartheta)\,. \tag{4.26}$$

For each index m*, define*

$$\begin{bmatrix} \xi_m(\vartheta) \\ \eta_m(\vartheta) \end{bmatrix} = \begin{bmatrix} \cos m\vartheta & -\sin m\vartheta \\ \sin m\vartheta & \cos m\vartheta \end{bmatrix} \cdot \begin{bmatrix} -kH_{m+1}^{(1)}(kR(\vartheta)) + \dfrac{m}{R(\vartheta)}\, H_m^{(1)}(kR(\vartheta)) \\ -\dfrac{m}{R(\vartheta)}\, H_m^{(1)}(kR(\vartheta))\psi(\vartheta) \end{bmatrix}. \tag{4.27}$$

Then, the coefficients A_m*,* B_m *in* (4.26) *are found to be the solution of the infinite linear system*

$$\sum_{m=0}^{+\infty} \begin{bmatrix} \mathrm{X}_{n,m}^{+} & \mathrm{Y}_{n,m}^{+} \\ \mathrm{X}_{n,m}^{-} & \mathrm{Y}_{n,m}^{-} \end{bmatrix} \cdot \begin{bmatrix} A_m \\ B_m \end{bmatrix} = \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}, \tag{4.28}$$

where

$$\mathrm{X}_{n,m}^{\pm} = \frac{\epsilon_n}{2\pi} \int_0^{2\pi} \xi_m(\vartheta) \left\{ \begin{array}{c} \cos n\vartheta \\ \sin n\vartheta \end{array} \right\} d\vartheta\,, \tag{4.29}$$

$$\mathrm{Y}_{n,m}^{\pm} = \frac{\epsilon_n}{2\pi} \int_0^{2\pi} \eta_m(\vartheta) \left\{ \begin{array}{c} \cos n\vartheta \\ \sin n\vartheta \end{array} \right\} d\vartheta\,, \tag{4.30}$$

with m*,* $n \in \mathbb{N}_0$*.*

Remark 3. Note that the above formulas still hold under the assumption that $R(\vartheta)$ is a piecewise continuous function, and the boundary data are described by square integrable, not necessarily continuous, function, so that the relevant Fourier coefficients α_m, β_m in equation (4.3) are finite quantities.

5. Numerical example

In the following numerical examples, we assume for the boundary $\partial\mathcal{D}$ a general polar equation of the type

$$R(\vartheta) = \left(\left| \frac{\cos(p\vartheta/4)}{\gamma_1} \right|^{\nu_1} + \left| \frac{\sin(q\vartheta/4)}{\gamma_2} \right|^{\nu_2} \right)^{-1/\nu_0}, \tag{5.1}$$

as introduced by J. Gielis in [7]. Very different shapes of the considered domain, including ellipses, Lamé curves (also called *Superellipses*), ovals, and $m-$fold symmetric figures can be obtained by assuming suitable values of parameters p, q, γ_1, γ_2, ν_0, ν_1, ν_2 in (5.1). It was noticed in [7] that many characteristic geometries

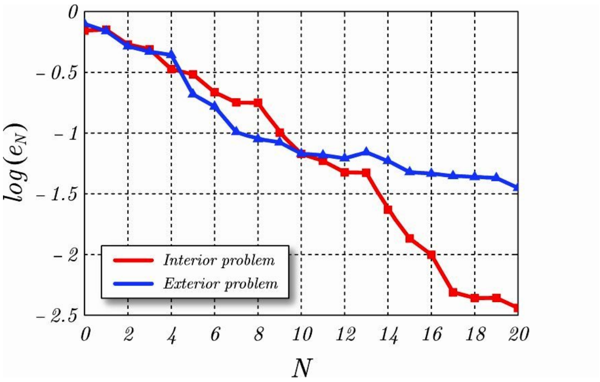

FIGURE 1. Relative boundary error e_N as function of the number N of terms in the expansion (5.3). The relevant domain $\mathcal{D}$ is described by the polar equation (5.1) with $\gamma_1 = \gamma_2 = 3/4$, $p = q = 5$, $\nu_0 = 10$, $\nu_1 = \nu_2 = 6$.

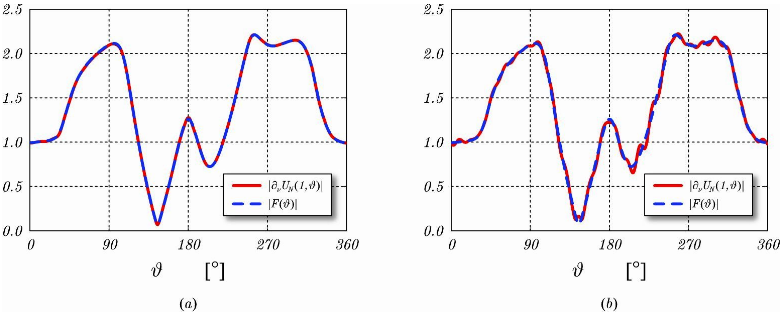

FIGURE 2. Angular behavior of the partial sum $U_N(1, \vartheta)$ of order $N = 20$ representing the solution of the interior (a) and exterior (b) problem for the Helmholtz equation. The relevant domain $\mathcal{D}$ is described by the polar equation (5.1) with $\gamma_1 = \gamma_2 = 3/4$, $p = q = 5$, $\nu_0 = 10$, $\nu_1 = \nu_2 = 6$.

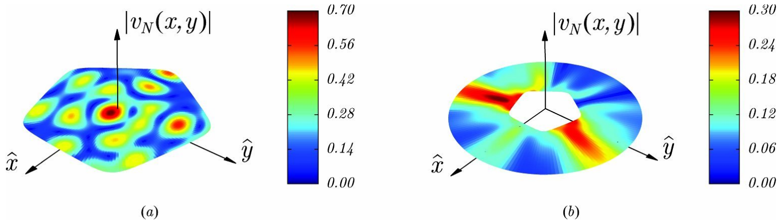

FIGURE 3. Spatial distribution of the partial sum $v_N(x, y)$ of order $N = 20$ representing the solution of the interior (a) and exterior (b) problem for the Helmholtz equation. The relevant domain $\mathcal{D}$ is described by the polar equation (5.1) with $\gamma_1 = \gamma_2 = 3/4$, $p = q = 5$, $\nu_0 = 10$, $\nu_1 = \nu_2 = 6$.

occuring in Nature (starfish, equisetum, raspberry, etc.) can be approximated in such a way. We emphasize that almost all two-dimensional normal-polar domains are described (or at least approximated in a close way) by the above mentioned curves.

To assess the performance of the proposed algorithm in terms of numerical accuracy and convergence rate, the relative boundary error has been evaluated as follows

$$e_N = \frac{\left\| \frac{\partial U_N}{\partial \nu}(1,\vartheta) - F(\vartheta) \right\|}{\|F(\vartheta)\|} , \tag{5.2}$$

$\| \cdot \|$ denoting the usual $L^2(\partial\mathcal{D})$ norm, and $U_N(\varrho^*, \vartheta)$ the partial sum of order N relevant to the Fourier-type series expansion representing the solution of the specific boundary-value problem for the Helmholtz equation, namely

$$U_N(\varrho^*, \vartheta) = \sum_{m=0}^{N} Z_m(km\varrho^* R(\vartheta))(A_m \cos m\vartheta + B_m \sin m\vartheta) , \tag{5.3}$$

where $Z_m(\cdot)$ are Bessel or Hankel functions of the first kind. By assuming in (5.1) $\gamma_1 = \gamma_2 = 3/4$, $p = q = 5$, $\nu_0 = 10$, $\nu_1 = \nu_2 = 6$, and $\vartheta \in [0, 2\pi]$, the domain $\mathcal{D}$ features a pentagon-like shape. Let $f(x, y) = x + 2iy^2 + \cos(2x^2 + iy)$ be the function describing the boundary data. Provided that the propagation constant is $k = 10$, the relative boundary error e_N as function of the number N of terms in the series expansion (5.3) exhibits the behavior shown in figure 1. As it appears from figure 2, the selection of the expansion order $N = 20$ leads to a very accurate Fourier representation of the solution, whose approximate spatial distribution is shown in figure 3.

Remark 4. We note that, where the boundary values have wide oscillations, it is necessary to increase the number of terms in the Fourier-like expansion approximating the solution of the problem in order to improve the numerical accuracy.

Remark 5. The L^2 norm of the difference between the exact solution and its approximate values is generally small. Point-wise convergence seems to be verified in the considered domains, with only exception of a set of measure zero, corresponding to quasi-cusped points. In these points oscillations of the approximate solution, recalling the classical Gibbs phenomenon, usually appear.

6. Conclusion

The use of stretched co-ordinate systems, reducing a starlike domain to a unit circle, allows the application of the classical Fourier method to a wide set of differential problems in complex two-dimensional normal-polar domains. In this way, closed-form solutions can be obtained by using suitable quadrature rules, so avoiding cumbersome numerical techniques such as finite-difference or finite-element methods.

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