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Two contiguous relations of Carlitz and Willett for balanced series

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Abstract[1]. The modified Abel lemma on summation by parts is employed to review two unusual contiguous relations discovered by Willett (1967) and Carlitz (1969), respectively, for classical ${}_4F_3$–series and basic ${}_4\phi_3$–series.

1. The modified Abel lemma on summation by parts

For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and Δ, respectively, by

$$\nabla\tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \Delta\tau_k = \tau_k - \tau_{k+1}$$

where Δ is adopted for convenience in the present paper, which differs from the usual operator Δ only in the minus sign.

Then *Abel's lemma* on summation by parts may be reformulated as

$$\boxed{\sum_{k=0}^{\infty} B_k \nabla A_k = [AB]_\infty - A_{-1}B_0 + \sum_{k=0}^{\infty} A_k \Delta B_k}$$

provided that the limit $[AB]_\infty := \lim_{m\to\infty} A_m B_{m+1}$ exists and one of the nonterminating series just displayed is convergent.

In fact, according to the definition of the backward difference, we have

$$\sum_{k=0}^{m} B_k \nabla A_k = \sum_{k=0}^{m} B_k\{A_k - A_{k-1}\} = \sum_{k=0}^{m} A_k B_k - \sum_{k=0}^{m} A_{k-1} B_k \ .$$

Replacing k by $1+k$ for the last sum, we derive the following expression:

$$\begin{aligned}\sum_{k=0}^{m} B_k \nabla A_k &= A_m B_{m+1} - A_{-1}B_0 + \sum_{k=0}^{m} A_k\{B_k - B_{k+1}\} = \\ &= A_m B_{m+1} - A_{-1}B_0 + \sum_{k=0}^{m} A_k \Delta B_k \ .\end{aligned}$$

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Letting $m \to \infty$, we confirm the modified Abel lemma on summation by parts.

Recently, several important summation formulae of classical and basic hypergeometric series have been reviewed by Chu [3, 4] through the modified Abel lemma on summation by parts. The same approach has further been employed by Chu and Wang [5] to derive numerous contiguous relations of classical ${}_3F_2$–series. There exist two very unusual relations on balanced series in literature. One is discovered by Willett [7, 1967] for the classical hypergeometric series. Another one is its q–analogue due to Carlitz [2, 1969]. However, their nonterminating forms derived by Carlitz [2] contain lacunae. The objective of the present paper is to establish two nonterminating contiguous relations via the modified Abel lemma on summation by parts, that correct the mistakes appeared in Carlitz' paper.

2. Willett's contiguous relation

For a complex x and a nonnegative integer n, define the shifted factorial by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}$$

with its multiparameter forms being abbreviated to

$$[\alpha, \beta, \cdots, \gamma]_n = (\alpha)_n\,(\beta)_n \cdots (\gamma)_n\,,$$

$$\begin{bmatrix} \alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C \end{bmatrix}_n = \frac{(\alpha)_n\,(\beta)_n \cdots (\gamma)_n}{(A)_n\,(B)_n \cdots (C)_n}\,.$$

From the study of Fuchsian system of two first–order linear differential equations with three regular singular points, Willett [7] (see Carlitz [2, equation 2] also) discovered the following terminating hypergeometric series identity.

Proposition 1 (Willett [7], equation 31)**.**

$${}_4F_3\left[\begin{matrix} a, b, 1+d-n, -n \\ 1+a-n, 1+b-n, d \end{matrix}\,\middle|\, 1\right] = \frac{n(n-1)(a-d)(b-d)}{(1+a-n)(1+b-n)d(d+1)} \times$$
$$\times\ {}_4F_3\left[\begin{matrix} 1+a, 1+b, 1+d-n, 2-n \\ 2+a-n, 2+b-n, 2+d \end{matrix}\,\middle|\, 1\right]\,.$$

Here and forth, according to Bailey [1, §2.1], the generalized hypergeometric series reads as

$${}_{1+r}F_s\left[\begin{matrix} a_0, a_1, \cdots, a_r \\ c_1, \cdots, c_s \end{matrix}\,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_r)_n}{n!(c_1)_n \cdots (c_s)_n} z^n\,.$$

Willett [7, equation 29], (see Carlitz [2, equation 1] also) noted also that the last contiguous relation corresponds to the following equivalent products of two Gaussian ${}_2F_1$–series:

$${}_2F_1\left[\begin{matrix} a, b \\ d \end{matrix}\,\middle|\, x\right]\ {}_2F_1\left[\begin{matrix} -a, -b \\ -d \end{matrix}\,\middle|\, x\right] = 1 + \frac{ab(a-d)(b-d)}{d^2(d+1)(d-1)}\, x^2 \times$$
$$\times\ {}_2F_1\left[\begin{matrix} 1+a, 1+b \\ 2+d \end{matrix}\,\middle|\, x\right]\ {}_2F_1\left[\begin{matrix} 1-a, 1-b \\ 2-d \end{matrix}\,\middle|\, x\right]\,.$$

Carlitz [2, equation 3], pointed out that there holds the following nonterminating form

$$ {}_4F_3\left[\begin{matrix} a,b,c,1+c+d \\ 1+a+c,1+b+c,d \end{matrix}\middle|\, 1\right] = \frac{(a-d)(b-d)c(1+c)}{(1+a+c)(1+b+c)d(1+d)}\times $$

$$ \times\, {}_4F_3\left[\begin{matrix} 1+a,1+b,2+c,1+c+d \\ 2+a+c,2+b+c,2+d \end{matrix}\middle|\, 1\right] . $$

Unfortunately, this result is false, which can be checked by letting $b=d$ and then applying the Gauss summation formula for ${}_2F_1$–series (cf. Bailey [1, §1.3])

$$ {}_2F_1\left[\begin{matrix} a,c \\ 1+a+c \end{matrix}\middle|\, 1\right] = \frac{\Gamma(1+a+c)}{\Gamma(1+a)\Gamma(1+c)} \neq 0 . $$

By means of Abel's lemma on summation by parts, we are going to establish the correct nonterminating version of Willett's identity.

Define the two sequences by

$$ A_k = \left[\begin{matrix} 1+a, & 1+c \\ 1, & 1+a+c \end{matrix}\right]_k \quad \text{and} \quad B_k = \left[\begin{matrix} b, & 1+c+d \\ d, & 1+b+c \end{matrix}\right]_k . $$

It is not hard to check the finite differences

$$ \nabla A_k = \left[\begin{matrix} a, & c \\ 1, & 1+a+c \end{matrix}\right]_k \quad \text{and} \quad \Delta B_k = \frac{(1+c)(d-b)}{d(1+b+c)} \left[\begin{matrix} b, & 1+c+d \\ 1+d, & 2+b+c \end{matrix}\right]_k $$

as well as the limiting relations

$$ A_{-1}B_0 = 0 \quad \text{and} \quad [AB]_\infty = \frac{\Gamma(1+a+c)\Gamma(1+b+c)\Gamma(d)}{\Gamma(1+a)\Gamma(b)\Gamma(1+c)\Gamma(1+c+d)} $$

where the last equality is justified by

$$ \begin{aligned} A_m B_{m+1} &= \frac{b(1+c+d)}{d(1+b+c)} \left[\begin{matrix} 1+a,1+b,1+c,2+c+d \\ 1,1+d,1+a+c,2+b+c \end{matrix}\right]_m = \\ &= \frac{b(1+c+d)}{d(1+b+c)} \frac{\Gamma(1+a+c)\Gamma(2+b+c)\Gamma(1+d)}{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)\Gamma(2+c+d)}\times \\ &\times \frac{\Gamma(1+a+m)\Gamma(1+b+m)\Gamma(1+c+m)\Gamma(2+c+d+m)}{\Gamma(1+m)\Gamma(1+d+m)\Gamma(1+a+c+m)\Gamma(2+b+c+m)} \end{aligned} $$

because the last line tends to one as $m \to \infty$ in view of the following asymptotic formula (cf. Rainville [6, Lemma 7 in §18])

$$ \Gamma(x+m) \approx (m-1)m^x . \tag{1} $$

According to the modified Abel lemma on summation by parts, we can manipulate the following balanced ${}_4F_3$–series:

$$
{}_4F_3\left[\begin{matrix}a,b,c,1+c+d\\1+a+c,1+b+c,d\end{matrix}\middle|1\right]=\sum_{k\geq 0}B_k\nabla A_k=[AB]_\infty-A_{-1}B_0+\sum_{k\geq 0}A_k\Delta B_k=
$$

$$
=\frac{\Gamma(1+a+c)\Gamma(1+b+c)\Gamma(d)}{\Gamma(1+a)\Gamma(b)\Gamma(1+c)\Gamma(1+c+d)}+
$$

$$
+\frac{(1+c)(d-b)}{d(1+b+c)}\sum_{k\geq 0}\left[\begin{matrix}1+a,b,1+c,1+c+d\\1,1+a+c,2+b+c,1+d\end{matrix}\right]_k.
$$

In terms of hypergeometric series, this can be expressed as the contiguous relation

$$
{}_4F_3\left[\begin{matrix}a,b,c,1+c+d\\1+a+c,1+b+c,d\end{matrix}\middle|1\right]=\frac{\Gamma(1+a+c)\Gamma(1+b+c)\Gamma(d)}{\Gamma(1+a)\Gamma(b)\Gamma(1+c)\Gamma(1+c+d)}+ \tag{2a}
$$

$$
+\frac{(1+c)(d-b)}{d(1+b+c)}\,{}_4F_3\left[\begin{matrix}1+a,b,1+c,1+c+d\\1+a+c,2+b+c,1+d\end{matrix}\middle|1\right]. \tag{2b}
$$

Alternatively, for the two sequences defined by

$$
C_k=\left[\begin{matrix}1+b, & 2+c\\1, & 2+b+c\end{matrix}\right]_k \quad\text{and}\quad D_k=\left[\begin{matrix}1+a, & 1+c+d\\1+d, & 1+a+c\end{matrix}\right]_k
$$

it is almost routine to calculate the finite differences

$$
\nabla C_k=\left[\begin{matrix}b, & 1+c\\1, & 2+b+c\end{matrix}\right]_k \quad\text{and}\quad \Delta D_k=\frac{c(d-a)}{(1+a+c)(1+d)}\left[\begin{matrix}1+a, & 1+c+d\\2+d, & 2+a+c\end{matrix}\right]_k
$$

as well as the limiting relations

$$
C_{-1}D_0=0 \quad\text{and}\quad [CD]_\infty=\frac{\Gamma(1+a+c)\Gamma(2+b+c)\Gamma(1+d)}{\Gamma(1+a)\Gamma(1+b)\Gamma(2+c)\Gamma(1+c+d)}
$$

where the last equality is justified by

$$
C_mD_{m+1}=\frac{(1+a)(1+c+d)}{(1+d)(1+a+c)}\left[\begin{matrix}2+a,1+b,2+c,2+c+d\\1,2+d,2+a+c,2+b+c\end{matrix}\right]_m=
$$

$$
=\frac{(1+a)(1+c+d)}{(1+d)(1+a+c)}\,\frac{\Gamma(2+a+c)\Gamma(2+b+c)\Gamma(2+d)}{\Gamma(2+a)\Gamma(1+b)\Gamma(2+c)\Gamma(2+c+d)}\times
$$

$$
\times\frac{\Gamma(2+a+m)\Gamma(1+b+m)\Gamma(2+c+m)\Gamma(2+c+d+m)}{\Gamma(1+m)\Gamma(2+d+m)\Gamma(2+a+c+m)\Gamma(2+b+c+m)}
$$

because the last line tends to one as $m\to\infty$ thanks again to the formula (1).

By means of the modified Abel lemma on summation by parts, we can similarly reformulate balanced series as follows:

$$
{}_4F_3\left[\begin{matrix}1+a,b,1+c,1+c+d\\1+a+c,2+b+c,1+d\end{matrix}\middle|1\right]=
$$

$$
=\sum_{k\geq 0}D_k\nabla C_k=[CD]_\infty-C_{-1}D_0+\sum_{k\geq 0}C_k\Delta D_k=
$$

$$= \frac{\Gamma(1+a+c)\Gamma(2+b+c)\Gamma(1+d)}{\Gamma(1+a)\Gamma(1+b)\Gamma(2+c)\Gamma(1+c+d)} +$$

$$+ \frac{c(d-a)}{(1+a+c)(1+d)} \sum_{k\geq 0} \left[\begin{matrix}1+a, 1+b, 1+c+d, 2+c\\ 1, 2+a+c, 2+b+c, 2+d\end{matrix}\right]_k .$$

This can be stated as another contiguous relation

$$\text{(3a)} \quad {}_4F_3\left[\begin{matrix}1+a, b, 1+c, 1+c+d\\ 1+a+c, 2+b+c, 1+d\end{matrix}\middle| 1\right] = \frac{\Gamma(1+a+c)\Gamma(2+b+c)\Gamma(1+d)}{\Gamma(1+a)\Gamma(1+b)\Gamma(2+c)\Gamma(1+c+d)} +$$

$$\text{(3b)} \quad + \frac{c(d-a)}{(1+a+c)(1+d)}\, {}_4F_3\left[\begin{matrix}1+a, 1+b, 2+c, 1+c+d\\ 2+a+c, 2+b+c, 2+d\end{matrix}\middle| 1\right] .$$

Substituting (3a)-(3b) into (2a)-(2b) and then simplifying the resulting equation, we get the following transformation formula.

Theorem 2 (Contiguous relation of ${}_4F_3$–series)**.**

$${}_4F_3\left[\begin{matrix}a, b, c, 1+c+d\\ 1+a+c, 1+b+c, d\end{matrix}\middle| 1\right] = \frac{\Gamma(1+a+c)\Gamma(1+b+c)\Gamma(1+d)}{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)\Gamma(1+c+d)} +$$

$$+ \frac{c(1+c)(a-d)(b-d)}{(1+a+c)(1+b+c)d(1+d)}\, {}_4F_3\left[\begin{matrix}1+a, 1+b, 2+c, 1+c+d\\ 2+a+c, 2+b+c, 2+d\end{matrix}\middle| 1\right] .$$

It is obvious that this theorem corrects the corresponding formula obtained by Carlitz [2, equation 3], where the term containing Γ–functions has been missing. When the series are terminated by $c = -n$ with $n \in \mathbb{N}$, this theorem reduces clearly to Willett's contiguous relation for terminating ${}_4F_3$–series displayed in Proposition 1.

3. Carlitz' contiguous relation

In comparison with the ordinary hypergeometric series, Carlitz [2, equation 8] derived a nonterminating q–analogue of Willett's identity, which is unfortunately false again.

We need to reproduce the notations of q–shifted factorial and basic hypergeometric series. For two indeterminate x and q, the shifted-factorial of x with base q is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \quad \text{for} \quad n \in \mathbb{N} .$$

When $|q| < 1$, we have two well-defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \quad \text{and} \quad (x;q)_n = (x;q)_\infty/(xq^n;q)_\infty .$$

The product and fraction of shifted factorials are abbreviated compactly to

$$[\alpha, \beta, \cdots, \gamma; q]_n = (\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n ,$$

$$\left[\begin{matrix}\alpha, \beta, \cdots, \gamma\\ A, B, \cdots, C\end{matrix}\middle| q\right]_n = \frac{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n(B;q)_n\cdots(C;q)_n} .$$

Following Bailey [1], the basic hypergeometric series is defined by

$$ {}_{1+r}\phi_s \left[\begin{matrix} a_0, a_1, \cdots, a_r \\ c_1, \cdots, c_s \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} \frac{(a_0;q)_n (a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (c_1;q)_n \cdots (c_s;q)_n} z^n $$

where the base q will be restricted to $|q| < 1$ for nonterminating q–series. Then the following q–analogue of Theorem 2 will be established.

Theorem 3 (Contiguous relation of ${}_4\phi_3$–series)**.**

$$ {}_4\phi_3 \left[\begin{matrix} a, b, c, qcd \\ qac, qbc, d \end{matrix} \middle| q; q \right] = \left[\begin{matrix} qa, qb, qc, qcd \\ q, qac, qbc, qd \end{matrix} \middle| q \right]_\infty + $$

$$ + \frac{q(1-c)(1-qc)(a-d)(b-d)}{(1-qac)(1-qbc)(1-d)(1-qd)} \, {}_4\phi_3 \left[\begin{matrix} qa, qb, q^2c, qcd \\ q^2ac, q^2bc, q^2d \end{matrix} \middle| q; q \right] . $$

We remark that the formula just displayed corrects the corresponding formula obtained by Carlitz [2, equation 8], where the quotient of infinite products has been missing. This theorem can analogously be proved through the modified Abel lemma on summation by parts.

Proof. For the two sequences given by

$$ A_k = \left[\begin{matrix} qa, & qc \\ q, & qac \end{matrix} \middle| q \right]_k \quad \text{and} \quad B_k = \left[\begin{matrix} b, & qcd \\ d, & qbc \end{matrix} \middle| q \right]_k $$

it is not difficult to verify the finite differences

$$ \nabla A_k = \left[\begin{matrix} a, c \\ q, qac \end{matrix} \middle| q \right]_k q^k \quad \text{and} \quad \Delta B_k = \frac{(b-d)(1-qc)}{(1-d)(1-qbc)} \left[\begin{matrix} b, qcd \\ qd, q^2bc \end{matrix} \middle| q \right]_k q^k $$

as well as the limiting relations

$$ A_{-1}B_0 = 0 \quad \text{and} \quad [AB]_\infty = \left[\begin{matrix} qa, b, qc, qcd \\ q, qac, qbc, d \end{matrix} \middle| q \right]_\infty . $$

In view of the modified Abel lemma on summation by parts, we can reformulate the following balanced ${}_4\phi_3$–series

$$ {}_4\phi_3 \left[\begin{matrix} a, b, c, qcd \\ qac, qbc, d \end{matrix} \middle| q; q \right] = \sum_{k\geq 0} B_k \nabla A_k = [AB]_\infty - A_{-1}B_0 + \sum_{k\geq 0} A_k \Delta B_k = $$

$$ = \left[\begin{matrix} qa, b, qc, qcd \\ q, qac, qbc, d \end{matrix} \middle| q \right]_\infty + \frac{(b-d)(1-qc)}{(1-d)(1-qbc)} \sum_{k\geq 0} \left[\begin{matrix} qa, b, qc, qcd \\ q, qac, q^2bc, qd \end{matrix} \middle| q \right]_k q^k $$

which leads us to the following contiguous relation

(4a) $$ {}_4\phi_3 \left[\begin{matrix} a, b, c, qcd \\ qac, qbc, d \end{matrix} \middle| q; q \right] = \left[\begin{matrix} qa, b, qc, qcd \\ q, qac, qbc, d \end{matrix} \middle| q \right]_\infty + $$

(4b) $$ + \frac{(b-d)(1-qc)}{(1-d)(1-qbc)} \, {}_4\phi_3 \left[\begin{matrix} qa, b, qc, qcd \\ qac, q^2bc, qd \end{matrix} \middle| q; q \right] . $$

Alternatively, with the two sequences defined by

$$C_k = \begin{bmatrix} qb, & q^2c \\ q, & q^2bc \end{bmatrix} q \Bigg]_k \quad \text{and} \quad D_k = \begin{bmatrix} qa, & qcd \\ qd, & qac \end{bmatrix} q \Bigg]_k$$

we can compute the finite differences

$$\nabla C_k = \left[\begin{matrix} b, qc \\ q, q^2bc \end{matrix}\middle| q\right]_k q^k \quad \text{and} \quad \Delta D_k = \frac{q(1-c)(a-d)}{(1-qac)(1-qd)} \left[\begin{matrix} qa, qcd \\ q^2d, q^2ac \end{matrix}\middle| q\right]_k q^k$$

as well as the limiting relations

$$C_{-1}D_0 = 0 \quad \text{and} \quad [CD]_\infty = \left[\begin{matrix} qa, qb, q^2c, qcd \\ q, qac, q^2bc, qd \end{matrix}\middle| q\right]_\infty .$$

Applying the modified Abel lemma on summation by parts, we can similarly manipulate the balanced ${}_4\phi_3$–series as follows:

$${}_4\phi_3 \left[\begin{matrix} qa, b, qc, qcd \\ qac, q^2bc, qd \end{matrix}\middle| q; q\right] = \sum_{k\geq 0} D_k \nabla C_k = [CD]_\infty - C_{-1}D_0 + \sum_{k\geq 0} C_k \Delta D_k =$$

$$= \left[\begin{matrix} qa, qb, q^2c, qcd \\ q, qac, q^2bc, qd \end{matrix}\middle| q\right]_\infty + \frac{q(1-c)(a-d)}{(1-qac)(1-qd)} \sum_{k\geq 0} \left[\begin{matrix} qa, qb, q^2c, qcd \\ q, q^2ac, q^2bc, q^2d \end{matrix}\middle| q\right]_k q^k .$$

This can be stated as another contiguous relation

(5a) $${}_4\phi_3 \left[\begin{matrix} qa, b, qc, qcd \\ qac, q^2bc, qd \end{matrix}\middle| q; q\right] = \left[\begin{matrix} qa, qb, q^2c, qcd \\ q, qac, q^2bc, qd \end{matrix}\middle| q\right]_\infty +$$

(5b) $$+ \frac{q(1-c)(a-d)}{(1-qac)(1-qd)} \, {}_4\phi_3 \left[\begin{matrix} qa, qb, q^2c, qcd \\ q^2ac, q^2bc, q^2d \end{matrix}\middle| q; q\right] .$$

Substituting (5a)-(5b) into (4a)-(4b) and then simplifying the resulting equation, we get the transformation formula displayed in Theorem 3.

□

When $c = q^{-n}$ with $n \in \mathbb{N}$, Theorem 3 reduces to the following contiguous relation for terminating ${}_4\phi_3$–series, which is attributed to Carlitz [2] even though his formula contains some slight misprints.

Proposition 4 (Carlitz [2], equation 11: $n \in \mathbb{N}$)**.**

$${}_4\phi_3 \left[\begin{matrix} a, b, q^{1-n}d, q^{-n} \\ q^{1-n}a, q^{1-n}b, d \end{matrix}\middle| q; q\right] = \frac{(a-d)(b-d)(1-q^n)(1-q^{n-1})}{(1-d)(1-qd)(a-q^{n-1})(b-q^{n-1})} \times$$

$$\times \, {}_4\phi_3 \left[\begin{matrix} qa, qb, q^{1-n}d, q^{2-n} \\ q^{2-n}a, q^{2-n}b, q^2d \end{matrix}\middle| q; q\right] .$$

As shown by Carlitz [2], this contiguous relation can alternatively be expressed as the following equivalent products of two ${}_2\phi_1$–series.

Corollary 5 (Carlitz [2], equation 15)**.**

$$
{}_2\phi_1\left[\begin{matrix}a,b\\ d\end{matrix}\,\middle|\, q;\frac{dx}{ab}\right]\;{}_2\phi_1\left[\begin{matrix}a^{-1},b^{-1}\\ d^{-1}\end{matrix}\,\middle|\, q;x\right]=
$$

$$
=1+\frac{(1-a^{-1})(1-b^{-1})(1-da^{-1})(1-db^{-1})}{(1-d)(1-qd)(1-d^{-1})(1-qd^{-1})}\;x^2\times
$$

$$
\times\;{}_2\phi_1\left[\begin{matrix}qa,qb\\ q^2d\end{matrix}\,\middle|\, q;\frac{dx}{ab}\right]\;{}_2\phi_1\left[\begin{matrix}qa^{-1},qb^{-1}\\ q^2d^{-1}\end{matrix}\,\middle|\, q;x\right]\;.
$$

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