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Approximation by polynomial solutions of PDEs of higher order and boundary value problems

Alberto Cialdea

Dedicated to Paolo Emilio Ricci, on the occasion of his 65th birthday

Abstract[1]. Let E be a partial differential operator of order $2m$ with real constant coefficients in any number of variables. Denote by $\{\omega_k\}$ a basis of polynomial solutions of the equation $Eu = 0$. We survey some recent results concerning the completeness of the system $\{(\omega_k, \partial_\nu \omega_k, \cdots, \partial_\nu^{m-1}\omega_k)\}$ on the boundary of a bounded domain in L^p and in C^0 norms. In particular necessary and sufficient conditions for the completeness to hold are given.

1. Introduction

Let E be an elliptic partial differential operator

$$Eu = \sum_{|\alpha|=0}^{m} a_\alpha(x)\, D^\alpha u \,, \tag{1}$$

where the coefficients a_α belong to $C^\infty(\mathbb{R}^n)$.

Let K be a compact set in $\mathbb{R}^n$ and Ω a domain containing K. Let f be a function which is continuous on K and such that $Ef = 0$ in the interior of K (if nonempty).

A classical problem in the approximation theory is to find conditions under which f can be approximated on K in a certain norm (for example the uniform norm) by a sequence of particular solutions of the equation $Eu = 0$ in Ω.

For example, if f is a holomorphic function of one complex variable, we may ask when f can be approximated in some norms by polynomials or by rational functions. The classical theorems of Weierstrass, Runge and Mergelyan are the main results in this area.

These kind of problems have been widely studied and extended to general elliptic partial differential equations.

Even if necessary and sufficient conditions are not known for general elliptic operators, several results are nowadays available for such operators (see, for example, [1, 16, 17]).

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[1]Author's address: A. Cialdea, Università degli Studi della Basilicata, Dipartimento di Matematica e Informatica, Via dell'Ateneo Lucano 10, I-85100 Potenza, Italy;
e-mail: `cialdea@email.it` .

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Theorem 1. *Let K be a compact set and A, B two open sets such that $K \subset A$, $\overline{A} \subset B \subset \mathbb{R}^n$. Suppose that*

(i) E and E^* satisfy the unique continuation property in B;
(ii) $K \setminus \partial K$ satisfies the restricted cone hypothesis;
(iii) $B \setminus K$ is connected.

Then $\overline{S} = \Omega(K)$, where S is one of the following spaces:

$$S = \left\{ p \ \mid \ p(x) = \int_H \varphi(y)\, s(x,y)\, dy,\ \varphi \in C^\infty(\overline{H}) \right\}$$

(H being a fixed compact set such that $\overline{H} \subset B \setminus \overline{A}$) ;

$$S = \left\{ p \ \mid \ \exists\, y_1, \cdots, y_m \in H : p(x) = \sum_{j=1}^{m} c_j s(x, y_j) \right\} .$$

If E has constant coefficients, Theorem 1 holds also with

$$S = \{ p \ \mid \ p \text{ is a polynomial}, \quad Ep = 0 \}$$

under some additional conditions (see [17]). For a short proof of Theorem 1 hinging on potential theory, we refer to [13].

The problem of completeness can be stated in a more sophisticated way, as proposed many years ago by Picone.

Let us consider a BVP for the operator E:

$$\begin{cases} Eu = 0 & \text{in } \Omega \\ B_h u = f_h & \text{on } \Sigma \quad (h = 1, \cdots, s) \end{cases} \tag{2}$$

which is supposed to be an index problem. Then there exists a solution of the problem (2) if and only if $(f_1, \cdots, f_s)$ satisfies a finite number of compatibility conditions:

$$\sum_{h=1}^{s} \int_\Sigma f_h \psi_h^{(k)}\, d\sigma = 0 \qquad k = 1, \cdots, r .$$

Let S be a family of particular solution of the equation $Eu = 0$ defined in a larger domain $A \supset \Omega$. The problem proposed by Picone is to prove that the system $\{(B_1\omega, \cdots, B_s\omega) \mid \omega \in S\}$ is complete in the subspace of $[L^p(\Sigma)]^s$ whose elements $(v_1, \cdots, v_s)$ are orthogonal to the vectors $(\psi_1^{(k)}, \cdots, \psi_s^{(k)})$

$$\sum_{h=1}^{s} \int_\Sigma v_h \psi_h^{(k)}\, d\sigma = 0 \qquad k = 1, \cdots, r .$$

It is worthwhile to remark that in the very particular case $n = 2$, $E = \Delta$, $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, $S = \{\mathscr{R}e\, z^k,\ \mathscr{I}m\, z^k \mid k = 0, 1, 2, \cdots\}$ $(z = x + iy)$, the completeness of the system $\{\omega|_\Sigma \mid \omega \in S\}$ in $C^0(\Sigma)$ (or in $L^p(\Sigma)$) is nothing but the completeness of the trigonometric system.

The first to prove some completeness theorems in the sense of Picone was Fichera, which proved the following results concerning the Dirichlet, the Neumann and the mixed problem for Laplace equation

Theorem 2 ([12])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary Σ belongs to C^2. Let $\{\omega_k\}$ denote the system of harmonic polynomials. Then*

(*i*) *the system* $\{\omega_k\}$ *is complete in* $L^2(\Sigma)$*;*
(*ii*) *the system* $\{\partial_\nu \omega_k\}$ *is complete in the space* $\{v \in L^2(\Sigma) \mid \int_\Sigma v\, d\sigma = 0\}$*;*
(*iii*) *the system* $\{\omega_k|_{\Sigma_1}, \partial_\nu \omega_k|_{\Sigma_2}\}$ *is complete in* $L^2(\Sigma_1) \times L^2(\Sigma_2)$*, where* Σ_1 *and* Σ_2 *are such that* $\Sigma_1 \cap \Sigma_2 = \emptyset$ *and* $\Sigma = \overline{\Sigma}_1 \cup \overline{\Sigma}_2$*,* Σ_k *having the same superficial measure as* $\overline{\Sigma}_k$ $(k = 1, 2)$.

We have to remark that one can repeat Fichera's proof with small changes and obtain the same completeness theorems (i) and (ii) in $L^p(\Sigma)$ $(1 \le p < \infty)$ or in $C^0(\Sigma)$, assuming that Σ is a Lyapunov boundary (i.e. $\Sigma \in C^{1,\lambda}$).

As far as (iii) is concerned, it is clear that completeness in $L^2(\Sigma_1) \times L^2(\Sigma_2)$ implies completeness in $L^p(\Sigma_1) \times L^p(\Sigma_2)$ for $1 \le p < 2$. However the completeness of the system $\{\omega_k|_{\Sigma_1}, \partial_\nu \omega_k|_{\Sigma_2}\}$ in $L^p(\Sigma_1) \times L^p(\Sigma_2)$ for $p > 2$ is still an open problem.

After Fichera's result, several other completeness theorems have been proved for particular BVPs. However while very general completeness theorems analogous to the Mergelyan one are known, this is not the case for the completeness theorems as proposed by Picone. In the literature there are several results which are connected to particular boundary value problems for the harmonic and the biharmonic equation, the Helmotz equation, the elasticity system, the heat equation, general 2nd order elliptic equations and higher order elliptic equations with constant coefficients in two variables. We refer to [13] for a list of references updated until 1979, to which we would like to add [2, 3, 4, 5, 6, 11, 18].

In the present paper we present some recent results ([8, 9, 10]) connected to the Dirichlet problem for general higher order elliptic equations. They are the only known completeness results for polynomial solutions of general partial differential operators of higher order in any number of variables.

Section 2 is devoted to some theorems in potential theory. They provide a key ingredient of the proof and seem to be interesting in themselves.

In Section 3 we describe the completeness results for an elliptic operator E with constant coefficients satisfying a Gårding inequality (see (19) below). In particular, we show that a certain algebraic condition on the characteristic polynomial of E is necessary and sufficient for the completeness theorems to hold.

It is interesting to remark that - generally speaking - the existence and uniqueness of solution of the Dirichlet problem (which is a consequence of (19)) does not imply the relevant completeness for polynomial solutions.

2. Some results in potential theory

Let Ω be a bounded domain in $\mathbb{R}^n$, whose boundary Σ is supposed to be C^1. By ν_x we denote the *inward* unit normal vector to Σ at the point $x \in \Sigma$.

Let $s(x-y)$ be the fundamental solution for Laplace equation

$$s(x-y) = \begin{cases} \dfrac{1}{2\pi} \log|x-y| & \text{if } n = 2 \\[2ex] \dfrac{-1}{(n-2)c_n} |x-y|^{2-n} & \text{if } n \ge 3 , \end{cases}$$

c_n being the hypersurface measure of the unit sphere in $\mathbb{R}^n$.

The following formulas are very well known

$$\lim_{\substack{x \to x_0 \\ x \in \nu^+_{x_0}}} \int_\Sigma \varphi(y)\, \partial_{\nu_y} s(x-y)\, d\sigma_y = -\frac{1}{2}\, \varphi(x_0) + \int_\Sigma \varphi(y)\, \partial_{\nu_y} s(x_0 - y)\, d\sigma_y , \tag{3}$$

$$\lim_{\substack{x\to x_0\\ x\in\nu^-_{x_0}}} \int_\Sigma \varphi(y)\,\partial_{\nu_y} s(x-y)\,d\sigma_y = \frac{1}{2}\,\varphi(x_0) + \int_\Sigma \varphi(y)\,\partial_{\nu_y} s(x_0-y)\,d\sigma_y\ , \tag{4}$$

$$\lim_{\substack{x\to x_0\\ x\in\nu^+_{x_0}}} \left(\int_\Sigma \varphi(y)\,\partial_{\nu_y} s(x-y)\,d\sigma_y - \int_\Sigma \varphi(y)\,\partial_{\nu_y} s(x'-y)\,d\sigma_y\right) = -\varphi(x_0) \tag{5}$$

where x_0 is a point on Σ, $x\in\nu^+_{x_0}$ ($x\in\nu^-_{x_0}$) means that x is taken on the inner (exterior) normal at x_0 and x' is its symmetric with respect to x_0. These formulas can be proved under different hypothesis for Σ and φ. For example, (3) and (4) hold if $\Sigma\in C^{1,\lambda}$, $\varphi\in L^1(\Sigma)$ and x_0 is a Lebesgue point for φ.

Under the same hypothesis formula (5) follows from (3) and (4). However (5) can be proved directly, supposing merely $\Sigma\in C^1$ (see the Appendix of [14]).

Let us consider now more general kernels. We recall that the function h is said to be essentially homogeneous of degree α, if $h(x)=h_1(x)\log|x|+h_2(x)$, where $h_2(\varrho x)=\varrho^\alpha h(x)$, $x\neq 0$, $\varrho>0$ and $h_1(x)$ is a homogeneous polynomial of degree α if α is a nonnegative integer, $h_1(x)\equiv 0$ otherwise.

The next result generalizes formulas (3) and (4)

Theorem 3 ([7]). *Let $h\in C^2(\mathbb{R}^n\setminus\{0\})$ be even ($h(-x)=h(x)$) and essentially homogeneous of degree $2-n$. Suppose that $\Sigma\in C^{1,\lambda}$, $\varphi\in L^1(\Sigma)$ and x_0 is a Lebesgue point for φ. Then*

$$\lim_{\substack{x\to x_0\\ x\in\nu^+_{x_0}}} \int_\Sigma \varphi(y)\,\partial_{x_k} h(x-y)\,d\sigma_y =$$

$$= \nu_k(x_0)\,\gamma(x_0)\,\varphi(x_0) + \int_\Sigma \varphi(y)\,\partial_{x_k} h(x_0-y)\,d\sigma_y\ , \tag{6}$$

$$\lim_{\substack{x\to x_0\\ x\in\nu^-_{x_0}}} \int_\Sigma \varphi(y)\,\partial_{x_k} h(x-y)\,d\sigma_y =$$

$$= -\nu_k(x_0)\,\gamma(x_0)\,\varphi(x_0) + \int_\Sigma \varphi(y)\,\partial_{x_k} h(x_0-y)\,d\sigma_y\ , \tag{7}$$

where $\gamma(x_0)$ is given by

$$\gamma(x_0) = \begin{cases} \pi h_1 - \dfrac{1}{2}\displaystyle\int_{|\xi|=1} \Delta h_2(\xi)\,\log|\xi\cdot\nu_{x_0}|\,d\sigma_\xi & \text{if } n=2\\[2ex] \dfrac{1}{2}\displaystyle\int_{|\xi|=1} [(2-n)h(\xi) - \Delta h(\xi)\,\log|\xi\cdot\nu_{x_0}|]\,d\sigma_\xi & \text{if } n\geq 3\ . \end{cases} \tag{8}$$

Note that the integrals in the right hand side of (6) and (7) are singular integrals, due to the strong singularity of the kernels.

As remarked in [11], the function γ can be expressed by means of the Fourier transform of the kernel h:

$$\gamma(x_0) = \frac{1}{2}\,\mathscr{F}(\Delta h)(\nu_{x_0}) = -2\pi^2\mathscr{F}(h)(\nu_{x_0}) \tag{9}$$

where the Laplacian Δ has to be understood in the sense of distributions and $\mathscr{F}$ denotes the Fourier transform

$$\mathscr{F}(h)(x) = \int_{\mathbb{R}^n} h(y)\,e^{-2\pi i x\cdot y}dy\ .$$

We have also

Theorem 4 ([7]). *Let $h \in C^2(\mathbb{R}^n \setminus \{0\})$ be even $(h(-x) = h(x))$ and essentially homogeneous of degree $2-n$. Suppose that $\Sigma \in C^1$, $\varphi \in L^1(\Sigma)$ and x_0 is a Lebesgue point for φ. Then*

$$\lim_{\substack{x \to x_0 \\ x \in \nu^+_{x_0}}} \left(\int_\Sigma \varphi(y)\, \partial_{x_k} h(x-y)\, d\sigma_y - \int_\Sigma \varphi(y)\, \partial_{x_k} h(x'-y)\, d\sigma_y \right) =$$

$$= 2\nu_k(x_0)\, \gamma(x_0)\, \varphi(x_0)\ , \tag{10}$$

where $\gamma(x_0)$ is given by (8) (*or* (9)).

If we try to prove a completeness theorem in the uniform norm, (10) is not sufficient anymore, and it is necessary to have a jump formula for a potential generated by a measure $\mu \in M(S)$, where $M(\Sigma)$ is the dual space of $C^0(\Sigma)$.

Theorem 5 ([10]). *Let $\Sigma \in C^1$. Let $h \in C^2(\mathbb{R}^n \setminus \{0\})$ be even and essentially homogeneous of degree $2-n$. If $\mu \in M(\Sigma)$, for any $p \in C^\lambda(\mathbb{R}^n)$ we have*

$$\lim_{\varrho \to 0^+} \left(\int_{\Sigma_\varrho} p(x_\varrho)\, d\sigma_{x_\varrho} \int_\Sigma \partial_{x_k} h(x_\varrho - y)\, d\mu_y - \right.$$

$$\left. - \int_{\Sigma_{-\varrho}} p(x_{-\varrho})\, d\sigma_{x_{-\varrho}} \int_\Sigma \partial_{x_k} h(x_{-\varrho} - y)\, d\mu_y \right) = 2 \int_\Sigma \nu_k \gamma p\, d\mu \tag{11}$$

where γ is given by (8) (*or* (9)).

Here $\{\Sigma_\varrho\}$ is a family of "parallel surfaces" contained in Ω if $\varrho > 0$ (in $\mathbb{R}^n \setminus \overline{\Omega}$ if $\varrho < 0$) and tending to Σ (as $\varrho \to 0$) in a certain sense. For all the details we refer to [10].

Let us consider now an elliptic operator E with real constant coefficients and no lower order term, i.e.

$$Eu = \sum_{|\alpha|=2m} a_\alpha D^\alpha u \tag{12}$$

with $a_\alpha \in \mathbb{R}$, such that

$$Q(\xi) = \sum_{|\alpha|=2m} a_\alpha\, \xi^\alpha > 0 \quad , \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}\ . \tag{13}$$

As shown by Fritz John [15], the functions

$$S_0(x-y) = \frac{1}{4(2\pi i)^{n-1}(2m-1)!}\, (\Delta_y)^{(n-1)/2} \int_{|\xi|=1} \frac{|(x-y)\cdot\xi|^{2m-1}}{Q(\xi)}\, d\sigma_\xi$$

for n odd, and

$$S_0(x-y) = \frac{-1}{(2\pi i)^n (2m)!}\, (\Delta_y)^{n/2} \int_{|\xi|=1} \frac{|(x-y)\cdot\xi|^{2m} \log|(x-y)\cdot\xi|}{Q(\xi)}\, d\sigma_\xi$$

for n even, provide a fundamental solution for (12).

Theorems 4 and 5 lead to the following results

Theorem 6 ([9])**.** *Let $\Sigma \in C^1$. Let $\varphi \in L^1(\Sigma)$ and $x_0 \in \Sigma$ be a Lebesgue point for φ. For any multi-index α with $|\alpha| = 2m-1$, we have*

$$\lim_{\substack{x\to x_0 \\ x\in\nu^+_{x_0}}} \left(\int_\Sigma \varphi(y)\, D^\alpha_y[S_0(x-y)]\, d\sigma_y - \int_\Sigma \varphi(y)\, D^\alpha_y[S_0(x'-y)]\, d\sigma_y \right) = = -\,\frac{\nu^\alpha(x_0)}{Q(\nu(x_0))}\,\varphi(x_0)\ . \tag{14}$$

Theorem 7 ([10])**.** *Let $\Sigma \in C^1$. Let α be a multi-index with $|\alpha| = 2m-1$. For any $p \in C^\lambda(\mathbb{R}^n)$ we have*

$$\lim_{\varrho\to 0^+} \left(\int_{\Sigma_\varrho} p(x_\varrho)\, d\sigma_{x_\varrho} \int_\Sigma D^\alpha_y S_0(x_\varrho - y)\, d\mu_y - \right.$$

$$\left. - \int_{\Sigma_{-\varrho}} p(x_{-\varrho})\, d\sigma_{x_{-\varrho}} \int_\Sigma D^\alpha_y S_0(x_{-\varrho} - y)\, d\mu_y \right) = -\int_\Sigma \frac{\nu^\alpha}{Q(\nu)}\, p\, d\mu\ . \tag{15}$$

Jump formulas (14) and (15) are useful also when we consider the elliptic operator (1) with real constant coefficients and lower order terms.

3. Completeness theorems for higher order operators in L^p and in C^0

Let us consider a higher order operator

$$Eu = \sum_{|\alpha|\le 2m} a_\alpha D^\alpha u \quad , \quad a_\alpha \in \mathbb{R}\ . \tag{16}$$

which is supposed to be elliptic (see (13)).

Moreover we suppose that

$$a_{(0,\cdots,0)} = 0\ . \tag{17}$$

The last condition is necessary and sufficient for the existence of polynomial solutions of the equation $Eu = 0$. In fact, if $a_{(0,\cdots,0)} \neq 0$ and p_k is a polynomial of degree k, $Ep_k = a_{(0,\cdots,0)}p_k + q_{k-1}$, q_{k-1} being a polynomial of degree less than or equal to $k-1$. This shows that there are no polynomial solutions of the equation $Ep = 0$.

Conversely, suppose that (17) is satisfied and let p_k be a polynomial of degree k. Denote by $c_{n,k}$ the number of the coefficients of a generic polynomial of degree k in n variables. Since Ep_k is a polynomial of degree at most $k-1$, the equation $Ep_k = 0$ is equivalent to a linear homogeneous system of $c_{n,k-1}$ equations with $c_{n,k}$ unknowns. Such a system admits eigensolutions, because $c_{n,k-1} < c_{n,k}$, and this proves that there are polynomial solutions of the equation $Ep = 0$.

Let us consider now the Dirichlet problem

$$\begin{cases} Eu = 0 & \text{in } \Omega \\ \partial^h_\nu u = f_h & \text{on } \Sigma \quad (h = 0,\cdots,m-1)\ . \end{cases}$$

Our aim is to give the relevant completeness theorems for polynomial solutions. More precisely, if $\{\omega_k\}$ is a basis of polynomial solutions of the equation $Eu = 0$[2],

[2]Saying that $\{\omega_k\}$ is a basis of polynomial solutions of the equation $Eu = 0$, we mean that $E\omega_k = 0$ and that any polynomial solution ω of the equation $E\omega = 0$ can be written as a finite linear combination of ω_k.

we want to prove that the system $\{(\omega_k, \partial_\nu \omega_k, \cdots, \partial_\nu^{m-1}\omega_k)\}$ is complete. The completeness will be considered in different spaces.

It is well known how to construct a basis of polynomial solutions when $E = \Delta^m$: if $\{Y_{hs}\}$ $(s = 1, \cdots, p_{nh}, h = 0, 1, \cdots)$ is a complete system of spherical harmonics, where $p_{nh} = (2h + n - 2)(h + n - 3)!/((n - 2)!h!)$, the system

$$|x|^{h+2j} Y_{hs}\left(\frac{x}{|x|}\right) \qquad (j = 0, \cdots, m-1,\ s = 1, \cdots, p_{nh},\ h = 0, 1, \cdots)$$

provides a basis of polyharmonic polynomials.

In [3] an explicit method is given for elliptic equations of higher order without lower order terms in two independent variables. For more general operators we refer to [19, 20, 21].

Let us consider now two different cases. At first, we consider elliptic operators of order $2m$ with no lower order terms, i.e. operators of the form (12), such that (13) holds.

The following result was proved in [8] (the particular case $E = \Delta^m$ was considered in [11])

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected and $\Sigma \in C^1$. Denote by $\{\omega_k\}$ a basis of polynomial solutions of the equation $Eu = 0$. The system $\{(\omega_k, \partial_\nu \omega_k, \cdots, \partial_\nu^{m-1}\omega_k)\}$ is complete in $[L^p(\Sigma)]^m$ $(1 \le p < \infty)$.*

When the operator E contains lower order terms, the situation is more delicate. We start recalling a condition introduced by Malgrange in [17]: we say that *a polynomial Q satisfies the M-condition* if all its irreducible factors over $\mathbb{C}$ vanish at the origin.

Let E be the operator (16) which now we prefer to write as

$$Eu = \sum_{|p|,|q|}^{0,m} (-1)^{|p|} a_{pq} D^p D^q u \qquad (a_{00} = 0) \tag{18}$$

and suppose that it satisfies Gårding inequality

$$\sum_{|p|,|q|}^{0,m} a_{pq} \int_\Omega D^q u\, D^p u\, dx \ge C\, \|u\|^2_{W^{m,2}(\Omega)} \qquad \forall\ u \in \overset{\circ}{C}{}^\infty(\Omega)\ . \tag{19}$$

We remark that Gårding inequality (19) implies the ellipticity of the operator E.

Generally speaking, Theorem 8 does not hold for such more general operators. However the next theorem provides a necessary and sufficient condition for the completeness result to hold.

Theorem 9 ([9])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected and $\Sigma \in C^1$. Let E be the operator* (18) *and suppose that the Gårding inequality* (19) *holds. Let us denote by Q the characteristic polynomial*

$$Q(\xi) = \sum_{|p|,|q|}^{0,m} (-1)^{|p|} a_{pq} \xi^{p+q}\ . \tag{20}$$

Let $\{\omega_k\}$ be a basis of polynomial solutions of the equation $Eu = 0$. The system $\{(\omega_k, \partial_\nu \omega_k, \cdots, \partial_\nu^{m-1}\omega_k)\}$ is complete in $[L^p(\Sigma)]^m$ if and only if the polynomial (20) *satisfies the M-condition.*

It is worthwhile to remark that if E does not contain lower order terms and it is elliptic, the Gårding inequality (19) and the M-condition are both satisfied. Therefore Theorem 9 implies Theorem 8.

Recently Theorem 9 (and then also Theorem 8) was extended to the case of uniform norms, where 11 played a key role.

Theorem 10 ([10]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected and $\Sigma \in C^1$. Let E be the operator* (18) *and suppose that the Gårding inequality* (19) *holds. Let $\{\omega_k\}$ be a basis of polynomial solutions of the equation $Eu = 0$. The system $\{(\omega_k, \partial_\nu \omega_k, \cdots, \partial_\nu^{m-1}\omega_k)\}$ is complete in $[C^0(\Sigma)]^m$ if and only if the polynomial* (20) *satisfies the M-condition.*

We remark that the Gårding inequality (19) and the M-condition are independent of each other, as some examples can show (see [9, p. 95]).

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