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The Euler legacy to modern physics

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Abstract¹. Particular families of special functions, conceived as purely mathematical devices between the end of XVIII and the beginning of XIX centuries, have played a crucial role in the development of many aspects of modern physics. This is indeed the case of the Euler Gamma function, which has been one of the key elements paving the way to string theories, furthermore the Euler-Riemann Zeta function has played a decisive role in the development of renormalization theories. The ideas of Euler and later those of Riemann, Ramanujan and of other, less popular, mathematicians have therefore provided the mathematical apparatus ideally suited to explore, and eventually solve, problems of fundamental importance in modern physics. The mathematical foundations of the theory of renormalization trace back to the work on divergent series by Euler and by mathematicians of two centuries ago. Feynman, Dyson, Schwinger ... rediscovered most of these mathematical “curiosities” and were able to develop a new and powerful way of looking at physical phenomena.

1. INTRODUCTION

Leonard Euler is universally recognized as one of the greatest mathematicians of all times. Albeit somebody believes him the greatest, the choice is however limited to him and Gauss and, just to make a comparison with great musicians, we can use the following equivalence:

$$\text{Euler} : \text{Mozart} = \text{Gauss} : \text{Beethoven} .$$

The work and life of Euler can be characterized by a simple statement, which has the flavour of an oximoron: Genius and Regularity.

More than other comments his “natural” greatness was recognized by Condorcet, who, commemorating him, stated “... il cessa de vivre et de calculer”. The calculus was as natural for him as for the other human beings breathing or for eagles flying. The Euler’s contribution to any branches of science have been seminal and any problem he touched has opened new fields of research in pure mathematics or

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physics. A good example is provided by the solution he gave to the popular problem of the seven bridges of Königsberg. The method proposed by him opened an entire new field of mathematics, known as Graph theory. His great attitude was perhaps that of giving a meaning to notions which, at first sight, could be defined absurdities. He correctly handled the theory of imaginary numbers, introduced the concepts of fractional derivative and defined the Gamma function, which is the extension to non integers of the operation of factorial.

In this paper we will describe the relevance of Euler heritage to modern science and in particular to physics. Our starting point will be Gamma function which, as a already stressed, was initially introduced to extend the operation of factorial to any real or complex number.

The definition proposed by Euler in 1729, in a letter to his friend Christian Goldbach, is reported below [2]

$$(1a) \quad \Gamma(x) = \int_0^1 [-\ln(\sigma)]^{x-1} d\sigma \quad , \quad (\Re(x) > 0) \quad ,$$

After setting, $\sigma = e^{-t}$, equation (1a) can be rewritten in the more familiar way

$$(1b) \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad , \quad (\Re(x) > 0) \quad .$$

An alternative form, yielding the Gamma in terms of the Gaussian function, is

$$(2) \quad \Gamma(x) = 2 \int_0^\infty e^{-t^2} t^{2x-1} dt \quad , \quad (\Re(x) > 0) \quad .$$

The extension to non positive values of the real part of the variable x , was proposed by Euler himself (1729, letter to Goldbach) and later by Gauss in 1811 [2], [13]. The definition, valid without any restriction on the real or imaginary part of the variable x , is

$$(3) \quad \Gamma(x) = \lim_{p \rightarrow \infty} \Gamma_p(x) \quad , \quad \Gamma_p(x) := \frac{p^x}{\prod_{n=0}^p (x + 1/n)} \quad .$$

It ensures that the function $\Gamma(x)$ has an infinite number of poles, located at $x = 0, -1, -2, \dots$ (see Fig. 1 for the relevant plot, drawn on the real axis and in the complex plane). An important step further in the theory of Gamma function was

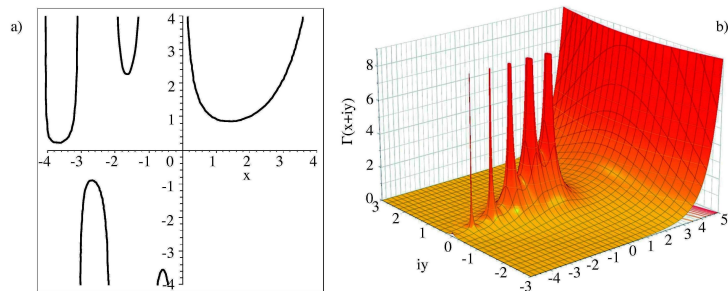


FIGURE 1. Plot of the Gamma function and of its modulus for a complex argument

due to Weierstrass [1], who proposed an infinite product formula, which determined significant successive developments. By setting indeed

$$(4) \quad p^x = e^{x \ln(p)} = \exp \left[x \left(\ln(p) - \sum_{n=1}^p \frac{1}{n} \right) \right] \exp \left(x \sum_{n=1}^p \frac{1}{n} \right)$$

in equation (3), we find

$$(5) \quad \frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n} \right) e^{-x/n} ,$$

where γ is the Euler-Mascheroni constant, given by

$$(6) \quad \gamma := \lim_{n \rightarrow \infty} \left[\sum_{s=1}^n \frac{1}{s} - \ln(n) \right] .$$

From the equation (6) it also follows that

$$(7a) \quad \frac{1}{\Gamma(x)\Gamma(-x)} = -x^2 \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} \right) ,$$

which, on account of the identities

$$(7b) \quad \sin(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} \right) , \quad \Gamma(-x) = -\frac{\Gamma(1-x)}{x} ,$$

yields the celebrated inversion formula

$$(8) \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} .$$

The Gamma function has other important properties which are reported below for the sake of completeness [1]

Expansion around a singularity

$$(9) \quad \begin{aligned} \Gamma(-n + \varepsilon) &= \frac{(-1)^n}{n!} \left[\frac{1}{\varepsilon} + \psi(n+1) + \right. \\ &\left. + \frac{\varepsilon}{2} \left(\frac{\pi^2}{3} + \psi^2(n+1) - \psi'(n+1) \right) \right] + o(\varepsilon^2) , \\ \psi(z) &:= \frac{\Gamma'(z)}{\Gamma(z)} . \end{aligned}$$

The duplication theorem (Legendre (1809))

$$(10) \quad \Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2x-1}} \Gamma(2x) .$$

Stirling formula

$$(11) \quad \lim_{x \rightarrow \infty} \Gamma(x) = \sqrt{2\pi x} x^x e^{-x} \left[1 + o\left(\frac{1}{x}\right) \right] .$$

Gauss multiplication theorem

$$(12) \quad \Gamma(nx) = \frac{n^{nx}}{\sqrt{(2\pi)^{n-1}n}} \prod_{r=0}^{n-1} \Gamma\left(x + \frac{r}{n}\right) .$$

Which for $x = 1/n$ yields the Euler product

$$(13) \quad \prod_{r=0}^{n-1} \Gamma\left(\frac{r}{n}\right) = \frac{\sqrt{(2\pi)^{n-1}}}{n}.$$

Let us now consider the following integral

$$(14) \quad I(x, y) = \int_0^{\infty} e^{-(x-1)\xi} (1 - e^{-\xi})^{y-1} d\xi,$$

representing a two variable generalization of the Gamma function. After the transformation of variable $e^{-\xi} = t$, it can be cast in the form

$$(15) \quad \int_0^1 t^{x-1} (1-t)^{y-1} dt = B(x, y).$$

The previous integral provides the more familiar form of the Euler B function, which can be considered the extension to the non integers of the binomial function. A straightforward manipulation shows indeed that

$$(16) \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{1}{x} \binom{x+y-1}{y-1}^{-1}.$$

The B function is evidently symmetric in the x and y variables and satisfies the further property

$$(17) \quad B(x, y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{x+n} \frac{\Gamma(y)}{\Gamma(y-n)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{1}{y+m} \frac{\Gamma(x)}{\Gamma(x-m)}.$$

In the hand of Euler the functions, named after him, became a powerful tool to make tremendous progress in any branches of pure and applied mathematics. Even though we will deeply in these revolutions in the forthcoming parts of the paper, we point out two important examples.

i) The formalism of derivatives of fractional order [17]. Euler noted indeed that ordinary derivatives satisfy the relationship

$$(18) \quad \left(\frac{d}{dx}\right)^m x^n = \frac{n!}{(n-m)!} x^{n-m}, \quad (m > n),$$

which suggests the following generalization to the non integer m

$$(19) \quad \left(\frac{d}{dx}\right)^\mu x^n = \frac{n!}{\Gamma(n-\mu+1)} x^{n-\mu}.$$

Accordingly the derivative of order $1/2$ of x writes

$$(20) \quad \left(\frac{d}{dx}\right)^{1/2} x = 2 \sqrt{\frac{x}{\pi}}.$$

ii) Derivation in closed form of different type of integrals [6]. The methods is simple and elegant and in these introductory remarks we will just refer to integrals we will use in the forthcoming parts of the paper. Integrals of the type

$$(21) \quad I = \int_{-\infty}^{\infty} \frac{1}{(1+ax^2)^\nu} dx, \quad \left(\Re(\nu) > \frac{1}{2}\right),$$

are usually treated using complex variable integration methods. It is however quite easy to obtain its explicit form in terms of Gamma functions, if one notes that the use of the Laplace transform identity

$$(22) \quad \frac{1}{a^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-sa} s^{\nu-1} ds ,$$

allows the reduction of (21) to the evaluation of a simple Gaussian integral. We find indeed

$$(23) \quad \begin{aligned} I &= \frac{1}{\Gamma(\nu)} \int_0^\infty ds e^{-s} s^{\nu-1} \left(\int_{-\infty}^\infty e^{-sax^2} dx \right) = \\ &= \frac{1}{\Gamma(\nu)} \sqrt{\frac{\pi}{a}} \int_0^\infty e^{-s} s^{\nu-3/2} ds = \sqrt{\frac{\pi}{a}} \frac{\Gamma(\nu-1/2)}{\Gamma(\nu)} . \end{aligned}$$

The same procedure can be exploited to get

$$(24) \quad \begin{aligned} I &= \int_{-\infty}^\infty \frac{1}{(1+ax^2+bx)^\nu} dx = \\ &= \sqrt{\frac{\pi}{a}} \frac{\Gamma(\nu-1/2)}{\Gamma(\nu)} \frac{1}{(1+b^2/4a)^\nu} , \quad \left(\Re(\nu) > \frac{1}{2} \right) . \end{aligned}$$

A generalization of the Laplace transform method is offered by the following identity due to Feynman² [3]

$$(25) \quad \frac{1}{A^\alpha B^\beta} = \frac{1}{B(\alpha, \beta)} \int_0^1 \frac{\xi^{\alpha-1} (1-\xi)^{\beta-1}}{[\xi A + (1-\xi)B]^{\alpha+\beta}} d\xi .$$

The use of equation (25) allows the derivation of even more complicated integrals in a fairly simple way

$$(26) \quad \begin{aligned} &\int_{-\infty}^\infty \frac{dx}{(1+\alpha_1 x^2 + \beta_1 x)^\nu (1+\alpha_2 x^2 + \beta_2 x)^\mu} = \\ &= \frac{1}{B(\alpha, \beta)} \int_0^1 d\xi \xi^{\alpha-1} (1-\xi)^{\beta-1} \int_0^\infty \frac{dx}{(1+Fx^2+Gx)^{\alpha+\beta}} = \\ &= \frac{\sqrt{\pi}}{B(\alpha, \beta)} \frac{\Gamma(\alpha+\beta-1/2)}{\Gamma(\alpha+\beta)} \int_0^1 d\xi \xi^{\alpha-1} (1-\xi)^{\beta-1} \frac{1}{\sqrt{F}} \frac{1}{(1+G^2/4F)^{\alpha+\beta}} , \\ &F := \xi\alpha_1 + (1-\xi)\alpha_2 \quad , \quad G := \xi\beta_1 + (1-\xi)\beta_2 . \end{aligned}$$

The examples given in these remarks yield an idea of how powerful and flexible are the methods associated with the properties of the Euler functions, in the following section we will discuss how the Beta function has played a key role in the evolution of the so called string theories.

²The proof is easily achieved by noting that

$$\frac{1}{AB} = \frac{1}{A-B} \int_B^A \frac{1}{z^2} dz = \int_0^1 \frac{dx}{(A-B)^2 [x-B/(A-B)]^2} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2} .$$

Equation (25) is obtained by keeping repeated derivatives with respect to A and B and then assuming that the derived expression holds for any real or complex value of the order of the derivative.

2. EULER B-FUNCTION AND STRING THEORY

In Fig. 2 we have reported the so called Chew and Frautschi (C-F) plot [18], proposed during the sixties of the last century, which showed the surprising feature according to which spins of elementary particles are proportional to the square of their masses. An analogous plot holds for baryon resonances and they (meson

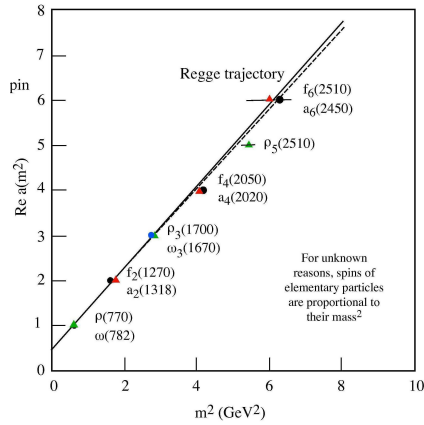


FIGURE 2. The Chew Frautschi plot yielding the spin of mesons vs. the square of the masses

and baryons) can be adjusted on linear trajectories with a universal slope, whose value can be fixed around 1 GeV^{-2} . These trajectories are usually referred as Regge trajectories. An almost natural question arising from the previous behavior is relevant to the nature of the potential which binds the particles forming the resonances. The physical argument suggesting the form of the potential is quite straightforward. Limiting ourselves to the case of mesons, we assume that they are composed by two quarks, linked by a string (see Fig. 3). The strings nowadays called gluon flux tubes, are characterized by a linear energy density k , so that we can conclude that

a) the total energy of a rotating string is

$$(27) \quad E = mc^2 = 2 \int_0^R \frac{kdr}{\sqrt{1 - (\nu/c)^2}} \quad , \quad \frac{\nu(r)}{c} = \frac{r}{R} \quad \Rightarrow$$

$$\Rightarrow 2 \int_0^R \frac{kdr}{\sqrt{1 - (\nu/c)^2}} = \pi kR .$$

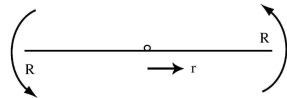


FIGURE 3. Sketch of meson resonance composed by a rotating string

b) The total angular momentum is

$$(28) \quad J = 2 \int_0^R \frac{kr\nu dr}{\sqrt{1 - (\nu/c)^2}} = \frac{1}{2} \pi k R^2 .$$

Comparing a) and b) we end up with

$$(29) \quad J = \alpha' m^2 \quad , \quad \alpha' = \frac{c^4}{2\pi k} .$$

The string-like potential is therefore adequate to account for the phenomenology underlying the $C - F$ plot, as a further remark we note that the force binding the quarks is independent on the radius and is just given by

$$(30) \quad |F_S| \cong \partial_R E = \frac{c^4}{2\alpha'} .$$

In Fig. 4 we have shown an elastic scattering process, in which two incoming particles (1, 2) form two outgoing particles (3, 4).

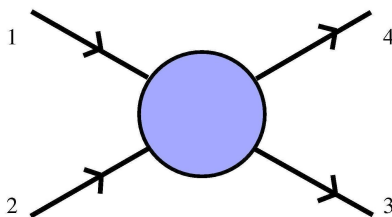


FIGURE 4. Elastic scattering of two ingoing mesons yielding two outgoing mesons and no other particle is produced

The kinematic variables, used to describe the process, are defined below, in terms of the particles four momenta

$$(31) \quad s = - \left(p_1^{(\mu)} + p_2^{(\mu)} \right)^2 \quad , \quad t = - \left(p_1^{(\mu)} - p_4^{(\mu)} \right)^2 \quad , \quad u = - \left(p_1^{(\mu)} - p_3^{(\mu)} \right)^2 .$$

They are the Mandelstam variables and are not all independent since

$$(32) \quad s + t + u = \sum_{i=1}^4 m_i^2 .$$

The variable s is essentially the center of mass energy, the physical meaning of the other variables is shown in Fig. 5 where we have reported a scattering processes in the three different “channels” s, t, u .

The scattering amplitudes can therefore be expressed in terms of these variables which link different processes. Even though we cannot describe more precisely these concepts, we can say that from the above description a kind of duality emerges between interactions described in the different channels and that the relevant amplitudes should reflect such idea. The scattering amplitude depends on the trajectories α , which can be defined in the various channels and the resonances appear as poles, the further constraint is that the poles can occur in e.g. the s or t channel *but not simultaneously*. Putting together what we have discussed so far and what we have

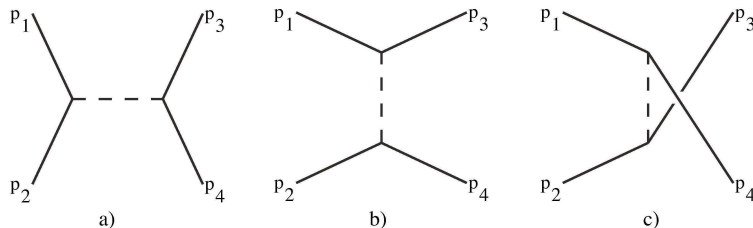


FIGURE 5. $s - u - t$ channel scattering processes: in the s -channel the particles 1, 2 couple to an unstable particle (a resonance) which eventually decays into 3, 4, in the t -channel 1 emits a particle and transform in 3, 2 absorbs the particle emitted by 1 and transforms in 4, the u -channel is the same as t with the role of 3, 4 interchanged

learned in the previous section, we can conclude that a possibility of constructing such an amplitude is the use of the following definition

$$(33) \quad A(s, t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}.$$

It might be argued that the product of the Γ functions at the denominator could be sufficient to account for the occurrence of the poles, it is therefore worth stressing that the denominator has been implanted just to avoid the occurrence of simultaneous poles, when both $\alpha(t), \alpha(s)$ are non negative integers.

The conclusion is therefore that the Euler B function appears an ideally suited candidate to reproduce the scattering amplitude in terms of the Regge trajectories. Equation (33) was first suggested by Veneziano³ [18], and this is one of the few examples of how an intuition relying on pure mathematical ground had such important impact on a physical theory. The use of the Stirling formula, discussed in the introductory section, yields e.g.⁴

$$(34) \quad \lim_{\substack{s \rightarrow \infty \\ \arg(s) > 0}} A(s, t) \cong \Gamma(-\alpha(t)) \frac{e^{\alpha(s)}[-\alpha(s)]^{-\alpha(s)-1/2}}{e^{\alpha(s)+\alpha(t)}[-\alpha(s)]^{-\alpha(s)-\alpha(t)-1/2}} \propto \\ \propto \Gamma(-\alpha(t)) e^{-\alpha(t)} [-s\alpha']^{-\alpha(t)}.$$

describing the amplitude behavior in the dual t -channel. It is remarkable that the Veneziano amplitude was derived as the result of an “educated” guess, based on general physical requirements fulfilled by a well known special function. The guess was later proved correct by Nambu and co-workers [18], who derived it from first principles, based on string potentials. It is even more amazing that a mathematical tool conceived in the XVII century was the opening key for the string theory, which, in the words of Daniele Amati, provides

“... *A theory of 21-th century fallen by chance in the 20-th century*”.

³In general the scattering amplitudes writes $T(s, t, u) = A(s, t) + A(s, u) + A(t, u)$ which states the invariance under the interchange of the variables.

⁴In deriving equation (34) it has been assumed that $\alpha(s) \equiv \alpha_0 + \alpha' s$ and the first term has been neglected for large s values.

3. THE EULER RIEMANN FUNCTION AND THE CASIMIR EFFECT

In the following we will discuss the methods employed by Euler to treat converging and diverging series. The first example is the sum of the series [11]

$$(35) \quad S_1 = 1 + 1 + 1 + 1 + \dots = \frac{1}{2} \quad , \quad S_2 = 1 - 2 + 3 - 4 + \dots = \frac{1}{4} .$$

The relevant proof is sketched below

$$(36) \quad \begin{aligned} S(\alpha, t) &= \sum_{n=0}^{\infty} (-\alpha)^n t^n = \frac{1}{1 + \alpha t} \quad \Rightarrow \quad S(1, 1) = \frac{1}{2} , \\ \partial_t S(1, t) &= \sum_{n=1}^{\infty} (-1)^n n t^{n-1} = -\frac{1}{(1+t)^2} \Big|_{t=1} \quad \Rightarrow \\ &\Rightarrow -\partial_t S(1, t)|_{t=1} = S_2 = 1 - 2 + 3 - 4 + \dots = \frac{1}{4} , \end{aligned}$$

where we have disregarded the fact that convergence, in classical sense, is ensured for $|t| > 1$.

If we trust the second of equations (35), we can derive even more disturbing identities. By denoting with O and E the infinite sums of even and odd positive integers, we find

$$(37) \quad \begin{aligned} \Sigma &:= O + E \quad O := 1 + 3 + 5 + \dots \quad , \quad E := 2 + 4 + 6 + \dots = 2\Sigma , \\ S_2 &= O - E \quad \Rightarrow \quad 2E = \Sigma - S_2 \quad \Rightarrow \quad 4\Sigma = \Sigma - S_2 \quad \Rightarrow \\ &\Rightarrow \quad \Sigma = -\frac{S_2}{3} \quad \Rightarrow \quad 1 + 2 + 3 + 4 + \dots = -\frac{1}{12} . \end{aligned}$$

Even though we have obtained the previous results as the consequence of dirty manipulations, they have a grain of truth as it will be shown in the following. To this aim we introduce the Euler-Riemann (E - R) function, defined by the infinite series

$$(38) \quad \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \quad , \quad (\Re(x) > 1) .$$

It is evident that if we relax the condition $\Re(x) > 1$ then $\zeta(1) = -1/12$, $\zeta(0) = 1/2$.

The use of the Laplace transform identity

$$(39) \quad \frac{1}{n^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} e^{-ns} s^{x-1} ds ,$$

allows the derivation of the associated integral representation

$$(40) \quad \zeta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \sum_{n=1}^{\infty} e^{-ns} s^{x-1} ds = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{e^s}{e^s - 1} s^{x-1} ds ,$$

($\Re(x) > 1$).

Its behavior, for positive real x -values, is reported in Fig. 6.

The E - R function plays a central role in a variety of physical problems, a very well known example is the black body radiation formula [15] (see Fig. 7), and the associated energy per unit volume, which, as easily verified, can be written as

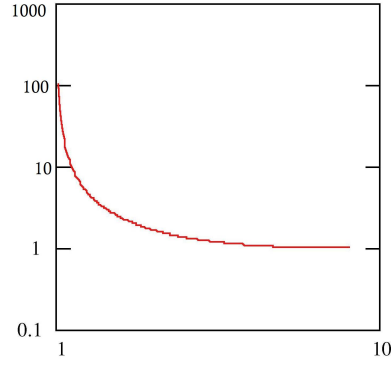
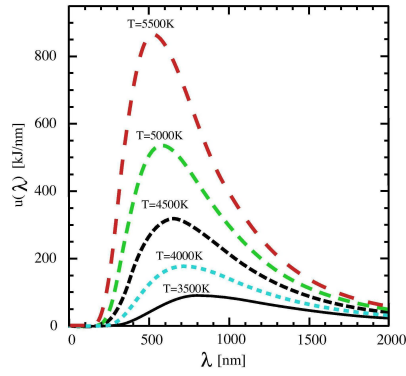
FIGURE 6. Log-plot of $\zeta(x)$ 

FIGURE 7. Black body radiation curve

$$(41) \quad u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/KT} - 1},$$

$$\int_0^\infty u(\lambda, T) d\lambda = \frac{U}{L^3} \Rightarrow \int_0^\infty \frac{x^3}{e^x - 1} dx = \zeta(4)\Gamma(4).$$

The values of the E - R function, for even values of the argument, can be evaluated using the properties of the Bernoulli numbers. To this aim we remind that the Bernoulli numbers are defined through the generating function

$$(42a) \quad \frac{t}{e^t - 1} = 1 + \sum_{k=2}^{\infty} \frac{t^k}{k!} B_k, \quad B_1 = -\frac{1}{2},$$

and that are explicitly defined as

$$(42b) \quad B_k = \frac{(-1)^k k}{2^k - 1} \sum_{i=1}^k 2^{-i} \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} (j+1)^{k-1}, \quad (k \geq 1),$$

$$B_0 = 1.$$

According to equation (42a) it is easily proved that the following expansion holds

$$(43a) \quad \cot(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} z^{2k-1} ,$$

Furthermore, since the following identities holds too

$$(43b) \quad \cot(z) - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{2z}{z^2 - (n\pi)^2} = -2 \sum_{k=0}^{\infty} \frac{z^{2k-1}}{\pi^{2k}} \zeta(2k) ,$$

we find, after comparing the like power z -coefficients,

$$(44) \quad \zeta(2m) = \frac{(-1)^{m-1} 2^{2m-1} B_{2m}}{(2m)!} \pi^{2m} .$$

The Euler Riemann function can be analytically continued to values of the variable such that $\Re(x) \leq 1$ (see Fig. 8), using the formula (due to Riemann)

$$(45) \quad \zeta(1-s) = 2^{1-s} \pi^{-s} \Gamma(s) \cos\left(\frac{s\pi}{2}\right) \zeta(s) .$$

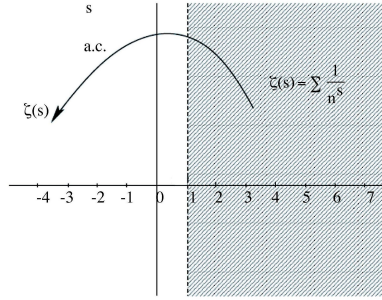


FIGURE 8. Analytic continuation of the Riemann function

Which confirms then divergent series obtained at the beginning of this section, we find indeed

$$(46) \quad s = 2 \quad \Rightarrow \quad \zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = -\frac{1}{2\pi^2} \Gamma(2) \zeta(2) = -\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{12} .$$

This, apparently foolish, conclusion has consequences of paramount importance in physics, in particular in the theory of the renormalization and allows the possibility of providing a meaning for divergent integrals in quantum electrodynamics, like those emerging in the study of the Casimir effect, which is a manifestation of forces of genuine quantum nature [14].

In Fig. 9 we have reported two conducting plates, separated by a distance d , for reasons of mode confinement not all the modes of the vacuum electromagnetic field will be contained within the conducting plates, those with longer wave length will be “confined” outside. The energy of the field outside will be therefore larger than

that inside; the plates are therefore experiencing an attractive force due to the electromagnetic pressure of radiation. The force pressure acting on the faces is just

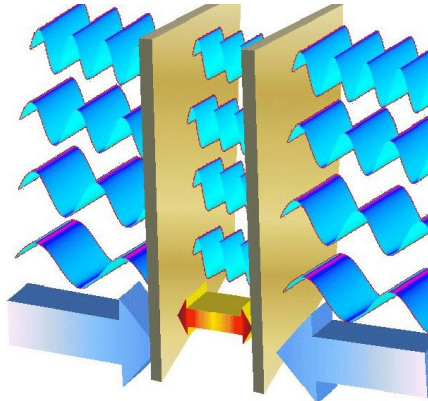


FIGURE 9. Conducting plates and Casimir effect

given by the derivative (with respect to the coordinate a) of the energy density inside, namely

$$(47) \quad \bar{F}_c = -\partial_a \frac{\langle E \rangle}{A} ,$$

where $\langle E \rangle/A$ is the average energy density (A is the plate's surface). Without entering into the details of the calculations, we note that it is provided by

$$(48) \quad \frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{6a^3} \sum_{n=1}^{\infty} n^3 \propto \zeta(-3) \quad , \quad \zeta(-3) = \frac{1}{8\pi^4} \Gamma(4)\zeta(4) .$$

It happens therefore that we “regularize” infinite sums using a mathematical procedure derived within the context of the analytical continuation of the E - R function.

4. THE EULER-RIEMANN FUNCTION AND THE THEORY OF RENORMALIZATION

The theory of the renormalization can be defined as the
*art of removing the infinities*⁵.

To better clarify what we mean, by “removing infinities” we consider an example from elementary calculus. It is well known that

$$(49) \quad I_n = \int x^n dx = \frac{1}{n+1} x^{n+1} + c ,$$

which diverges if we take the limit $n \rightarrow -1$, we can obtain a finite result if we “subtract” the infinite by properly choosing the constant c . By setting indeed⁶

$$(50) \quad I_n = \frac{1}{n+1} x^{n+1} - \frac{1}{n+1} ,$$

⁵Even though this definition is more appropriate for the technique known as “regularization” we will continue to use it to provide a suitable conceptual reference.

⁶This argument was developed during an informal discussion of one of the Authors (G. D.) with T. E. O. Hermesen.

we obtain (see Fig. 10)

$$(51) \quad \lim_{n \rightarrow \infty} I_n = \ln(x),$$

as it must be, since $\int 1/x = \ln(x) dx$. Let us now consider the following infinite

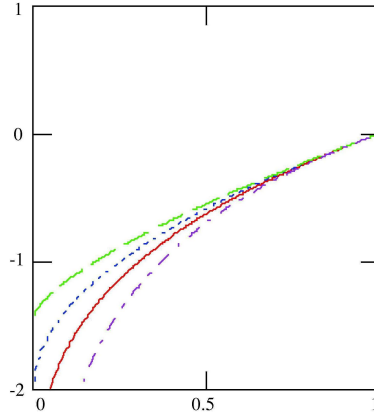


FIGURE 10. I_n vs. x for different values of n ($=-0.3$ dash, $=-0.5$ dot, $=-0.7$ continuous $=-1$ dash-dot)

sum

$$(52) \quad \Sigma(x) = \sum_{n=0}^{\infty} f(x+n),$$

which on account of the identity $e^{\lambda \partial_x} f(x) = f(x+\lambda)$ can be written as

$$(53) \quad \Sigma(x) = \sum_{n=0}^{\infty} e^{n \partial_x} f(x) = -\frac{\partial_x}{e^{\partial_x} - 1} F(x) \quad , \quad f(x) = \partial_x F(x).$$

The use of the Bernoulli numbers generating function (42a) yields

$$(54) \quad \Sigma(x) = -\sum_{n=0}^{\infty} \frac{B_n}{n!} \partial_x^n F(x).$$

If for example $f(x) = x^m$, we obtain that $\Sigma(0) = -B_{m+1}/(m+1)$, which is finite result and coincides with $\zeta(-m)$. Also in this case we have subtracted from an expression giving rise to a divergent term an other divergence of the same order. In more crude terms we state that we have managed to reduce our problem to an operation of the type $\Re = (\infty + a) - \infty$ Talking of dirty tricks!!!

In an attempt of providing a more rigorous formulation for the subtraction of infinities, we define the following Ramanujan, renormalized sum of a diverging series as

$$(55) \quad \sum_n^{\Re} a(n) = \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N a(n) - \int_1^N a(t) dt \right).$$

that actually $\zeta(-s) \neq \sum_{n=1}^{\infty} 1/n^{-s}$, but it represent its “renormalized counterpart. In general the integrals which should be cured in *QED* are of the type

$$(59) \quad I(m, \Lambda) = \int_0^{\Lambda} p^m dp .$$

The parameter Λ appearing in equation (59) is called *the regulator* and is an essential element in the theory of renormalization. The integral (59) is diverging for $\Lambda \rightarrow \infty$, therefore much care should be devoted in dealing with these integrals and then take the limit to eliminate the divergences. The quantity p is in general linked to the momentum of the exchanged particle, if it is a photon $p \propto 1/\lambda$ and depending on the sign of m we may have infrared (low momentum) or ultra violet (large momentum) divergences.

The first who suggested a way out to the problem of infinities by the use of the *E-R* function was Schwinger [10], who noted that if we transform the above integral into a sum, we find

$$(60) \quad \lim_{\Lambda \rightarrow \infty} I(m, \Lambda) \cong \zeta(-m) .$$

Which is finite for any positive m . However the approximation is too crude to get always consistent results and cannot be extended to any real (positive or negative) value of m .

The use of the trapezoid integral method allows a more rigorous handling of the integrals of equation (59). We find indeed [18]

$$(61) \quad I(m, \Lambda) = \int_0^{\Lambda} p^m dp \cong \frac{\Lambda^m}{2} + \sum_{n=1}^{\Lambda} n^m - \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} a_{m,r} \Lambda^{m-2r-1} ,$$

$$a_{m,r} := \frac{\Gamma(m+1)}{\Gamma(m-2r+2)} .$$

By noting that

$$(62) \quad \Lambda^s = s \int_0^{\Lambda} p^{s-1} dp = sI(s-1, \Lambda) ,$$

we can cast equation (61) in the form of the following recurrence

$$(63) \quad I(m, \Lambda) = \frac{m}{2} I(m-1, \Lambda) + \zeta(-m) - \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} a_{m,r} (m-2r+1) I(m-2r, \Lambda) ,$$

which is a refinement of the original Schwingers suggestion, proposed by Elizalde, Garcia and Hartle [10]. In this way all the integrals of the type can be written as linear combination of $\zeta(-2r-1)$.

We find from equation (62)

$$(64) \quad I(0, \Lambda) \cong \zeta(0) = \frac{1}{2} \quad , \quad I(1, \Lambda) \cong \frac{1}{3} I(0, \Lambda) + \zeta(-1) ,$$

$$I(3, \Lambda) \cong \frac{1}{2} [I(0, \Lambda) - a_{2,1} B_2] + \zeta(-1) \quad , \quad \dots .$$

We consider now the case of negative m values, we introduce a second regulator ε and write (see equations (21)-(24))

$$(65) \quad \begin{aligned} I(m, \Lambda, \varepsilon) &= \int_0^\Lambda \frac{1}{(\varepsilon + p)^m} dp = \frac{1}{\Gamma(m)} \int_0^\Lambda dp \int_0^\infty e^{-s(\varepsilon+p)} s^{m-1} ds = \\ &= \frac{\varepsilon^{-(m-1)}}{\Gamma(m)} \int_0^\infty \sigma^{m-1} e^{-\sigma} d\sigma \left[\int_0^{\Lambda\varepsilon^{-1}} e^{\sigma\omega} d\omega \right]. \end{aligned}$$

By expanding the integral in square brackets, we eventually end up with

$$(66) \quad I(m, \Lambda, \varepsilon) = \frac{\varepsilon^{-(m-1)}}{\Gamma(m)} \sum_{n=0}^{\infty} \frac{\Gamma(m+n)}{n!} I\left(n, \frac{\Lambda}{3}\right),$$

which again yields the integral in terms of E - R functions. One can clearly rise doubts on the convergence of the series on the left of equation (65) and we will further comment these aspects of the problem in the concluding section.

5. THE EULER GAMMA FUNCTION AND RENORMALIZATION TECHNIQUES

As we have already remarked the evaluation of Feynman diagrams involving loop corrections, like that reported in Fig. 13 [18], requires the evaluation of integrals

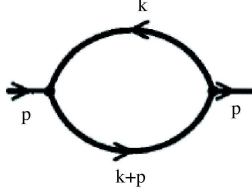


FIGURE 13. Feynman diagram for one loop corrections

of the type

$$(67) \quad I(d, n) = \int \frac{d^d k}{[k^2 + 1]^n},$$

where d denotes the dimensionality of the space, where we are performing the integration.

The integral (66), even though apparently complicated, can be easily worked out, using the methods discussed in the introductory section, which yield

$$(68) \quad \begin{aligned} I(d, n) &= \frac{1}{\Gamma(n)} \int d^d k \int_0^\infty e^{-s} e^{-sk^2} s^{n-1} ds = \\ &= \frac{1}{\Gamma(n)} \int_0^\infty e^{-s} s^{n-1} ds \left[\int d^d k e^{-sk^2} \right]. \end{aligned}$$

The integral in the square brackets, over the d -dimensional volume, is easily worked out as

$$(69) \quad \int d^d k e^{-sk^2} = \left[\int_{-\infty}^{\infty} dk_x e^{-sk_x^2} \right]^d = \left(\frac{\pi}{s} \right)^{d/2}.$$

We can finally express $I(d, n)$ in terms of the Gamma functions as follows

$$(70) \quad I(d, n) = \frac{\pi^{d/2}}{\Gamma(n)} \Gamma\left(n - \frac{d}{2}\right) = \frac{\pi^{d/2}}{\Gamma(d/2)} B\left(n - \frac{d}{2}, \frac{d}{2}\right).$$

The integral in equation (68) converges only if $d < 2n$, we note however that, by setting $\varepsilon = 1 - d/2$, we can write

$$(71) \quad I(d, 1) = 4\pi^{1-\varepsilon} \varepsilon \Gamma(\varepsilon).$$

The other integrals for $n > 1$ are all proportional to equation (71) as easily verified, we find e.g. for $n = 2$

$$(72) \quad I(d, 2) = \varepsilon I(d, 1).$$

While, for larger n values, we can establish straightforward recurrences. The conclusion is that we have individuated the source of the divergences, which manifest themselves as pole in the complex plane and can be therefore subtracted by direct methods.

Just to give an explicit example, we consider the first order loop integral appearing in φ^4 perturbative theories, (λ is a coupling constant)

$$(73) \quad \text{⊙} = -\lambda \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2}.$$

Can accordingly be written in terms of $I(d, 1)$ the relevant divergences located at positive even d values and can be subtracted in various way, which will be later touched on. The second loop integrals

$$(74) \quad \text{>⊙<} = \lambda^2 \int \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2} \frac{1}{(\mathbf{p} + \mathbf{k})^2 + m^2}.$$

can be evaluated by using analogous tricks, involving the properties of the Gamma function too, they reduce indeed to (see equations (21)-(26))

$$(75) \quad \frac{\lambda^2}{(4\pi)^{d/2}} \frac{\Gamma(2 - d/2)}{\Gamma(2)} \int_0^1 dx \frac{1}{[k^2 x(1-x) + m^2]^{2-d/2}},$$

after using the already quoted identity (see equation (25) and footnote 1)

$$(76) \quad (AB)^{-1} = \int_0^1 \frac{dx}{[xA + (1-x)B]}.$$

We will further comment on the previous results in the concluding section.

6. MATHEMATICS OR MATHEMAGICS

Some mathematicians (and even physicists) including Euler, Ramanujan and Feynman are some times called mathemagicians, for their attitude in giving a meaning to absurd. Symbols in mathematics are like the words of the ordinary language. These symbols are usually entangled to form a phrase with a definite meaning. Words, if put together without a proper organization, loses any meaning, this happens for mathematical symbols too. However meaning is not absolute but relative since in many cases it needs a key of interpretation, in some cases the absence of meaning may be only apparent. Mathemagicians have the ability of interpreting the key of apparently disconnected symbols, thus yielding a meaning

for seemingly foolish expressions. Just to give an example we consider the following relation [5]

$$(77) \quad F(x) = \hat{E}_{\partial_x} f(x) \quad , \quad \hat{E}_{\partial_x} = [\partial_x]!$$

Which is, at first sight, a meaningless operation, since we are not acquainted with the definition of an operator, which is the factorial of the derivative operator. We remind however that the use of the Gamma function has provided a meaning for the extension of the factorial operation to any real or complex number and to the definition of the derivatives of fractional orders. This last extension has required two distinct logical steps

- 1) *recognize the fact that the derivative can be treated as a common algebraic quantity*
- 2) *the use of the properties of the factorial for any real number.*

The same point of view can be adopted to define the operator appearing in equation (77), we can therefore using the Gamma function formally write

$$(78) \quad \hat{E}_{\partial_x} = \Gamma(\partial_x + 1) = \int_0^\infty e^{-\xi} \xi^{\partial_x} d\xi$$

and, by setting $\xi^{\partial_x} = e^{\ln(\xi)\partial_x}$, we find

$$(79) \quad \hat{E}_{\partial_x} f(x) = \int_0^\infty e^{-\xi} f(x + \ln(\xi)) d\xi$$

and conclude that the action of the factorial derivative on a given function can be expressed in terms of the logarithmic convolution reported in equation (78), the meaning of such operation is therefore well defined, even though apparently crazy. Other identities, disturbing as the Munk scream (Fig. 14), may be reconciled with a non esoteric interpretation. In the previous sections we have seen that the regularization of divergent series is a consequence of analytic continuation techniques, but Euler obtained the same result without any recourse to methods of complex analysis which were largely unknown to him. Just repeating the previously outlined procedure we can adopt the Euler method to evaluate sums of the type [19]

$$(80) \quad \sum_{n=1}^{\infty} (-1)^n n^k = \partial_\xi^k \sum_{n=1}^{\infty} (-1)^n e^{-n\xi} \Big|_{\xi=0} = -\partial_\xi^k \frac{1}{e^\xi + 1} \Big|_{\xi=0} .$$

By noting that (see equation (42b))

$$(81) \quad \frac{1}{e^\xi + 1} = \frac{1}{e^\xi - 1} - \frac{2}{e^{2\xi} - 1} = \sum_{k=1}^{\infty} \frac{B_{k+1}}{(k+1)!} \xi^k + \frac{1}{\xi} ,$$

$$\sum_{n=1}^{\infty} (-1)^k n^k = \frac{1 - 2^{k+1}}{k+1} B_{k+1} .$$

The use of the techniques discussed in the paper yields other identities as e.g.

$$(82) \quad \infty! = \sqrt{2\pi} ,$$

whose proof goes as sketched below

$$(83a) \quad \lim_{n \rightarrow \infty} n! = \lim_{n \rightarrow \infty} \exp \left(\sum_{s=1}^{\infty} \ln(s) \right) .$$

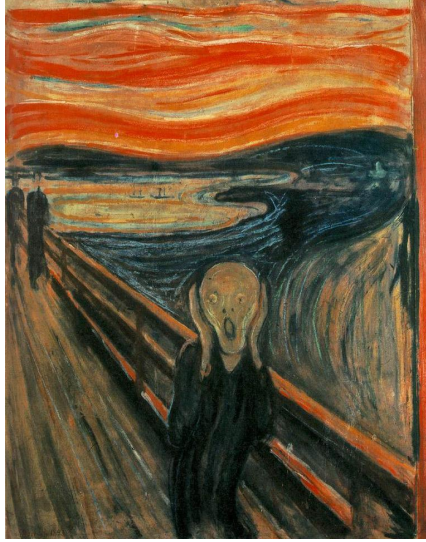


FIGURE 14. The Munk Scream (according to the Authors interpretation it can be understood as the reaction of a rigorist to identities of the type (77) and (81))

Being

$$(83b) \quad \sum_{s=1}^{\infty} \ln(s) = \zeta'(0) = \ln(\sqrt{2\pi}) ,$$

we obtain the identity (81). Regarding equation (83b), we note that it just follows from a renormalization procedure of the type

$$(83c) \quad \sum_n^{\Re} \ln(n) = \lim_{n \rightarrow \infty} \left[\sum_{n=1}^N - \int_1^N \ln(t) dt \right] = \lim_{n \rightarrow \infty} [\ln(N!) - (N \ln(N) - N)] ,$$

whose graphical interpretation is reported in Fig. 15. We can exploit the concept

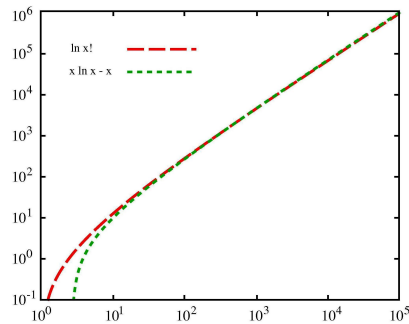


FIGURE 15. A graphical interpretation of the Stirling approximation

of factorial derivative to get a meaning for the “wildly” divergent series

$$(84a) \quad \Delta = \sum_{n=1}^{\infty} (-1)^n n! ,$$

which, according to our formalism, can be cast in the form of the following (convergent) integral

$$(84b) \quad \Delta = (\partial_x!) \sum_{n=1}^{\infty} (-1)^n e^{nx} |_{x=0} = \int_0^{\infty} \frac{e^{-t}}{t+1} dt .$$

The research on the above series was inspired to Euler by the solution of the following differential equation

$$(85) \quad x^2 \partial_x y + y = x \quad , \quad y(0) = 0 ,$$

whose series expansion solution

$$(86) \quad y(x) = \sum_{n=1}^{\infty} (-1)^n n! x^n$$

does not converge for any x value. On the other side, its solution written in integral form

$$(87) \quad y(x) = x \int_0^{\infty} \frac{e^{-t}}{1+xt} dt$$

is well behaved for any positive x . We will see in the forthcoming section that the previous result can be framed within the more general context of the Borel transform.

7. INTEGRAL TRANSFORMS AND E - R FUNCTIONS

Before concluding our tour we will discuss how the link between Zeta function and particular types of integral transforms (like the Mellin-Laplace transform) follows from the Euler manipulations. To this aim we consider the function

$$(88) \quad \zeta(x, p) = \sum_{n=1}^{\infty} \frac{x^n}{n^p} ,$$

which can also be viewed as of a kind of generalization of the logarithmic function. On account of the identity

$$(89) \quad (x\partial_x)^n x^m = m^n x^m ,$$

we can write equation (88) as the following finite expression [7]

$$(90) \quad \zeta(x, p) = (x\partial_x)^{-p} \sum_{n=1}^{\infty} n^n = (x\partial_x)^{-p} \left(\frac{1}{1-x} - 1 \right) .$$

The function (80) can therefore be written in operational terms as (see equation (22))

$$(91) \quad \zeta(x, p) = (x\partial_x)^{-p} \frac{x}{1-x} .$$

Furthermore, by exploiting the identities

$$(92) \quad (x\partial_x)^{-p} = \frac{1}{\Gamma(p)} \int_0^\infty e^{-\xi\hat{D}_x} \xi^{p-1} d\xi ,$$

$$e^{\lambda\hat{D}_x} f(x) = f(e^\lambda x) \quad , \quad \hat{D}_x := x\partial_x ,$$

we end up with the following integral representation

$$(93) \quad \zeta(x, p) = \frac{1}{\Gamma(p)} \int_0^\infty \xi^{p-1} \frac{e^{-\xi} x}{1 - e^{-\xi} x} d\xi .$$

The exponential operator appearing in the first of equations (92) is called the dilation operator and its use was originally suggested by Euler himself. It plays a central role in the theory of the *E-R* function and the associated operator [7]

$$(94) \quad \hat{Z}_p = \frac{1}{\Gamma(p)} \int_0^\infty e^{-\xi\hat{D}_x} \xi^{p-1} d\xi$$

is the generator of a new transform, in between the Laplace and Mellin transforms [7] and indeed its application on a given function yields

$$(95) \quad \hat{Z}_p f(x) = \frac{1}{\Gamma(p)} \int_0^\infty f(e^{-\xi} x) \xi^{p-1} d\xi .$$

Furthermore, once acting on the *E-R* function, provides the following variable

$$(96) \quad \hat{Z}_p \zeta(x, m)|_{x=1} = \zeta(m + p) .$$

Before closing this section, let us discuss the link of the topics we have treated so far with the so called Borel transform. We consider the series

$$(97) \quad S(x, a_n) = \sum_{n=0}^\infty a_n x^n ,$$

which can be rearranged as it follows

$$(98) \quad S(x, a_n) = \sum_{n=0}^\infty (n! a_n) \frac{x^n}{n!} = \int_0^\infty e^{-t} \sum_{n=0}^\infty a_n \frac{(tx)^n}{n!} dt .$$

If

$$(99) \quad \sum_{n=0}^\infty \frac{x^n}{n!} a_n = g(x) ,$$

we obtain

$$(100) \quad S(x, a_n) = \int_0^\infty e^{-t} g(tx) dt ,$$

thus transforming a (possibly) divergent sum in a (possibly) converging integral. The concepts associated with the Borel transform methods can easily be linked to the discussion we have developed so far, it is easily seen for example that

$$(101) \quad S(x, (-1)^n n!) = \int_0^\infty \frac{e^{-t}}{1 + xt} dt ,$$

while the Borel transform of the exponential can be associated with Bessel type function

$$(102) \quad S\left(x, \frac{(-1)^n}{n!}\right) = \int_0^\infty e^{-t} C_0(tx) dt ,$$

where $C_0(x) = \sum_{r=0}^{\infty} (-x)^r / (r!)^2$ is the 0-th order Tricomi function linked to the 0-th order Bessel function by the relation

$$(103) \quad C_0(x) = J_0(2\sqrt{x}) .$$

We cannot go deeper into the theory of Borel transform, which would open new speculations on the link between symbolic calculus and physics.

8. CONCLUDING REMARKS

This review paper has dealt with some old research in pure mathematics, which have provided a decisive impact on the formulation of most recent theories in physics. The legacy of Euler to modern science has been so wide and profound that we have just grasped on a few outcome of a prodigious mind, which is still a source of inspiration and is influencing the modern thought. Ramanujan, the famous Indian self thought mathematician, was one of the main interpreter of this heritage and his work on series (divergent or not) opened new scenarios. Feynman, some time referred as the modern Euler, elaborated new methods to evaluate physical processes, in which the point of view of Euler (and Ramanujan) becomes crucial. The renormalization methods are now used in different branch of physics, but some time without the necessary awareness. We consider therefore useful to explain once again that by “renormalizable” theories it is meant theories free from divergences. Within their framework, infinities can be removed, or better they can be controlled and in some sense they can be absorbed into the free parameters of the theory itself. In *QED* the infinities are absorbed into the quantities reported in Fig. 16. The meaning of all this is that the sensible quantities are not the bare

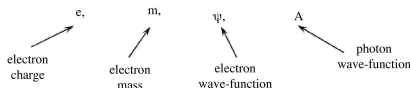


FIGURE 16. Renormalized quantities of *QED*

(without any interaction) mass, charge ... but the “dressed” ones, which contain all the contributions from all the different diagrams. This fact has further profound consequences, we consider the diagrams reported in Fig. 17, which account for the quantum corrections to the charge, (the coupling constant of *QED*). Denoting by the contribution from a single loop, we find [16] (see Fig. 18)

$$(104) \quad e(Q) = e_0 \sum_{n=0}^{\infty} (-1)^n [I(Q)]^n = \frac{e_0}{1 + I(Q)} .$$

It is evident that it can be evaluated at a different energy scale in such a way that

$$(105) \quad e(\mu) = \frac{e(Q)}{1 + I(\mu)} .$$

Which ensures that charge (or the *QED* coupling constant) changes with energy, analogous conclusion can be drawn for the *QCD*. This last observation opens consideration associated with the Khallen-Szymanzyk renormalization equation, which is out of the interest of this paper. We have emphasized the usefulness of the methods associated with “*the theory of the subtraction of infinities*” but we have also stressed that, in spite of its success, it does not lie on solid mathematical grounds.

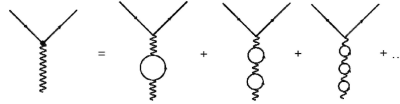


FIGURE 17. Quantum loop corrections to the electron charge



FIGURE 18. Single loop contribution

If such a criticism would come from pure mathematicians it could be just considered as the consequence of the perennial dialectic between different ways of conceiving mathematics. If the criticisms come from one of the founders of the theory itself, they should be taken more seriously. We quote therefore Feynman who declared [12]

The shell game that we play . . . is technically called “renormalization”. But no matter how clever the word, it is still what I would call a dippy process! Having to resort to such hocus-pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent. It’s surprising that the theory still hasn’t been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate.

We consider appropriate these criticisms and those persistently expressed by Dirac, according to whom

Most physicists are very satisfied with the situation. They say: “Quantum electrodynamics is a good theory and we do not have to worry about it any more”. I must say that I am very dissatisfied with the situation, because this so-called “good theory” does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small not neglecting it just because it is infinitely great and you do not want it!

We are not legitimate to draw any further comment, the best conclusion we may offer to the reader is Fig. 19, reporting the famous Gauguin painting

Where are we coming from?

We are we?

Where are we going?

We would just to add an other question

Where are we?

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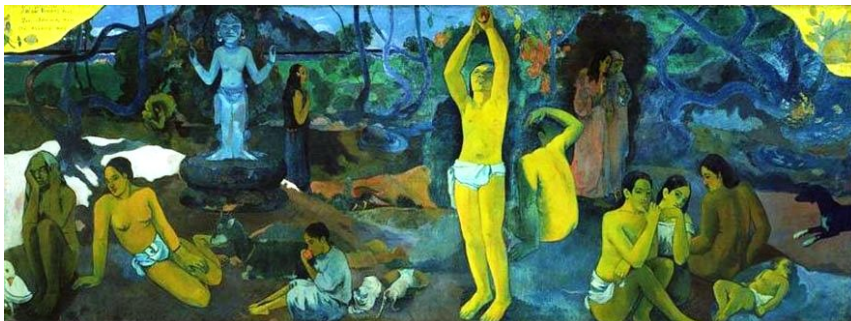


FIGURE 19. Paul Gauguin: D'où venons nous? Que sommes nous?
Où allons nous?

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