Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 9(2010), pp. 153–161.

On generating relations involving phase-space Hermite polynomials

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Abstract[1]. The phase-space Hermite polynomials (PSHP) are framed within the context of the representation $\uparrow_{\omega,\mu}$ of the Lie algebra $\mathcal{G}(0,1)$. The analysis has been carried out by generalizing the formalism relevant to Hermite polynomials. New results involving PSHP are derived.

1. INTRODUCTION

The Hermite polynomials $H_n(x)$ specified by the generating function

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \ , \tag{1.1}$$

are set of orthogonal polynomials over the domain $(-\infty, \infty)$ with respect to the weight function e^{-x^2}

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} \, dx = \delta_{m,n} 2^n n! \sqrt{\pi} \ . \tag{1.2}$$

These polynomials arise in probability, such as the Edgeworth series; in combinatorics; as an example of an Appell sequence, obeying the umbral calculus; and in physics, as the eigenstates of the quantum harmonic oscillator.

Further, we observe that the Hermite functions

$$\psi_n(x) = \frac{1}{\sqrt{n! 2^n \sqrt{\pi}}} \, e^{-x^2/2} H_n(x) \ , \tag{1.3}$$

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This work has been done under a Major Research Project No. F.33-110/2007 (SR) sanctioned to the second author by the University Grants Commission, Government of India, New Delhi.

Keywords. Phase-space Hermite polynomials, Lie algebra, generating relations.

AMS Subject Classification. 33C45, 33C50, 33C80.

contains the square root of the weight function and have been scaled appropriately, they are orthonormal:

$$\int_{-\infty}^{\infty} \psi_n(x)\psi_m(x)\,dx = \delta_{n,m}\ . \tag{1.4}$$

This equation is equivalent to the Schrödinger equation for a harmonic oscillator in quantum mechanics. The Hermite functions $\psi_n(x)$ satisfy the following differential equation:

$$\psi_n''(x) + (2n+1-x^2)\psi_n(x) = 0\ . \tag{1.5}$$

The phase-space formalism has revealed its relevance for formulating quantum and classical problems and provides an appropriate framework for stating the correspondence principle between classical and quantum mechanics [11].

The phase-space approach to classical and quantum systems requires advanced analytical tools. Such an approach characterizes the evolution of a physical system through a set of variables, reducing to the canonically conjugate variables in the classical limit. A significant analytical tool to treat these problems may come from the generalized many variables Hermite polynomials. They form an orthonormal system in many dimensions and seems to be the natural tool for treating the harmonic oscillator dynamics in phase-space.

The generalized Hermite polynomials and the associated orthogonal functions play a crucial role within the framework of phase-space formulation of classical or quantum mechanics. Indeed they play the same role as the ordinary Hermite polynomials and related harmonic oscillator functions play in the Schrödinger picture.

Dattoli et al. [4] and Dattoli and Torre [7] introduced and discussed the theory of phase-space Hermite polynomials (PSHP) using an operator formalism. These polynomials play a crucial role within the framework of phase-space formalism of classical and quantum mechanics.

The PSHP $H_{m,n}(x,p)$ are defined by the generating relation [7, p. 1637 (1)]

$$e^{\underline{u}^T \widehat{M} \underline{z} - (1/2)\underline{u}^T \widehat{M} \underline{u}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} H_{m,n}(x,p)\,\frac{h^m}{m!}\,\frac{t^n}{n!}\ , \tag{1.6a}$$

where T denotes the transpose and

$$\underline{z} = \begin{pmatrix} x \\ p \end{pmatrix}\ ,\quad \underline{u} = \begin{pmatrix} h \\ t \end{pmatrix}\ ,\quad |h| < \infty,\ |t| < \infty,\ h \neq 0,\ t \neq 0\ . \tag{1.6b}$$

The 2×2 matrix $\widehat{M}$ is

$$\widehat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\ ,\quad \Delta := ac - b^2 > 0,\ a > 0,\ c > 0\ . \tag{1.6c}$$

The study of special functions from the Lie-algebraic point of view has witnessed a significant evolution during the recent years. The applications of the theory of representations of Lie groups and their Lie algebras allow interpretations of many familiar one variable special functions, see for example Miller [12]. Recently a fundamental relationship has been established between certain generalized special functions of mathematical physics with various Lie groups and Lie algebras, see for example [3, 9, 10].

In this paper, we consider the problem of framing PSHP $H_{m,n}(x,p)$ into the context of the representation $\uparrow_{\omega,\mu}$ of a four-dimensional Lie algebra $\mathcal{G}(0,1)$ [12,

p. 85]. In section 2, we give the basic properties of PSHP $H_{m,n}(x,p)$ and their special cases. In section 3, generating relations involving PSHP $H_{m,n}(x,p)$ and associated Laguerre polynomials $L_l^{(n)}(x)$ [1] are obtained. In section 4, we consider some special cases which would yield inevitably to many new and known generating relations for the polynomials related to PSHP.

2. PROPERTIES OF PHASE-SPACE HERMITE POLYNOMIALS $H_{m,n}(x,p)$

The Rodrigues type formula for PSHP $H_{m,n}(x,p)$ is given as [7, p. 1637 (4)]

$$H_{m,n}(x,p) = (-1)^{m+n} e^{(1/2)\underline{z}^T \widehat{M} \underline{z}} \, \frac{\partial^{m+n}}{\partial x^m \partial p^n} \, e^{(-1/2)\underline{z}^T \widehat{M} \underline{z}} \, . \tag{2.1}$$

The PSHP $H_{m,n}(x,p)$ satisfy the following differential and pure recursion relations

$$\begin{aligned} \frac{\partial}{\partial x} H_{m,n}(x,p) &= am H_{m-1,n}(x,p) + bn H_{m,n-1}(x,p) \, , \\ \frac{\partial}{\partial p} H_{m,n}(x,p) &= bm H_{m-1,n}(x,p) + cn H_{m,n-1}(x,p) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} H_{m+1,n}(x,p) &= (ax+bp) H_{m,n}(x,p) - am H_{m-1,n}(x,p) - \\ &\quad - bn H_{m,n-1}(x,p), \\ H_{m,n+1}(x,p) &= (bx+cp) H_{m,n}(x,p) - bm H_{m-1,n}(x,p) - \\ &\quad - cn H_{m,n-1}(x,p), \end{aligned} \tag{2.3}$$

respectively.

The differential equation for PSHP $H_{m,n}(x,p)$ is given as

$$\left(\frac{\partial^2}{\partial x^2} - \frac{b}{c} \, \frac{\partial^2}{\partial x \partial p} - (ax+bp) \left(\frac{\partial}{\partial x} - \frac{b}{c} \, \frac{\partial}{\partial p} \right) + \frac{m\Delta}{c} \right) H_{m,n}(x,p) = 0 \, , \tag{2.4a}$$

$$\left(\frac{\partial^2}{\partial p^2} - \frac{b}{a} \, \frac{\partial^2}{\partial p \partial x} - (bx+cp) \left(\frac{\partial}{\partial p} - \frac{b}{a} \, \frac{\partial}{\partial x} \right) + \frac{n\Delta}{a} \right) H_{m,n}(x,p) = 0 \, , \tag{2.4b}$$

or equivalently

$$\begin{aligned} &\left(\frac{1}{\Delta} \left(-c \, \frac{\partial^2}{\partial x^2} + 2b \, \frac{\partial^2}{\partial x \partial p} - a \, \frac{\partial^2}{\partial p^2} \right) + \right. \\ &\left. + \left(x \, \frac{\partial}{\partial x} + p \, \frac{\partial}{\partial p} \right) - (m+n) \right) H_{m,n}(x,p) = 0 \, . \end{aligned}$$

The PSHP $H_{m,n}(x,p)$ are closely related to the harmonic oscillator phase-space function (HOPSF) $\mathcal{H}_{m,n}(x,p)$ according to [7, p. 1638 (11)]

$$\mathcal{H}_{m,n}(x,p) = \sqrt{\frac{\Delta^{1/2}}{2\pi}} \, \frac{1}{\sqrt{m!n!}} \, H_{m,n}(x,p) e^{-(1/4)\underline{z}^T \widehat{M} \underline{z}} \, . \tag{2.5}$$

We note that for $a = c = 2$, $b = s = p = 0$, $x = ix/2\sqrt{y}$ and $h = -i\sqrt{y}\,t$, the PSHP $H_{m,n}(x,p)$ reduce to 2-variable Hermite-Kampé de Fériet polynomials

(2VHKdFP) $H_n(x,y)$ which are defined as [2, p. 341 (23)]:

$$H_n(x,y) = n! \sum_{r=0}^{[n/2]} \frac{x^{n-2r} y^r}{(n-2r)!r!} . \tag{2.6}$$

In terms of classical Hermite polynomials $H_n(x)$ or $He_n(x)$ [1], it is easily seen from the definition (2.5) that

$$H_n(2x,-1) = H_n(x) , \tag{2.7a}$$

and

$$H_n\left(x, -\frac{1}{2}\right) = He_n(x) . \tag{2.7b}$$

Also, there exists the following close relationship [2, p. 341 (21)]:

$$H_n(x,y) = (-i)^n \; y^{n/2} \; H_n\left(\frac{ix}{2\sqrt{y}}\right) = i^n (2y)^{n/2} He_n\left(\frac{x}{i\sqrt{2y}}\right) \tag{2.8}$$

with the classical Hermite polynomials, where $He_n(x)$ are linked to $H_n(x)$ by the relation

$$H_n(x) = 2^{n/2} He_n(\sqrt{2}x) .$$

We observe that for $a = c = 2$, $b = 0$, $p = y$, the PSHP $H_{m,n}(x,p)$ reduce to special Hermite 2D polynomials $H_{m,n}(I;x,y)$ [13, p. 1607, (3.11)], where

$$H_{m,n}(I;x,y) = H_m(x) H_n(y) . \tag{2.9}$$

The four-dimensional local Lie group $G(0,1)$, is a multiplicative matrix group with elements [12, p. 9]

$$g(a,b,c,\tau) = \begin{pmatrix} 1 & ce^{\tau} & a & \tau \\ 0 & e^{\tau} & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad a,b,c,\tau \in \mathbb{C} . \tag{2.10}$$

The group $G(0,1)$ is called the complex harmonic oscillator group [12, Chapter 10]. The basis for the Lie algebra $\mathcal{G}(0,1) = L[G(0,1)]$ is provided by the matrices

$$\mathcal{J}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathcal{J}^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$\mathcal{J}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathcal{E} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{2.11}$$

with commutation relations

$$[\mathcal{J}^3, \mathcal{J}^{\pm}] = \pm \mathcal{J}^{\pm} \quad , \quad [\mathcal{J}^+, \mathcal{J}^-] = -\mathcal{E} \quad , \quad [\mathcal{E}, \mathcal{J}^+] = [\mathcal{E}, \mathcal{J}^-] = [\mathcal{E}, \mathcal{J}^3] = 0 . \tag{2.12}$$

3. REPRESENTATION $\uparrow_{\omega,\mu}$ OF $\mathcal{G}(0,1)$ AND GENERATING RELATIONS

Miller [12] have defined the irreducible representation $\uparrow_{\omega,\mu}$ of $\mathcal{G}(0,1)$ for each $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0$. The spectrum S of this representation is the set $\{-\omega+n;$ n a non negative integer$\}$ and the representation space V has a basis $(f_m)_{m\in S}$, such that

$$J^3 f_m = m f_m \ , \ E f_m = \mu f_m \ , \ J^+ f_m = \mu f_{m+1} \ , \ J^- f_m = (m+\omega) f_{m-1} \ ,$$

$$C_{0,1} f_m = (J^+J^- - EJ^3) f_m = \mu\omega f_m \ . \tag{3.1}$$

The operators $J^\pm, J^3, E$ satisfy the commutation relations

$$\begin{aligned} &[J^3, J^+] = J^+ \quad , \quad [J^3, J^-] = -J^- \ , \\ &[J^+, J^-] = -E \quad , \quad [J^\pm, E] = [J^3, E] = 0 \ . \end{aligned} \tag{3.2}$$

In particular, we look for the functions

$$f_{m,n}(x,p;q,s) = Z_{m,n}(x,p) q^m s^n \ , \tag{3.3}$$

such that

$$\begin{aligned} &K^3 f_{m,n} = m f_{m,n} \ , \ E f_{m,n} = \mu f_{m,n} \ , \ K^+ f_{m,n} = \mu f_{m+1,n} \ , \\ &K^- f_{m,n} = (m+\omega) f_{m-1,n} \ , \\ &C_{0,1} f_{m,n} = (K^+K^- - EK^3) f_{m,n} = \mu\omega f_{m,n} \qquad (\omega \neq 0;\ m \in S) \end{aligned} \tag{3.4}$$

and also

$$\begin{aligned} &K^{3'} f_{m,n} = n f_{m,n} \ , \ E' f_{m,n} = \mu f_{m,n} \ , \ K^{+'} f_{m,n} = \mu f_{m,n+1} \ , \\ &K^{-'} f_{m,n} = (n+\omega) f_{m,n-1} \ , \\ &C_{0,1} f_{m,n} = (K^{+'}K^{-'} - E'K^{3'}) f_{m,n} = \mu\omega f_{m,n} \qquad (\omega \neq 0;\ n \in S) \ . \end{aligned} \tag{3.5}$$

The sets of operators $\{K^\pm, K^3, E\}$ and $\{K^{\pm'}, K^{3'}, E'\}$ satisfy the commutation relations identical to (3.2).

There are numerous possible solutions of equation (3.2). We assume that the sets of linear differential operators $\{K^\pm, K^3, E\}$ and $\{K^{\pm'}, K^{3'}, E'\}$ take the forms

$$\begin{aligned} &K^+ = (ax+bp)q - q\frac{\partial}{\partial x} \quad , \quad K^- = \frac{1}{\Delta q}\left(c\,\frac{\partial}{\partial x} - b\,\frac{\partial}{\partial p}\right) \ , \\ &K^3 = \frac{\partial}{\partial q} \quad , \quad E = 1 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &K^{+'} = (bx+cp)s - s\,\frac{\partial}{\partial p} \quad , \quad K^{-'} = \frac{-1}{\Delta s}\left(b\,\frac{\partial}{\partial x} - a\,\frac{\partial}{\partial p}\right) \ , \\ &K^{3'} = s\,\frac{\partial}{\partial s} \quad , \quad E' = 1 \ , \end{aligned} \tag{3.7}$$

respectively. The operators in equations (3.6) and (3.7) satisfy the commutation relations (3.2).

In terms of the functions $Z_{m,n}(x,p)$ and using operators (3.6) and (3.7), relations (3.4) and (3.5) reduce to

$$
(3.8)\qquad
\begin{aligned}
&(i) && \left((ax+bp)-\frac{\partial}{\partial x}\right) Z_{m,n}(x,p)=\mu Z_{m+1,n}(x,p)\ ,\\
&(ii) && \frac{1}{\Delta}\left(c\,\frac{\partial}{\partial x}-b\,\frac{\partial}{\partial p}\right) Z_{m,n}(x,p)=(m+\omega)Z_{m-1,n}(x,p)\ ,\\
&(iii) && \left(\frac{\partial^2}{\partial x^2}-\frac{b}{c}\,\frac{\partial^2}{\partial x\partial p}-(ax+bp)\left(\frac{\partial}{\partial x}-\frac{b}{c}\,\frac{\partial}{\partial p}\right)+\right.\\
& && \left.+\frac{(m+\mu\omega)\Delta}{c}\right) Z_{m,n}(x,p)=0 \quad , \quad m,n=0,1,2,\cdots
\end{aligned}
$$

and

$$
(3.9)\qquad
\begin{aligned}
&(i) && \left((bx+cp)-\frac{\partial}{\partial p}\right) Z_{m,n}(x,p)=\mu Z_{m,n+1}(x,p)\ ,\\
&(ii) && \frac{-1}{\Delta}\left(b\,\frac{\partial}{\partial x}-a\,\frac{\partial}{\partial p}\right) Z_{m,n}(x,p)=(n+\omega)Z_{m,n-1}(x,p)\ ,\\
&(iii) && \left(\frac{\partial^2}{\partial p^2}-\frac{b}{a}\,\frac{\partial^2}{\partial p\partial x}-(bx+cp)\left(\frac{\partial}{\partial p}-\frac{b}{a}\,\frac{\partial}{\partial x}\right)+\right.\\
& && \left.+\frac{(n+\mu\omega)\Delta}{a}\right) Z_{m,n}(x,p)=0 \quad , \quad m,n=0,1,2,\cdots
\end{aligned}
$$

respectively. We can take $\omega = 0$ and $\mu = 1$ without any loss of generality for special function theory . We observe that for $\omega = 0$ and $\mu = 1$ (iii) of equations (3.8) and (3.9) coincide with the differential equations (2.4a) and (2.4b) respectively of PSHP $H_{m,n}(x,p)$. In fact, for all $m,n \in S$ the choice for $Z_{m,n}(x,p) = H_{m,n}(x,p)$ satisfy equations (3.8) and (3.9). Thus we conclude that the functions $f_{m,n}(x,p;q,s) = H_{m,n}(x,p)q^m s^n$, $m,n \in S$ form a basis for a realization of the representation $\uparrow_{0,1}$ of $\mathcal{G}(0,1)$. By using [12, p. 18 (Theorem 1.10)], this representation of $\mathcal{G}(0,1)$ can be extended to a local multiplier representation [12, p. 17] of $G(0,1)$. Using operators (3.6), the local multiplier representation $T(g)$, $g \in G(0,1)$ defined on $\mathcal{F}$, the space of all functions analytic in a neighborhood of the point $(x^0,p^0,q^0,s^0) = (1,1,1,1)$, takes the form

$$
(3.10)\qquad
\begin{aligned}
&[T(\exp a_1\mathcal{E})f](x,p;q,s)=\exp(a_1)f(x,p;q,s)\ ,\\
&[T(\exp b_1\mathcal{J}^+)f](x,p;q,s)=\exp\left(\frac{b_1q}{2}(2ax+2bp-b_1aq)\right)\cdot\\
&\qquad\qquad\cdot f\left(x(1-\frac{b_1q}{x}),p;q,s\right)\ ,\\
&[T(\exp c_1\mathcal{J}^-)f](x,p;q,s)=f\left(x(1+\frac{c_1c}{\Delta xq}),p(1-\frac{c_1b}{\Delta pq});q,s\right)\ ,\\
&[T(\exp \tau_1\mathcal{J}^3)f](x,p;q,s)=f(x,p;qe^{\tau_1},s)\ ,
\end{aligned}
$$

for $f \in \mathcal{F}$. If $g \in G(0,1)$ has parameters (a_1, b_1, c_1, τ_1), then

$$T(g) = T(\exp a_1 \mathcal{E}) T(\exp b_1 \mathcal{J}^+) T(\exp c_1 \mathcal{J}^-) T(\exp \tau_1 \mathcal{J}^3)$$

and therefore we obtain

$$\begin{aligned}[T(g)f](x,p;q,s) &= \exp\left(a_1 + \frac{b_1 q}{2}(2ax + 2bp - b_1 aq)\right) \cdot \\ &\cdot f\left(x(1 - \frac{b_1 q}{x} + \frac{c_1 c}{\Delta xq}), p(1 - \frac{c_1 b}{\Delta pq}); qe^{\tau_1}, s\right) .\end{aligned} \tag{3.11}$$

The matrix elements of $T(g)$ with respect to the analytic basis $f_{m,n}(x,p;q,s) = H_{m,n}(x,p)q^m s^n$ are the functions $A_{lk}(g)$ uniquely determined by $\uparrow_{\omega,\mu}$ of $\mathcal{G}(0,1)$ and are defined by

$$[T(g)f_{k,n}](x,p;q,s) = \sum_{l=0}^{\infty} A_{lk}(g) f_{l,n}(x,p;q,s) \quad , \quad k = 0,1,2,3\cdots . \tag{3.12}$$

Therefore, we prove the following result:

Theorem 3.1. *The following generating equation holds:*

$$\begin{aligned}&\exp\left(\frac{b_1 q}{2}(2ax + 2bp - b_1 aq)\right) \cdot \\ &\cdot H_{k,n}\left(x\left(1 + \frac{c_1 c}{\Delta xq} - \frac{b_1 q}{x}\right), p\left(1 - \frac{c_1 b}{\Delta pq}\right)\right) = \\ &= \sum_{l=0}^{\infty} (c_1)^{k-l} L_l^{k-l}(-b_1 c_1) H_{l,n}(x,p) q^{l-k} \quad , k = 0,1,2,\cdots .\end{aligned} \tag{3.13}$$

Proof. Using (3.11), we obtain

$$\begin{aligned}&\exp\left(a_1 + \frac{b_1 q}{2}(2ax + 2bp - b_1 aq) + \tau_1 k\right) \cdot \\ &\cdot H_{k,n}\left(x\left(1 + \frac{c_1 c}{\Delta xq} - \frac{b_1 q}{x}\right), p\left(1 - \frac{c_1 b}{\Delta pq}\right)\right) = \\ &= \sum_{l=0}^{\infty} A_{lk}(g) H_{l,n}(x,p) q^{l-k} \quad , \quad k = 0,1,2,\cdots\end{aligned} \tag{3.14}$$

and the matrix elements $A_{lk}(g)$ are given by [12, p. 87 (4.26)],

$$A_{lk}(g) = \exp(a_1 + k\tau_1)(c_1)^{k-l} L_l^{k-l}(-b_1 c_1) \quad , \quad k,l \geq 0 . \tag{3.15}$$

Substituting the value of $A_{lk}(g)$ given by (3.15) into (3.14) and simplifying we obtain result (3.13).

Similarly, for the operators (3.7), we have the following result:

Theorem 3.2. *The following generating equation holds:*

$$\exp\left(\frac{b_2 s}{2}(2bx+2cp-b_2cs)\right)\cdot$$

$$\cdot H_{m,r}\left(x\left(1-\frac{c_2 b}{\Delta xs}\right),p\left(1-\frac{b_2 s}{p}+\frac{c_2 a}{\Delta ps}\right)\right)= \tag{3.16}$$

$$=\sum_{i=0}^{\infty}(c_2)^{r-i}L_i^{r-i}(-b_2c_2)H_{m,i}(x,p)s^{i-r} \quad , \quad r=0,1,2,\cdots .$$

4. Special cases

We consider some special cases of generating relations (3.13) and (3.16).

I. Taking $a=c=2$, $p=y$, $b=0$ and making use of (2.9) in (3.13), we obtain

$$\exp\left(2b_1xs-b_1^2s^2\right)H_k\left(x(1-\frac{b_1 s}{x}+\frac{c_1}{2xs})\right)H_n(y)=$$

$$=\sum_{l=0}^{\infty}(c_1)^{k-l}L_l^{k-l}(-b_1c_1)H_l(x)H_n(y)s^{l-k}\ , \tag{4.1}$$

which on taking $q=0$ and replacing b and c by $-b$ and $-c$ respectively, reduces to [12, p. 106 (4.76)],

$$\exp\left(-2b_1xs-b_1^2s^2\right)H_k\left(x\left(1+\frac{b_1 s}{x}-\frac{c_1}{2xs}\right)\right)=$$

$$=\sum_{l=0}^{\infty}(-c_1)^{k-l}L_l^{k-l}(-b_1c_1)H_l(x)s^{l-k}\ , \tag{4.2}$$

Similar corresponding results can be obtained for generating relation (3.16).

II. Taking $a=c=2$, $b=s=p=0$, $q=-i\sqrt{y}s$, $x=ix/2\sqrt{y}$ and making use of (2.8) in (3.13), we get

$$\exp\left(b_1xs+b_1^2ys^2\right)H_k\left(x\left(1+\frac{2b_1ys}{x}+\frac{c_1}{2xs}\right),y\right)=$$

$$=\sum_{l=0}^{\infty}(c_1)^{k-l}L_l^{k-l}(-b_1c_1)H_l(x,y)s^{l-k}\ . \tag{4.3}$$

which for $s=1$ reduces to [6, p. 84, (21)]. Further using (2.7b) in above equation, we get

$$\exp\left(b_1xs-\frac{b_1^2s^2}{2}\right)He_k\left(x\left(1-\frac{b_1 s}{x}+\frac{c_1}{2xs}\right)\right)=$$

$$=\sum_{l=0}^{\infty}(c_1)^{k-l}L_l^{k-l}(-b_1c_1)He_l\left(x\right)s^{l-k}\ . \tag{4.4}$$

Similarly, we can obtain results corresponding to generating relation (3.16).

III. Taking $a = p = 0$, $b = 1$, $q = -ys$ and $c = -2zs$, we obtain

$$H_k^{(3,2)}\left(x\left(1+\frac{b_1ys}{x}-\frac{2c_1z}{yx}\right),y,z\right)=$$

$$=\sum_{l=0}^{\infty}(c_1)^{k-l}L_l^{k-l}(-b_1c_1)(-y)^{l-k}H_l^{(3,2)}(x,y,z)s^{l-k}\ , \tag{4.5}$$

where $H_l^{(3,2)}(x,y,z)$ denotes the Bell type polynomial of three variables defined by the generating function [8, p. 403, (26)]

$$\exp(xt+yt^2+zt^3)=\sum_{n=0}^{\infty}H_n^{(3,2)}(x,y,z)\frac{t^n}{n!}\ . \tag{4.6}$$

Similar results can be obtained for generating relation (3.16).

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