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Bounds for the classical orthogonal polynomials and related special functions

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Abstract[1]. The main object of this lecture is first to survey a number of convenient two-sided inequalities for various families of classical orthogonal polynomials, which were derived for specific ranges of values of their arguments by using matrix and other methods, and then to present several bounding inequalities for the *classical* Jacobi function of the first kind. A number of closely-related or analogous inequalities for numerous other classes of special functions (such as the *classical* Laguerre function) are also considered.

1. Introduction, definitions and preliminaries

A square matrix

$$A = [a_{ij}] \qquad (i, j = 1, \cdots, n)$$

of *complex* numbers a_{ij} $(i, j = 1, \cdots, n)$ is said to have (*strictly*) dominant *main* (or principal) diagonal if the following inequality holds true:

$$|a_{kk}| > \sum_{i \neq k} |a_{ki}| =: \mathcal{R}_k \qquad (\forall k \in \{1, \cdots, n\}) \ , \tag{1.1}$$

that is, if the diagonal element dominates in each row. Indeed, if the matrix A has dominant diagonal, then there exist constants $\sigma_1, \cdots, \sigma_n$ defined by

$$\sigma_k |a_{kk}| = \mathcal{R}_k \qquad (k \in \{1, \cdots, n\}) \tag{1.2}$$

such that

$$0 \leq \sigma_k < 1 \qquad (k \in \{1, \cdots, n\}) \ . \tag{1.3}$$

We begin by recalling here the following results (Theorem 1 and Theorem 2 below), which provide upper and lower bounds on $\det A$ when the matrix A has (strictly) dominant main (or principal) diagonal.

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Theorem 1 (Price [31]). *Let A be an $n \times n$ matrix of complex numbers and let* (1.1) *hold true. Then*

$$\prod_{k=1}^{n} \mathcal{L}_k \leq |\det A| \leq \prod_{k=1}^{n} \mathcal{U}_k \ , \tag{1.4}$$

where

$$\mathcal{L}_k := |a_{kk}| - \sum_{j>k} |a_{kj}| \quad \textit{and} \quad \mathcal{U}_k := |a_{kk}| + \sum_{j>k} |a_{kj}| \tag{1.5}$$

$$(k \in \{1, \cdots, n\}) \ .$$

Theorem 2 (Brenner [2]). *Let A be an $n \times n$ matrix of complex numbers and let* (1.2) *and* (1.3) *hold true. Then*

$$\prod_{k=1}^{n} \mathcal{L}_k^* \leq |\det A| \leq \prod_{k=1}^{n} \mathcal{U}_k^* \tag{1.6}$$

where

$$\mathcal{L}_k^* := |a_{kk}| - \sum_{j>k} \sigma_j |a_{kj}| \quad \textit{and} \quad \mathcal{U}_k^* := |a_{kk}| + \sum_{j>k} \sigma_j |a_{kj}| \tag{1.7}$$

$$(0 \leq \sigma_k < 1 \quad ; \quad k \in \{1, \cdots, n\}) \ .$$

By applying Theorem 1 and Theorem 2, Brenner [3] gave upper and lower bounds for such classical orthogonal polynomials as the Chebyshev polynomials of the first kind:

$$T_n(x) := \sum_{k=0}^{[n/2]} \binom{n}{2k} x^{n-2k} (x^2 - 1)^k \ , \tag{1.8}$$

the Hermite polynomials:

$$H_n(x) := \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \frac{(2k)!}{k!} \ (2x)^{n-2k} \ , \tag{1.9}$$

the Laguerre polynomials:

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \ , \tag{1.10}$$

and the Legendre (or spherical) polynomials:

$$P_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k} \ . \tag{1.11}$$

Motivated essentially by the work of Brenner [3], Srivastava and Brenner [37] subsequently extended the results of Brenner [3] to hold true for a general class of orthogonal polynomials. With a view to presenting these general results of Srivastava and Brenner [37], we introduce some further definitions and notations.

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials, where $p_n(x)$ is a polynomial of degree *precisely* n in x, and let the *inner product*:

$$(p_m, p_n) := \int_a^b p_m(x) p_n(x) \, d\mu(x) = h_n \delta_{m,n} \tag{1.12}$$

$$(m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad ; \quad \mathbb{N} := \{1, 2, 3, \cdots\}) ,$$

where $\delta_{m,n}$ is the Kronecker delta, (a, b) is a finite, one-sided infinite, or two-sided infinite interval on the real axis $\mathbb{R}$, and $d\mu(x)$ is a distribution along that interval. Here

$$h_n = \|p_n\|^2 := (p_n, p_n) \qquad (n \in \mathbb{N}_0) \tag{1.13}$$

and $\mu(x)$ is a non-decreasing function in (a, b). In case $\mu(x)$ is also absolutely continuous on the interval (a, b), we may set

$$\mu'(x) = w(x) ,$$

and then refer to $w(x)$ as the *weight function* of the orthogonal system $\{p_n(x)\}_{n=0}^{\infty}$.

The family of the classical orthogonal polynomials includes, for example, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by

$$P_n^{(\alpha,\beta}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k} \tag{1.14}$$

or, equivalently, by

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1\left(-n, \alpha+\beta+n+1; \alpha+1; \frac{1-x}{2}\right) \tag{1.15}$$

in terms of the Gauss hypergeometric function ${}_2F_1$ which corresponds to a special case

$$p - 1 = q = 1$$

of the generalized hypergeometric function ${}_pF_q$ (with p numerator and q denominator parameters) defined by

$$\begin{aligned} &{}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z) = \\ &= {}_pF_q \left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ \beta_1, \cdots, \beta_q; \end{matrix} \; z \right] := \sum_{h=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \end{aligned} \tag{1.16}$$

$$\big(p, q \in \mathbb{N}_0 \quad ; \quad p \leq q+1 \quad ; \quad p < q+1 \quad \text{and} \quad |z| < \infty \quad ; \quad p = q+1 \quad \text{and}$$

$$z \in \mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\} \quad ; \quad p = q+1 \quad , \quad z \in \partial\mathbb{U} \quad \text{and} \quad \Re(\omega) > 0\big) ,$$

where, for convenience,

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0\,; \;\; \lambda \neq 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}\,; \;\; \lambda \in \mathbb{C}) , \end{cases} \tag{1.17}$$

in terms of Gamma functions, and

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j , \tag{1.18}$$

provided (of course) that no zeros appear in the denominator of (1.16).

Clearly, since

$$(-N)_n = \begin{cases} \dfrac{(-1)^n N!}{(N-n)!} & (n = 0, 1, \cdots, N) \\ 0 & (n = N+1, N+2, N+3, \cdots) \end{cases} \tag{1.19}$$

the series in (1.16) would terminate when one (or more) of the numerator parameters $\alpha_1, \cdots, \alpha_p$ is zero or a negative integer, and then the question of convergence of the series in (1.16) will obviously not arise. Thus, if one of the numerator parameters $\alpha_1, \cdots, \alpha_p$ is a nonpositive integer $-N$, and there are no zeros in the denominator of (1.16), the function ${}_pF_q$ would reduce to what may be called a *hypergeometric polynomial* of degree N in x. For such a hypergeometric polynomial, it is fairly easy to observe from the definition (1.16) that

$$
\begin{aligned}
{}_{p+1}F_q\left[\begin{matrix} -n, & \alpha_1, \cdots, \alpha_p; \\ & \beta_1, \cdots, \beta_q; \end{matrix}\; z\right] &= \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}\,(-z)^n \cdot \\
\cdot\, {}_{q+1}F_p\left[\begin{matrix} -n, & 1-\beta_1-n, \cdots, 1-\beta_q-n; \\ & 1-\alpha_1-n, \cdots, 1-\alpha_p-n; \end{matrix}\; \frac{(-1)^{p+q}}{z}\right] & \qquad (n \in \mathbb{N}_0)
\end{aligned} \tag{1.20}
$$

which can be applied to rewrite the hypergeometric representation (1.15) in another *equivalent* form:

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \binom{\alpha+\beta+2n}{n}\left(\frac{x-1}{2}\right)^n \cdot \\
&\cdot\, {}_2F_1\left(-n, -\alpha-n; -\alpha-\beta-2n; \frac{2}{1-x}\right) .
\end{aligned} \tag{1.21}
$$

When

$$\min\{\Re(\alpha), \Re(\beta)\} > -1 ,$$

the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal with respect to the Beta distribution on $[-1,1]$ as follows:

$$
\begin{aligned}
&\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\, dx = \\
&= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}\, \delta_{m,n}
\end{aligned} \tag{1.22}
$$

$$(m, n \in \mathbb{N}_0 \quad ; \quad \min\{\Re(\alpha), \Re(\beta)\} > -1) .$$

Various other members of the family, which are *special* cases of the Jacobi polynomials, include the Gegenbauer (or ultraspherical) polynomials $C_n^\nu(x)$, where

$$
\begin{aligned}
C_n^{\alpha+1/2}(x) &= \binom{\alpha+n}{n}^{-1}\binom{2\alpha+n}{n} P_n^{(\alpha,\alpha)}(x) = \\
&= \sum_{k=0}^{n} \frac{(\alpha+1/2)_k(\alpha+1/2)_{n-k}}{k!(n-k)!}\, e^{i(n-2k)\theta} \qquad (x = \cos\theta) ,
\end{aligned} \tag{1.23}
$$

the relatively more familiar Legendre (or spherical) polynomials [cf. Equation (1.11)]:

$$P_n(x) = P_n^{(0,0)}(x) = C_n^{1/2}(x) , \tag{1.24}$$

and the Chebyshev polynomials (of the first and second kind) [cf. Equation (1.8)]:

$$\begin{cases} T_n(x) = \dbinom{n-1/2}{n}^{-1} P_n^{(-1/2,-1/2)}(x) = \dfrac{1}{2}\, nC_n^0(x) \\[2ex] U_n(x) = \dfrac{1}{2}\dbinom{n+1/2}{n+1}^{-1} P_n^{(1/2,1/2)}(x) = C_n^1(x)\ , \end{cases} \tag{1.25}$$

where, by definition,

$$C_n^0(x) = \lim_{\lambda\to 0}\left\{\lambda^{-1} C_n^\lambda(x)\right\}\ . \tag{1.26}$$

Two other important members of the family of the classical orthogonal polynomials are the Hermite polynomials [cf. Equation (1.9)]:

$$\begin{aligned} H_n(x) &= \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}\,(2x)^{n-2k} = \\ &= (2x)^n\ {}_2F_0\left[\begin{matrix} -\frac{1}{2}\,n,\ -\frac{1}{2}\,n+\frac{1}{2}; \\ \rule{3cm}{0.4pt}\ ; \end{matrix}\ -\frac{1}{x^2}\right] \end{aligned} \tag{1.27}$$

and the Laguerre polynomials [cf. Equation (1.10)]:

$$L_n^{(\alpha)}(x) = \binom{\alpha+n}{n}\ {}_1F_1\left[\begin{matrix} -n; \\ \alpha+1; \end{matrix}\ x\right]\ . \tag{1.28}$$

Indeed, since

$$H_n(x) = (-1)^n 2^{n/2} n! \lim_{\alpha\to\infty}\left\{\alpha^{-n/2} L_n^{(\alpha)}(\alpha + x\sqrt{2\alpha})\right\}\ , \tag{1.29}$$

$$H_n(x) = n! \lim_{\lambda\to\infty}\left\{\lambda^{-n/2} C_n^\lambda\left(\frac{x}{\sqrt{\lambda}}\right)\right\}\ , \tag{1.30}$$

and

$$L_n^{(\alpha)}(x) = \lim_{\beta\to\infty}\left\{P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)\right\}\ , \tag{1.31}$$

and also since

$$H_{2n+\epsilon}(x) = (-1)^n 2^{2n+\epsilon} n! x^\epsilon L_n^{(\epsilon-1/2)}(x^2) \qquad (\epsilon = 0 \ \text{ or } \ 1)\ , \tag{1.32}$$

many of the properties of the Hermite and Laguerre polynomials can eventually be deduced from those involving the classical Jacobi polynomials.

Another interesting class of orthogonal polynomials is provided by the generalized Bessel polynomials $y_n(x,\alpha,\beta)$ defined by

$$\begin{aligned} y_n(x,\alpha,\beta) &:= \sum_{k=0}^{n} \binom{n}{k}\binom{\alpha+n+k-2}{k} k! \left(\frac{x}{\beta}\right)^k = \\ &= {}_2F_0\left[\begin{matrix} -n, \alpha+n-1; \\ \rule{2.5cm}{0.4pt}\ ; \end{matrix}\ -\frac{x}{\beta}\right]\ , \end{aligned} \tag{1.33}$$

which were studied systematically by Krall and Frink [18] (and, subsequently, by Grosswald [15]). In view of the following limit relationship:

$$y_n(x,\alpha,\beta)=\lim_{\lambda\to\infty}\left\{\frac{n!}{(\lambda)_n}\,P_n^{(\lambda-1,\alpha-\lambda-1)}\left(1+\frac{2\lambda x}{\beta}\right)\right\} \tag{1.34}$$

and of the following easy consequence of (1.20):

$$y_n(x,\alpha,\beta)=n!\left(-\frac{x}{\beta}\right)^n L_n^{(1-\alpha-2n)}\left(\frac{\beta}{x}\right)\ , \tag{1.35}$$

the Bessel polynomials are also recoverable from the classical Jacobi and Laguerre polynomials.

The so-called classical Jacobi, Laguerre, and Hermite polynomials, and many of their aforementioned relatives, are often characterized by one or the other of a number of interesting and useful properties which they are known to have in common (see, for details, Szegö [39] and Erdélyi et al. [8, Vol. II]; see also Srivastava and Manocha [38]).

In the general case of the polynomial system $\{p_n(x)\}_{n=0}^{\infty}$ involved in (1.12) and (1.13), let k_n denote the coefficient of x^n, and k'_n the coefficient of x^{n-1}, in $p_n(x)$. Also let

$$r_n=\frac{k'_n}{k_n}\qquad(n\in\mathbb{N}_0)\ . \tag{1.36}$$

Then it is known that the polynomial system $\{p_n(x)\}_{n=0}^{\infty}$ satisfies a *three-term* recurrence relation in the following form (cf., e.g., [8, Vol. II, p. 158]):

$$p_{n+1}(x)=(A_nx+B_n)p_n(x)-C_np_{n-1}(x)\qquad(n\in\mathbb{N}_0)\ , \tag{1.37}$$

where (by definition and normalization)

$$p_{-1}(x)=0\quad\text{and}\quad p_0(x):=1\ , \tag{1.38}$$

and

$$A_n=\frac{k_{n+1}}{k_n}\ ,\ \ B_n=A_n(r_{n+1}-r_n)\ \text{ and }\ C_n=\frac{k_{n+1}\,k_{n-1}h_n}{k_n^2h_{n-1}}\qquad(n\in\mathbb{N}_0)\ , \tag{1.39}$$

h_n and r_n being given by (1.13) and (1.36), respectively.

The *three-term* recurrence relation (1.37) is readily seen to yield the following family of determinantal representations for $p_{n+1}(x)$:

$$p_{n+1}(x)=\det\begin{bmatrix} A_nx+B_n, & C_n^{1/r}, & \cdots & \cdots \\ C_n^{(r-1)/r}, & A_{n-1}x+B_{n-1}, & C_{n-1}^{1/r}, & \cdots \\ & \ddots & & \\ \cdots & C_2^{(r-1)/r}, & A_1x+B_1, & C_1^{1/r} \\ \cdots & \cdots & C_1^{(r-1)/r}, & A_0x+B_0 \end{bmatrix} \tag{1.40}$$

$$(n\in\mathbb{N}\quad;\quad r\in\mathbb{N})\ ,$$

where (*and throughout this presentation*) it is tacitly assumed that

$$C_j^{1/r}\cdot C_j^{(r-1)/r}=C_j\qquad(j=1,\cdots,n\ ;\ \ r\in\mathbb{N})\ , \tag{1.41}$$

each of the matrix elements not displayed in (1.40) being zero.

In the simple cases when $r = 1$ and $r = 2$, if we apply Theorem 1 to the determinantal representations resulting from (1.40), we are led to Theorem 3 and Theorem 4, respectively (cf. [37]).

Theorem 3. *For all x such that*

$$\left\{\begin{array}{l} |A_0x + B_0| > 1 \\ |A_jx + B_j| > 1 + |C_j| \qquad (j = 1, \cdots, n-1) \\ |A_nx + B_n| > |C_n| \ , \end{array}\right. \tag{1.42}$$

the polynomial $p_{n+1}(x)$ satisfies the following inequalities:

$$\begin{aligned} &|A_0x + B_0| \prod_{j=1}^{n} \{|A_jx + B_j| - |C_j|\} \leq |p_{n+1}(x)| \leq \\ &\leq |A_0x + B_0| \prod_{j=1}^{n} \{|A_jx + B_j| + |C_j|\} \qquad (n \in \mathbb{N}) \ . \end{aligned} \tag{1.43}$$

Theorem 4. *For all x such that*

$$\left\{\begin{array}{l} |A_0x + B_0| > \sqrt{|C_1|} \\ |A_jx + B_j| > \sqrt{|C_j|} + \sqrt{|C_{j+1}|} \qquad (j = 1, \cdots, n-1) \\ |A_nx + B_n| > \sqrt{|C_n|} \ , \end{array}\right. \tag{1.44}$$

the polynomial $p_{n+1}(x)$ satisfies the following inequalities:

$$\begin{aligned} &|A_0x + B_0| \prod_{j=1}^{n} \{|A_jx + B_j| - \sqrt{|C_j|}\} \leq |p_{n+1}(x)| \leq \\ &\leq |A_0x + B_0| \prod_{j=1}^{n} \{|A_jx + B_j| + \sqrt{|C_j|}\} \qquad (n \in \mathbb{N}) \ . \end{aligned} \tag{1.45}$$

In the general case of the determinantal expression given by (1.40), Theorem 1 would similarly yield the following unification (and generalization) of Theorem 3 and Theorem 4.

Theorem 5. *For all x such that*

$$\left\{\begin{array}{l} |A_0x + B_0| > |C_1|^{(r-1)/r} \\ |A_jx + B_j| > |C_j|^{1/r} + |C_{j+1}|^{(r-1)/r} \qquad (j = 1, \cdots, n-1) \\ |A_nx + B_n| > |C_n|^{1/r} \qquad (r \in \mathbb{N}) \ , \end{array}\right. \tag{1.46}$$

the polynomial $p_{n+1}(x)$ satisfies the following inequalities:

$$\begin{aligned} &|A_0x + B_0| \prod_{j=1}^{n} \left\{|A_jx + B_j| - |C_j|^{1/r}\right\} \leq |p_{n+1}(x)| \leq \\ &\leq |A_0x + B_0| \prod_{j=1}^{n} \left\{|A_jx + B_j| + |C_j|^{1/r}\right\} \qquad (n, r \in \mathbb{N}) \ . \end{aligned} \tag{1.47}$$

Remark 1. In its *special* cases when $r = 1$ and $r = 2$, Theorem 5 would reduce immediately to Theorem 3 and Theorem 4, respectively.

Remark 2. Significantly better bounds for the polynomials $p_{n+1}(x)$ $(n \in \mathbb{N})$ than those provided by Theorems 3, 4 and 5 above can be obtained by applying Theorem 2 instead of Theorem 1.

Remark 3. The constants A_n, B_n and C_n, occurring in the *three-term* recurrence relation (1.37), depend upon the normalization of the polynomials $p_n(x)$ $(n \in \mathbb{N})$. Hence, for different normalizations of $p_n(x)$ $(n \in \mathbb{N})$, the correspondingly different inequalities (1.42), (1.45) and (1.47) are obtained.

2. DIRECT DEMONSTRATIONS OF THEOREMS 3, 4 AND 5

In this section we give a *direct* proof of each of Theorems 3, 4 and 5 *without* using their matrix derivation based heavily upon the upper and lower bounds provided by Theorem 1 (see also [32]).

Let the polynomial $p_n(x)$ be defined by the recurrence relation (1.37) and let x satisfy each of the constrainst in (1.42). Also, for convenience, let us suppose that

$$C_0 = 0\ . \tag{2.1}$$

Then we shall first prove (by the principle of mathematical induction) that

$$\begin{aligned} &\{|A_n x + B_n| - |C_n|\}|p_n(x)| \leq |p_{n+1}(x)| \leq \\ &\quad \leq \{|A_n x + B_n| + |C_n|\}|p_n(x)| \qquad (n \in \mathbb{N}_0) \end{aligned} \tag{2.2}$$

which obviously holds true when $n = 0$ by virtue of (1.37) for $n = 0$.

Suppose that the inequalities in (2.2) hold true for $n \in \mathbb{N}_0$. Then, since

$$|A_n x + B_n| - |C_n| > 0\ , \tag{2.3}$$

in view of the last inequality in (1.42), it follows immediately from (2.2) that

$$|p_n(x)| \leq |p_{n+1}(x)| \qquad (n \in \mathbb{N}_0)\ . \tag{2.4}$$

Now, by making use of the recurrence relation (1.37) with n replaced by $n+1$, we find that

$$\begin{aligned} &|\,|A_{n+1}x + B_{n+1}|\,|p_{n+1}(x)| - |C_{n+1}|\,|p_n(x)|\,| \leq |p_{n+2}(x)| \leq \\ &\quad \leq |A_{n+1}x + B_{n+1}|\,|p_{n+1}(x)| + |C_{n+1}|\,|p_n(x)|\ , \end{aligned} \tag{2.5}$$

which readily yields

$$\begin{aligned} &\{|A_{n+1}x + B_{n+1}| - |C_{n+1}|\}|p_{n+1}(x)| \leq |p_{n+2}(x)| \leq \\ &\quad \leq \{|A_{n+1}x + B_{n+1}| + |C_{n+1}|\}|p_{n+1}(x)|\ , \end{aligned} \tag{2.6}$$

by means of the inequalities in (2.3) and (2.4).

The assertion (2.6) shows that (2.2) holds true also when n is replaced by $n+1$, thus completing the proof of (2.2) by the principle of mathematical induction on n. Finally, Theorem 3 is proven by applying (2.2) *iteratively.*

Theorem 4 and Theorem 5 can be proved in an analogous manner. In particular, in the case of Theorem 4, the induction hypothesis:

$$\begin{aligned} &\{|A_n x + B_n| - \sqrt{|C_n|}\}|p_n(x)| \leq |p_{n+1}(x)| \leq \\ &\quad \leq \{|A_n x + B_n| + \sqrt{|C_n|}\}|p_n(x)| \qquad (n \in \mathbb{N}_0)\ , \end{aligned} \tag{2.7}$$

implies the following inequalities:

$$\sqrt{|C_{n+1}|}\ |p_n(x)| \leq |p_{n+1}(x)| \qquad (n \in \mathbb{N}_0)\ . \tag{2.8}$$

3. SIGNIFICANT IMPROVEMENTS OF THE UPPER BOUNDS IN THEOREMS 3, 4 AND 5

With a view to deriving significantly sharper upper bounds for $|p_{n+1}(x)|$ $(n \in \mathbb{N}_0)$ than those given by Theorems 3, 4 and 5, let the orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ in Section 1 be normalized so that

$$k_n > 0 \qquad (n \in \mathbb{N}) \ . \tag{3.1}$$

Then, clearly, we see that

$$A_n > 0 \quad \text{and} \quad C_n > 0 \qquad (n \in \mathbb{N}_0) \ . \tag{3.2}$$

Next, according to a well-known result (cf., e.g., Szegö [39, p. 44, Theorem 3.3.1]), the n zeros of $p_n(x)$ are located in the *interior* of the interval $[a,b]$. Hence we have

$$p_n(x) > 0 \qquad (x \in [b,\infty) \ ; \ \ n \in \mathbb{N}_0) \tag{3.3}$$

and

$$(-1)^n p_n(x) > 0 \qquad (x \in (-\infty,a] \ ; \ \ n \in \mathbb{N}_0) \ . \tag{3.4}$$

Furthermore, it follows from (3.3) and the recurrence relation (1.37) that

$$0 < p_{n+1}(x) \leq (A_n x + B_n) p_n(x) \qquad (x \in [b,\infty) \ ; \ \ n \in \mathbb{N}) \ . \tag{3.5}$$

We also have

$$A_n x + B_n \leq 0 \qquad (x \in [b,\infty) \ ; \ \ n \in \mathbb{N}_0) \ ; \tag{3.6}$$

otherwise $p_{n+1}(x)$ would become negative for $x \in [b,\infty)$.

By iterating the inequality in (3.5), it is easily proved that

$$p_{n+1}(x) \leq \prod_{j=0}^{n} (A_j x + B_j) \qquad (x \in [b,\infty) \ ; \ \ n \in \mathbb{N}_0) \ . \tag{3.7}$$

In a similar manner, we can obtain the following inequality:

$$|p_{n+1}(x)| \leq \prod_{j=0}^{n} (A_j x + B_j) \qquad (x \in (-\infty,a] \ ; \ \ n \in \mathbb{N}_0) \ . \tag{3.8}$$

4. APPLICATIONS AND CONSEQUENCES INVOLVING THE JACOBI AND RELATED POLYNOMIALS

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which are defined already by (1.14) or (1.15), satisfy the following three-term recurrence relation [8, Vol. II, p. 169]:

$$\begin{aligned} &2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x) = \\ &= (2n+\alpha+\beta+1)[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x + \alpha^2 - \beta^2]P_n^{(\alpha,\beta)}(x) - \\ &-2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(x) \qquad (n \in \mathbb{N}_0) \ , \end{aligned} \tag{4.1}$$

where (by definition and normalization)

$$P_0^{(\alpha,\beta)}(x) = 1 \quad \text{and} \quad P_{-1}^{(\alpha,\beta)}(x) = 0 \ . \tag{4.2}$$

Thus, if we let

$$D_n = 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) \qquad (n \in \mathbb{N}_0) \ , \tag{4.3}$$

then we have

$$\left\{\begin{aligned} D_nA_n &= (2n+\alpha+\beta)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)\\ D_nB_n &= (\alpha^2-\beta^2)(2n+\alpha+\beta+1)\\ D_nC_n &= 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2) \qquad (n\in\mathbb{N}_0)\ . \end{aligned}\right. \tag{4.4}$$

With A_n, B_n and C_n defined by (4.4), the inequalities in (1.43) (that is, in Theorem 3) are satisfied when

$$|x| \geq R + S_1\ ,$$

while those in (1.45) (that is, in Theorem 4) are satisfied when

$$|x| \geq R + S_2\ ,$$

where

$$R := \max_{0\leq j\leq n}\left|\frac{\beta^2-\alpha^2}{(2j+\alpha+\beta)(2j+\alpha+\beta+2)}\right| \tag{4.5}$$

$$\begin{aligned} S_1 := \max_{0\leq j\leq n-1}&\left|\frac{2(j+1)(j+\alpha+\beta+1)}{(2j+\alpha+\beta+1)(2j+\alpha+\beta+2)}\right| +\\ &+ \max_{1\leq j\leq n}\left|\frac{2(j+\alpha)(j+\beta)}{(2j+\alpha+\beta)(2j+\alpha+\beta+1)}\right|\ , \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} S_2 := \max_{0\leq j\leq n-1}&\left|\frac{2(j+1)(j+\alpha+\beta+1)}{(2j+\alpha+\beta+1)(2j+\alpha+\beta+2)}\ \cdot\right.\\ &\left.\cdot\sqrt{\frac{(j+\alpha+1)(j+\beta+1)(2j+\alpha+\beta+4)}{(j+2)(j+\alpha+\beta+2)(2j+\alpha+\beta+2)}}\right| +\\ + \max_{1\leq j\leq n}&\left|\frac{2}{(2j+\alpha+\beta+1)}\sqrt{\frac{(j+1)(j+\alpha)(j+\beta)(j+\alpha+\beta+1)}{(2j+\alpha+\beta)(2j+\alpha+\beta+2)}}\right|\ . \end{aligned} \tag{4.7}$$

We are thus led from Theorems 3 and 4 to the following result for the classical Jacobi polynomials.

Theorem 6. *Let R, S_1 and S_2 be defined by* (4.5), (4.6) *and* (4.7), *respectively. Then the Jacobi polynomial $P_{n+1}^{(\alpha,\beta)}(x)$ $(n\in\mathbb{N}_0)$ satisfies the following two-sided inequalities:*

$$\begin{aligned} \prod_{j=1}^{n}\{|A_jx+B_j|-|C_j|\} &\leq \frac{1}{2}\,|P_{n+1}^{(\alpha,\beta)}(x)|\cdot|(\alpha+\beta+2)x+(\alpha-\beta)|^{-1} \leq\\ &\leq \prod_{j=1}^{n}\{|A_jx+B_j|+|C_j|\} \quad (n\in\mathbb{N}\ ;\ |x|\geq R+S_1) \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \prod_{j=1}^{n}\{|A_jx+B_j|-\sqrt{|C_j|}\} &\leq \frac{1}{2}\,|P_{n+1}^{(\alpha,\beta)}(x)||(\alpha+\beta+2)x+(\alpha-\beta)|^{-1} \leq\\ &\leq \prod_{j=1}^{n}\{|A_jx+B_j|+\sqrt{|C_j|}\} \quad (n\in\mathbb{N}\ ;\ |x|\geq R+S_2)\ , \end{aligned} \tag{4.9}$$

where A_j, B_j *and* C_j $(j = 1, \cdots, n)$ *are defined by* (4.4).

For suitable *special* values of the parameters α and β, Theorem 6 would, in view of such relationships as those given in Section 1, yield upper and lower bounds for the Gegenbauer (or ultraspherical) polynomials, the Legendre (or spherical) polynomials, the Chebyshev polynomials of the first and second kind, and so on. In particular, if we suppose that

$$\min\{\Re(\alpha), \Re(\beta)\} > -1 \quad \text{and} \quad \Re(\alpha+\beta) \geq 0 \ , \tag{4.10}$$

then it can be seen from the definitions (4.5), (4.6) and (4.7) that

$$R = \left| \frac{\beta - \alpha}{\alpha + \beta + 2} \right| \quad , \quad S_1 \leq 3 \quad , \quad \text{and} \quad S_2 \leq \surd(2|\alpha + \beta + 4|) \ . \tag{4.11}$$

Thus we obtain the following corollary.

Corollary 1. *Let each of the inequalities in* (4.10) *be satisfied. Then the bounds in* (4.8) *and* (4.9) *hold true when*

$$|x| \geq 3 + \left| \frac{\beta - \alpha}{\alpha + \beta + 2} \right| \tag{4.12}$$

and

$$|x| \geq \surd(2|\alpha + \beta + 4|) + \left| \frac{\beta - \alpha}{\alpha + \beta + 2} \right| \ , \tag{4.13}$$

respectively.

It may be remarked in passing that the condition (4.12) is satisfied for all *admissible* values of the parameters α and β if, for instance, $|x| \geq 4$. Furthermore, when

$$\alpha = \beta = \nu - \frac{1}{2} \quad \text{and} \quad \Re(\nu) \geq \frac{1}{2} \ , \tag{4.14}$$

Equations (4.8) and (4.9) would provide us with bounds for the Gegenbauer (or ultraspherical) polynomial $C_{n+1}^{\nu}(x)$ $(n \in \mathbb{N})$ for

$$|x| \geq 3 \quad \text{and} \quad |x| \geq \surd(2|2\nu + 3|) \ , \tag{4.15}$$

respectively.

Applications of Theorems 3 and 4 (and also those of Theorem 5) to other classical orthogonal polynomials can be derived along the lines detailed above. Moreover, the sharper upper bounds given by (3.7) and (3.8) can also be suitably specialized for the cases of the Jacobi polynomials and their such related polynomials as (for example) the Lagrange polynomials in two and more variables (see, for details, [7]).

5. TWO-SIDED INEQUALITIES FOR OTHER FAMILIES OF SPECIAL FUNCTIONS AND POLYNOMIALS

The matrix methods of Srivastava and Brenner [37] for finding upper and lower bounds for a general class of orthogonal polynomials were subsequently applied by a number of authors in order to obtain upper and lower bounds for other families of special functions and polynomials. In this section we choose to recall some of these developments which were motivated essentially by the aforementioned work of Srivastava and Brenner [37].

If, for convenience, we set

$$\mathcal{F}(a) := {}_2F_1(a, b; c; x) \tag{5.1}$$

in terms of the Gauss hypergeometric function defined by (1.16) with, of course,

$$p - 1 = q = 1 \ ,$$

then it is known that (cf., e.g., Luke [23, p. 259, Equation 6.2.2(2)])

$$\mathcal{F}(a + n + 1) = A_n \mathcal{F}(a + n) + B_n \mathcal{F}(a + n - 1) \ , \tag{5.2}$$

where

$$A_n = \frac{(a + n)(2 - x) + bx - c}{(a + n)(1 - x)} \quad \text{and} \quad B_n = \frac{c - a - n}{(a + n)(1 - x)} \ . \tag{5.3}$$

Making use of the three-term recurrence relation (5.2), just as Srivastava and Brenner [37] had applied (1.37) earlier, Buschman [5] derived upper and lower bounds for $\mathcal{F}(a + n)$ and, by the familiar principle of confluence, for Kummer's confluent hypergeometric function:

$${}_1F_1(a + n + 1; c; x) \ ,$$

which corresponds to (1.16) when

$$p = q = 1$$

(cf. Buschman [5, p. 304, Theorem 1 and Theorem 2]).

Inequalities for the Gauss hypergeometric function were indeed given by earlier authors, but (as also remarked by Buschman [5, p. 304]) the principle of confluence cannot usefully be applied to the inequalities of Flett [10], whereas the results of Ross and Bordelon [33] treat denominator parameters (and do not cover numerator parameters).

Analogous procedure was applied by Buschman [5, p. 305, Theorem 3 and Theorem 4] in order to obtain upper and lower bounds for Tricomi's confluent hypergeometric function (cf., e.g., Erdélyi et al. [8, Vol. I, p. 255 et seq.]):

$$\Psi(a;\ c + n + 1;\ x)$$

and for the modified Bessel function (cf. Watson [42, p. 78 et seq.]):

$$K_{\nu+n+1}(x) \ .$$

Several further improvements and refinements in some of the afore cited results of Buschman [5, p. 304, Theorem 1 and Theorem 2] were given subsequently by Joshi and Arya [16]. A. Srivastava [34], on the other hand, made use of the aforementioned matrix methods of Srivastava and Brenner [37] with a view to obtaining upper and lower bounds for the following generalization of Pasternack's polynomials (cf., e.g., [38, p. 183, Problem 43]):

$$\mathcal{P}_n(x; \alpha, \beta, \gamma) := {}_3F_2(-n, n + \alpha, x; \beta, \gamma; 1) \qquad (n \in \mathbb{N}_0) \ , \tag{5.4}$$

in terms of the Clausen hypergeometric function defined by (1.16) with, of course,

$$p - 1 = q = 2 \ .$$

6. BOUNDS FOR THE CLASSICAL JACOBI FUNCTION AND RELATED SPECIAL FUNCTIONS

In the usual notation, the *classical* Jacobi function $P_\nu^{(\alpha,\beta)}(z)$ $(\nu \in \mathbb{C})$ of the *first kind* is defined by (see, for example, [23, p. 433])

$$\begin{aligned} P_\nu^{(\alpha,\beta)}(z) &:= \sum_{k=0}^{\infty} \binom{\nu+\alpha}{\nu-k}\binom{\nu+\beta}{k}\left(\frac{z-1}{2}\right)^k\left(\frac{z+1}{2}\right)^{\nu-k} = \\ &= \binom{\nu+\alpha}{\nu}\,{}_2F_1\left(-\nu,\alpha+\beta+\nu+1;\alpha+1;\frac{1-z}{2}\right) \qquad (\nu\in\mathbb{C}) \end{aligned} \tag{6.1}$$

in terms of the Gauss hypergeometric function ${}_2F_1$. Here, *just as in the preceding sections*, we make use of a generalized binomial coefficient given by

$$\binom{\kappa}{\mu} := \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\mu+1)\Gamma(\mu+1)} =: \binom{\kappa}{\kappa-\mu} \qquad (\kappa,\mu\in\mathbb{C}) \;. \tag{6.2}$$

Together with the *classical* Jacobi function $Q_\nu^{(\alpha,\beta)}(z)$ $(\nu \in \mathbb{C})$, of the *second kind*, which possesses a hypergeometric representation given by (cf. [23, p. 449]; see also [38, p. 453, Problem 26])

$$\begin{aligned} Q_\nu^{(\alpha,\beta)}(z) &= 2^{\alpha+\beta+\nu}B(\alpha+\nu+1,\beta+\nu+1)(z-1)^{-\alpha-\nu-1}(z+1)^{-\beta}\cdot \\ &\cdot {}_2F_1\left(\nu+1,\alpha+\nu+1;\alpha+\beta+2\nu+2;\frac{2}{1-z}\right) \qquad (\nu\in\mathbb{C}) \;, \end{aligned} \tag{6.3}$$

these *classical* Jacobi functions $P_\nu^{(\alpha,\beta)}(z)$ $(\nu \in \mathbb{C})$ and $Q_\nu^{(\alpha,\beta)}(z)$ $(\nu \in \mathbb{C})$ are known to satisfy the following differential equation:

$$(1-z^2)\,\frac{d^2w}{dz^2} + [\beta-\alpha-(\alpha+\beta+2)z]\,\frac{dw}{dz} + (\alpha+\beta+\nu+1)\nu w = 0 \tag{6.4}$$

$$\left(w \equiv P_\nu^{(\alpha,\beta)}(z)\right) \;,$$

$B(\alpha,\beta)$ being the familiar Beta function defined by

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{6.5}$$

$$(\min\{\Re(\alpha),\Re(\beta)\}>0) \;.$$

Now, for the *Riemann-Liouville fractional derivative operator* $\mathcal{D}_z^\mu$ of (real or complex) order μ defined by (cf. [9, Vol. II, p. 181 et seq.]; see also [17] and [36])

$$\mathcal{D}_z^\mu\{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^z (z-t)^{-\mu-1}f(t)\,dt & (\Re(\mu)<0) \\[2ex] \dfrac{d^m}{dz^m}\left\{\mathcal{D}_z^{\mu-m}\{f(z)\}\right\} & (m-1\le\Re(\mu)<m\;\;(m\in\mathbb{N})) \;, \end{cases} \tag{6.6}$$

it is known that

$$\mathcal{D}_z^\mu\{z^\lambda\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\,z^{\lambda-\mu} \qquad (\Re(\lambda)>-1) \tag{6.7}$$

and that

$$\mathcal{D}_z^\mu\{f(z)\cdot g(z)\} = \sum_{j=0}^{\infty}\binom{\mu}{j}\mathcal{D}_z^{\mu-j}\{f(z)\}\cdot D_z^j\{g(z)\} \qquad (\mu\in\mathbb{C}) \;, \tag{6.8}$$

which, in the special case when

$$\mu = m \qquad (m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad ; \quad \mathbb{N} := \{1, 2, 3, \cdots\}) \ ,$$

yields the familiar Leibniz rule of calculus, D_z^j being the *ordinary* derivative operator of order $j \in \mathbb{N}_0$ with respect to z.

By *correctly* applying these last properies (6.7) and (8.8), it is fairly straightforward to observe that the *first-kind* classical Jacobi function $P_\nu^{(\alpha,\beta)}(z)$ $(\nu \in \mathbb{C})$, would satisfy the following Rodrigues formula:

$$\begin{aligned} P_\nu^{(\alpha,\beta)}(z) = \frac{(-2)^{-\nu}}{\Gamma(\nu+1)}\,(1-z)^{-\alpha}(1+z)^{-\beta}\cdot \\ \cdot\mathcal{D}_z^\nu\{(1-z)^{\alpha+\nu}(1+z)^{\beta+\nu}\} \qquad (\nu \in \mathbb{C}) \ , \end{aligned} \tag{6.9}$$

only in the case of the *classical* Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ $(n \in \mathbb{N}_0)$, that is, *only* when

$$\nu = n \qquad (n \in \mathbb{N}_0) \ .$$

The obviously erroneous formula (6.9) *with* $\nu \in \mathbb{C}$ was interpreted as the definition of the so-called *fractional Jacobi function* in a recent seemingly invalid rederivation of some of the familiar properties of the well-known (rather *classical*) Jacobi function $P_\nu^{(\alpha,\beta)}(z)$ $(\nu \in \mathbb{C})$, by Gogovcheva and Boyadjiev [12, p. 433, Definition 2].

In our presentation here, we aim now at deriving several bounding inequalities for the Jacobi function $|P_\nu^{(\alpha,\beta)}(z)|$ $(\nu \in \mathbb{C})$, which is defined by (6.1) above. Our method is based largely upon some results derived in a recent work by Pogány and Srivastava [30] (see also [35]).

7. A SET OF USEFUL LEMMAS AND OTHER PRELIMINARY RESULTS

For the classical Laguerre function $L_\nu^{(\mu)}(z)$ $(\nu \in \mathbb{C})$, defined, in terms of the confluent hypergeometric function ${}_1F_1$, by

$$L_\nu^{(\mu)}(z) := \sum_{k=0}^\infty \binom{\mu+\nu}{\nu-k}\frac{(-z)^k}{k!} = \binom{\mu+\nu}{\nu}\,{}_1F_1(-\nu;\mu+1;z) \qquad (\nu \in \mathbb{C}) \ , \tag{7.1}$$

a bounding inequality (asserted by Lemma 1 below) was proven by Eric Russell Love (1912–2001) [21] by making use of the following well-known integral representation:

$$L_\nu^{(\mu)}(x) = \frac{e^x x^{-\mu/2}}{\Gamma(\nu+1)}\int_0^\infty e^{-t}t^{\nu+\mu/2}J_\mu(2\sqrt{xt})\,dt \tag{7.2}$$

$$(x \geq 0 \ ; \ \Re(\mu+\nu) > -1)$$

involving the first-kind Bessel function $J_\nu(z)$ of order ν, defined by

$$J_\nu(z) := \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{\nu+2n} \tag{7.3}$$

$$(z \in \mathbb{C} \setminus (-\infty, 0) \ ; \ \nu \in \mathbb{C}) \ .$$

Lemma 1. *The following bounding inequality holds true for the Laguerre function* $L_\nu^{(\mu)}(x)$*:*

$$|L_\nu^{(\mu)}(x)| \leq \frac{\Gamma(\Re(\mu+\nu)+1)\Gamma(\Re(\mu)+(1/2))}{|\Gamma(\nu+1)||\Gamma(\Re(\mu)+1)||\Gamma(\mu+(1/2))|}\,e^x \ , \tag{7.4}$$

$$\left(x > 0 \ ; \ \Re(\mu) > -\frac{1}{2} \ ; \ \Re(\mu+\nu) > -1\right) .$$

Recently, Pogány and Srivastava [30] applied some inequalities due to Yudell Leo Luke (1918–1983) [22], Landau [20], Olenko [26] and Krasikov [19] with a view to presenting several remarkable improvements over Love's inequality (7.2). We recall here the bounding inequalities of Pogány and Srivastava [30] in the form of the following lemmas.

Lemma 2. *The following inequality holds true for the Laguerre function* $L_\nu^{(\mu)}(x)$*:*

$$\left|L_\nu^{(\mu)}(x)\right| \leq \frac{(\mu+\nu)e^x}{\nu(\mu-|\nu|)B(\mu,\nu)(1+x)} \tag{7.5}$$

$$(x \geq 0 \ ; \ \mu > |\nu| \ ; \ \nu > -1) ,$$

where $B(\alpha,\beta)$ *denotes the familiar Beta function defined already by* (6.5)*.*

Lemma 3. *The following bounding inequality holds true for the Laguerre function* $L_\nu^{(\mu)}(x)$*:*

$$\left|L_\nu^{(\mu)}(x)\right| \leq m_\nu^\mu(x) \frac{e^x x^{-\mu/2}}{\Gamma(\nu+1)} \tag{7.6}$$

$$\left(x > 0 \ ; \ \mu > 0 \ ; \ \nu > -1 \ ; \ \mu + 2\nu > -\frac{3}{2}\right) ,$$

where

$$\begin{aligned} m_\nu^\mu(x) := \min_{x,\mu,\nu} \Bigg\{ & \frac{b_L \Gamma(\mu/2 + \nu + 1)}{\mu^{1/3}} , \\ & \frac{c_L \Gamma(\mu/2 + \nu + 5/6)}{\sqrt[3]{2}\, x^{1/6}} , \frac{d_O \Gamma(\mu/2 + \nu + 3/4)}{\sqrt{2}\, x^{1/4}} \Bigg\} , \end{aligned} \tag{7.7}$$

and the coefficients b_L*,* c_L *and* d_O *are given, respectively, by*

$$b_L := \sqrt[3]{2} \sup_{x \in \mathbb{R}^+} \{\mathrm{Ai}(x)\} , \tag{7.8}$$

$$c_L := \sup_{x \in \mathbb{R}^+} \left\{ x^{1/3} J_0(x) \right\} \tag{7.9}$$

and

$$d_O := b_L \sqrt{\mu^{1/3} + \frac{\alpha_1}{\mu^{1/3}} + \frac{3\alpha_1^2}{10\mu}} \qquad (\mu > 0) \tag{7.10}$$

in terms of the Bessel function $J_\nu(z)$ *defined by* (7.3) *and the familiar Airy function* $\mathrm{Ai}(x)$ *defined by*

$$\mathrm{Ai}(x) := \frac{\pi}{3} \sqrt{\frac{x}{3}} \left[J_{-1/3}\left(2\left(\frac{x}{3}\right)^{3/2}\right) + J_{1/3}\left(2\left(\frac{x}{3}\right)^{3/2}\right) \right] . \tag{7.11}$$

Lemma 4. *The following bounding inequality holds true for the Laguerre function* $L_\nu^{(\mu)}(x)$ *when* $x>0$, $\mu>0$ *and* $r\in(0,2)$*:*

$$\left|L_\nu^{(\mu)}(x)\right| \leqq \frac{\sqrt{\Gamma(\mu+2\nu+1/2)}e^x x^{-\mu/2}}{\Gamma(\nu+1)(2-r)^{(\mu/2)-\nu+1/4}}\cdot$$

$$\cdot\left(\frac{d_O^2}{2\sqrt{x}}\left[1-\exp\left(-\frac{r}{16x}\left[\lambda+(\lambda+1)^{2/3}\right]\right)\right]+\right.$$

$$\left.+\frac{4\mathcal{K}_\mu}{\pi r^{3/2}}\,\Gamma\left(\frac{3}{2},\frac{r}{16x}[\lambda+(\lambda+1)^{2/3}]\right)\right)^{1/2}, \tag{7.12}$$

where

$$\lambda:=(2\mu+1)(2\mu+3) \quad \textit{and} \quad \mathcal{K}_\mu:=[(2\mu+1)(2\mu+3)+1]^{2/3}-2(2\mu+1)\ ,$$

$\Gamma(z,\kappa)$ *being the incomplete Gamma function of the second kind, defined by*

$$\Gamma(z,\kappa):=\int_\kappa^\infty t^{z-1}e^{-t}\,dt \qquad (\kappa\in\mathbb{C}\ ;\ \Re(z)>0 \quad \textit{when} \quad \kappa=0)\ . \tag{7.13}$$

8. BOUNDING INEQUALITIES FOR THE JACOBI FUNCTION OF THE FIRST KIND

First of all, in light the hypergeometric representations in (6.1) and (7.1), we find from the Eulerian integral [cf. Equation (7.13)]:

$$\Gamma(z,0)\equiv\Gamma(z):=\int_0^\infty t^{z-1}\,e^{-t}\,dt \qquad (\Re(z)>0) \tag{8.1}$$

that

$$\Gamma(\alpha+\beta+\nu+1)P_\nu^{(\alpha,\beta)}(z)=\int_0^\infty t^{\alpha+\beta+\nu}e^{-t}L_\nu^{(\alpha)}\left(\frac{1}{2}(1-z)t\right)dt \tag{8.2}$$

$$(\Re(\alpha+\beta+\nu)>-1)\ ,$$

which, in the special case when

$$\nu=n \qquad (n\in\mathbb{N}_0)\ , \tag{8.3}$$

happens to be a well-known result (cf., e.g., [38, p. 94, Problem 24]).

By writing (8.2) in the following (*relatively simpler*) form:

$$\left|P_\nu^{(\alpha,\beta)}(1-2x)\right| \leqq \frac{1}{\Gamma(\alpha+\beta+\nu+1)}\int_0^\infty t^{\alpha+\beta+\nu}e^{-t}\left|L_\nu^{(\alpha)}(xt)\right|dt \tag{8.4}$$

$$(\alpha,\beta,\nu\in\mathbb{R}\ ;\ \alpha+\beta+\nu>-1)\ ,$$

and then appealing to the corresponding version of Love's inequality (7.4), we obtain Theorem 7 below.

Theorem 7. *The following bounding inequality holds true for the classical Jacobi function of the first kind:*

$$\left|P_\nu^{(\alpha,\beta)}(1-2x)\right| \leqq \binom{\nu+\alpha}{\nu}(1-x)^{-\alpha-\beta-\nu-1} \tag{8.5}$$

$$(0<x<1\ ;\ \alpha>-1\ ;\ \nu+\alpha>-1\ ;\ \alpha+\beta+\nu>-1)\ .$$

In a similar manner, we can apply Lemmas 2 and 3 with a view to deducing the results asserted by Theorems 8 and 9 below. In particular, in our proof of Theorem 8, we make use *also* of the following known result [9, p. 137, Entry 4.3(7)]:

$$\int_0^\infty t^\sigma (t+\kappa)^{-1} e^{-st}\,dt = \kappa^\sigma e^{\kappa s}\Gamma(\sigma+1)\Gamma(-\sigma,\kappa s) \tag{8.6}$$

$$(\Re(s) > 0 \ ; \ \Re(\sigma) > -1 \ ; \ |\arg(\kappa)| < \pi) \ ,$$

where $\Gamma(z,\kappa)$ is the incomplete Gamma function of the second kind defined by (7.13). The details involved are being left as an exercise for the interested reader.

Theorem 8. *The following bounding inequality holds true for the classical Jacobi function of the first kind:*

$$\left|P_\nu^{(\alpha,\beta)}(1-2x)\right| \leqq \frac{\alpha+\nu}{\nu(\alpha-|\nu|)B(\alpha,\nu)}\; x^{-\alpha-\beta-\nu-1}e^{(1-x)/x} \,.$$

$$\cdot\Gamma\left(-\alpha-\beta-\nu,\frac{1-x}{x}\right) \tag{8.7}$$

$$(0 < x < 1 \ ; \ \alpha > |\nu| \ ; \ \nu > -1 \ ; \ \alpha+\beta+\nu > -1) \ .$$

Theorem 9. *The following bounding inequality holds true for the classical Jacobi function of the first kind:*

$$\left|P_\nu^{(\alpha,\beta)}(1-2x)\right| \leqq \mathcal{M}_\nu^{\alpha,\beta}(x)\frac{x^{-\alpha/2}(1-x)^{-(\alpha/2)-\beta-\nu-1}}{\Gamma(\nu+1)\Gamma(\alpha+\beta+\nu+1)} \tag{8.8}$$

$$\Bigg(0 < x < 1 \ ; \ \alpha > 0 \ ; \ \alpha+2\nu > -\frac{3}{2} \ ;$$

$$\alpha+2(\beta+\nu) > -\frac{3}{2} \ ; \ \min\{\nu,\alpha+\beta+\nu\} > -1\Bigg) \ ,$$

where

$$\mathcal{M}_\nu^{\alpha,\beta}(x) := \min_{x,\alpha,\beta,\nu}\left\{\frac{b_L}{\alpha^{1/3}}\,\Gamma\left(\frac{\alpha}{2}+\nu+1\right)\Gamma\left(\frac{\alpha}{2}+\beta+\nu+1\right)\right. \ ,$$

$$\frac{c_L}{\sqrt[3]{2}}\,\Gamma\left(\frac{\alpha}{2}+\nu+\frac{5}{6}\right)\Gamma\left(\frac{\alpha}{2}+\beta+\nu+\frac{5}{6}\right)\left(\frac{x}{1-x}\right)^{-1/6} \ ,$$

$$\left.\frac{d_O}{\sqrt{2}}\,\Gamma\left(\frac{\alpha}{2}+\nu+\frac{3}{4}\right)\Gamma\left(\frac{\alpha}{2}+\beta+\nu+\frac{3}{4}\right)\left(\frac{x}{1-x}\right)^{-1/4}\right\} \ , \tag{8.9}$$

the coefficients b_L, c_L and d_O being given by (7.8), (7.9) *and* (7.10), *respectively.*

9. Concluding remarks and observations

In this concluding section of our presentation, we choose to briefly indicate some of the many possibilities in which the various two-sided inequalities (as well as the methodology and techniques) discussed in the foregoing sections can be suitably applied in order to derive analogous two-sided inequalities involving several other closely-related polynomials and special functions.

First of all, in recent years, a great deal of attention seems to have been paid to an obvious variant of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, especially in the context of the familiar group-thoeretic (Lie algebraic) method of Louis Weisner (1899–1988), which is described fairly adequately in the works of Miller [25], McBride [24, Chapter 2 and Chapter 3], and Srivastava and Manocha [38, Chapter 6] (see, for details [13] and [28], and the *numerous* references already cited in *each* of these works). These so-called *extended* Jacobi polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$, studied by (among others) Izuru Fujiwara (1928–1985) [11] in an attempt to give a unified presentation of the classical orthogonal polynomials (especially Jacobi, Laguerre, and Hermite polynomials), are defined by the Rodrigues formula:

$$\begin{aligned}F_n^{(\alpha,\beta)}(x;a,b,c) := \frac{(-c)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta}\cdot\\ \cdot\frac{d^n}{dx^n}\{(x-a)^{\alpha+n}(b-x)^{\beta+n}\}\qquad (n\in\mathbb{N}_0)\end{aligned}\tag{9.1}$$

and are orthogonal over the interval (a,b) with respect to the weight function [cf. Equation (1.22)]:

$$w(x;a,b) := (x-a)^{\alpha}(b-x)^{\beta}\ .\tag{9.2}$$

In fact, the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ are essentially those that were considered earlier by Szegö [39, p. 58], who showed (by means of a *simple* linear transformation) that these polynomials are just a constant multiple of the classical Jacobi polynomials. For the sake of ready reference, we rewrite Szegö's observation [39, p. 58, Equation (4.1.2)] in the form (cf., e.g., Srivastava and Manocha [38, p. 388, Problem 11]):

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \{c(a-b)\}^{-n}P_n^{(\alpha,\beta)}\left(\frac{2(x-a)}{a-b}+1\right)\tag{9.3}$$

or, equivalently,

$$P_n^{(\alpha,\beta)}(x) = \{c(a-b)\}^{-n}F_n^{(\alpha,\beta)}\left(\frac{1}{2}\{a+b+(a-b)x\};a,b,c\right)\ .\tag{9.4}$$

Thus, as already pointed out by Srivastava and Manocha [38], the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ may be looked upon as being equivalent to (and *not* as a generalization of) the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. More importantly, by simply appealing to the relationships (9.3) and (9.4), one can easily translate the two-sided inequalities of section 4 in terms of the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$.

Next, as observed by Pittaluga et al. [29], the so-called *modified* Laguerre polynomials $L_{a,b,c,n}(x)$ defined by (cf. [14]; see also many other references cited already by Pittaluga et al. [29]):

$$L_{a,b,c,n}(x) := \frac{b^n(c)_n}{n!}\ {}_1F_1\left(-n;c;\frac{ax}{b}\right)\tag{9.5}$$

$$(b \neq 0 \ ; \ \ c \neq 0, -1, -2, \cdots)$$

are simply the Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by (1.28), since

$$L_{a,b,c,n}(x) = b^n L_n^{(c-1)} \left(\frac{ax}{b} \right) \tag{9.6}$$

or, equivalently,

$$L_n^{(c)}(x) = b^{-n} L_{a,b,c+1,n} \left(\frac{bx}{a} \right) . \tag{9.7}$$

More generally, a similarly modified version of the classical Laguerre function (cf., e.g., Buchholz [4, p. 212]; see also Miller [25, p. 328]):

$$\begin{aligned} L_\nu^{(\mu)}(x) &:= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\ {}_1F_1(-\nu;\mu+1;x) = \\ &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)}\ e^x\ {}_1F_1(\mu+\nu+1;\mu+1;-x) \ , \end{aligned} \tag{9.8}$$

which was considered recently by Pathan and Khan [27] in the following (*corrected*) form (cf. [27, p. 1, Equation (1.1)]):

$$\begin{aligned} L_{\alpha,\beta,\mu,\nu}(x) &:= \frac{\beta^\nu \Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu+1)}\ {}_1F_1\left(-\nu;\mu;\frac{\alpha x}{\beta}\right) = \\ &= \frac{\beta^\nu \Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu+1)}\ \exp\left(\frac{\alpha x}{\beta}\right)\ {}_1F_1\left(\mu+\nu;\mu;-\frac{\alpha x}{\beta}\right) \ , \end{aligned} \tag{9.9}$$

satisfies each of the following *obvious* relationships:

$$L_{\alpha,\beta,\mu,\nu}(x) = \beta^\nu L_\nu^{(\mu-1)} \left(\frac{\alpha x}{\beta} \right) \tag{9.10}$$

and

$$L_\nu^{(\mu)}(x) = \beta^{-\nu} L_{\alpha,\beta,\mu+1,\nu} \left(\frac{\beta x}{\alpha} \right) . \tag{9.11}$$

Thus, by simply making use of the aforementioned results of Brenner [3] and Buschman [5], one can apply the relationships (9.6) and (9.7) [or, alternatively, the relationships (9.10) and (9.11)] with a view to deriving the corresponding bounds for these so-called *modified* Laguerre polynomials $L_{a,b,c,n}(x)$ and the *modified* Laguerre function $L_{\alpha,\beta,\mu,\nu}(x)$.

Finally, we consider the Tricomi-Carlitz polynomials $\mathcal{T}_n^{(\alpha)}(x)$ defined by (cf. [6] and [41]; see also [1] and [40])

$$\mathcal{T}_n^{(\alpha)}(x) := \sum_{k=0}^{n} (-1)^k \binom{x-\alpha}{k} \frac{x^{n-k}}{(n-k)!} \ , \tag{9.12}$$

which are related to the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ as follows:

$$\mathcal{T}_n^{(\alpha)}(x) = (-1)^n L_n^{(x-\alpha-n)}(x) \ . \tag{9.13}$$

Tricomi [41] observed that the polynomial system $\left\{ \mathcal{T}_n^{(\alpha)}(x) \right\}_{n=0}^{\infty}$ is not a system of orthogonal polynomials. However, as pointed out subsequently by Carlitz [6], the polynomials $\mathcal{L}_n^{(\alpha)}(x)$ defined by

$$\mathcal{L}_n^{(\alpha)}(x) := x^n \mathcal{T}_n^{(\alpha)}(x^{-2}) = (-x)^n L_n^{(x^{-2}-\alpha-n)}(x^{-2}) \tag{9.14}$$

are orthogonal over the interval $(-\infty, \infty)$ with respect to a certain *step* function as the weight function, and satisfy the following three-term recurrence relation:

$$(9.15) \qquad (n+1)\mathcal{L}_{n+1}^{(\alpha)}(x) = (n+\alpha)x\mathcal{L}_n^{(\alpha)}(x) - \mathcal{L}_{n-1}^{(\alpha)}(x) \qquad (n \in \mathbb{N}_0) \ ,$$

which obviously is of the form (1.37) leading *eventually* to the general two-sided inequalities asserted by Theorem 3, Theorem 4, and Theorem 5. By appealing appropriately to the relationships (9.13) and (9.14), and the three-term recurrence relation (9.15), one can similarly obtain the corresponding two-sided inequalities involving the Tricomi-Carlitz polynomials $\mathcal{T}_n^{(\alpha)}(x)$ and $\mathcal{L}_n^{(\alpha)}(x)$ by means of the results, methodology, and techniques discussed in this presentation. The details involved in all these (and other analogous) derivations are being left as an exercise for the interested reader.

Each of the following *further* remarks are potentially useful in *future* investigations on the subject of our presentation here.

Remark 4. Our method of proof of Theorems 7, 8 and 9 above, which is based heavily upon the integral representation (8.2) for the Jacobi function, does not seem to apply easily to the bounding inequality (7.12) asserted by Lemma 4.

Remark 5. The matrix methods (described and applied, among others, by Rassias and Srivastava [32]) require the use of a three-term recurrence relation which is satisfied by a fairly large family of special functions including (for example) such classical orthogonal polynomials as the Jacobi polynomials and their many relatives. Consequently, in the absence of an appropriate three-term recurrence relation, it does not seem to be possible to apply these matrix methods to the classical Jacobi function $P_\nu^{(\alpha,\beta)}(z)$ $(\nu \in \mathbb{C})$, the classical Laguerre function $L_\nu^{(\mu)}(z)$ $(\nu \in \mathbb{C})$, and so on.

Remark 6. The bounding inequalities for the first-kind Jacobi function, which are presented in the preceding section, are consequences of several potentially useful and reasonably sharp inequalities for the classical Laguerre function (see Section 7). Direct and alternative derivations of such and other bounding inequalities for the first-kind Jacobi function and other special functions, without using the results of Section 7, might be a worthwhile direction for further investigation.

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