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Some families of combinatorial series identities and associated rational sums

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Abstract[1]. Several interesting combinatorial series identities and rational sums were derived in a number of recent papers by using different methods and techniques. The main object of this lecture is to present a brief survey of these earlier works and to demonstrate how some much more general combinatorial series identities can be obtained by means of certain summation theorems for hypergeometric series. We also propose to indicate various relevant connections of the results presented here with those in many of the aforementioned investigations.

1. Introduction and motivation

Throughout this presentation, we shall denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively. Making use of the familiar Gamma function $\Gamma(z)$ defined, for $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$, by

$$\Gamma(z) = \begin{cases} \displaystyle\int_0^\infty e^{-t}\, t^{z-1}\, dt & (\Re(z) > 0) \\ \dfrac{\Gamma(z+n)}{\prod_{j=0}^{n-1}(z+j)} & \left(z \in \mathbb{C} \setminus \mathbb{Z}_0^- \;;\; n \in \mathbb{N}\right) \end{cases} \tag{1}$$

$$\left(\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} \quad ; \quad \mathbb{Z}^- := \{-1,-2,-3,\cdots\} \quad ; \quad \mathbb{N} := \{1,2,3,\cdots\}\right) ,$$

a generalized binomial coefficient $\binom{\lambda}{\mu}$ may be defined (for real or complex parameters λ and μ) by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)} =: \binom{\lambda}{\lambda-\mu} \qquad (\lambda, \mu \in \mathbb{C}) , \tag{2}$$

so that, in the special case when

$$\mu = n \qquad (n \in \mathbb{N}_0 \;;\; \mathbb{N}_0 := \mathbb{N} \cup \{0\}) ,$$

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we have

$$\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n\,(-\lambda)_n}{n!} \qquad (n \in \mathbb{N}_0)\ , \tag{3}$$

where $(\lambda)_n$ denotes the Pochhammer symbol given by

$$(\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) \qquad (n \in \mathbb{N})\ . \tag{4}$$

Binomial coefficients play an important rôle in many diverse areas of the mathematical, physical and statistical sciences, including (for example) number theory, probability and statistics. Single, double and multiple sums involving products of binomial coefficients, rational functions, harmonic numbers, and so on, are usually referred to as combinatorial series identities (see, for examples, [4], [8] and [16]). Recently, motivated essentially by the following series identity [1]:

$$\sum_{k=1}^{n} \left[(-1)^{k+1} \binom{n}{k} \sum_{1 \le i \le j \le k} \frac{1}{ij} \right] = \frac{1}{n^2}\ , \tag{5}$$

Díaz Barrero et al. [2] presented a procedure to derive generalizations and extensions of the series identity (5) given by Theorems 1 and 2 below.

Theorem 1. *Let n be a positive integer. Then, for*

$$x \in \mathbb{R} \setminus \{-1, \cdots, -n\} \qquad (n \in \mathbb{N})\ ,$$

$$\sum_{k=1}^{n} \left[(-1)^{k+1} \binom{n}{k} \left(\sum_{1 \le i \le j \le k} \frac{1}{x^2 + (i+j)x + ij} \right) \binom{x+k}{k}^{-1} \right] = \frac{n}{(x+n)^3}\ . \tag{6}$$

Theorem 2. *Let n be a positive integer. Then, for*

$$x \in \mathbb{R} \setminus \{-1, \cdots, -n\} \qquad (n \in \mathbb{N})\ ,$$

$$\begin{aligned} &\sum_{k=1}^{n} \left[(-1)^{k+1} \binom{n}{k} \left(\sum_{j=1}^{k} \frac{1}{(x+j)^3} + \sum_{1 \le i < j \le k} \frac{1}{(x+i)(x+j)(2x+i+j)} + \right.\right. \\ &\qquad \left.\left. + \sum_{1 \le i < j < \ell \le k} \frac{1}{(x+i)(x+j)(x+\ell)} \right) \binom{x+k}{k}^{-1} \right] = \frac{n}{(x+n)^4}\ . \end{aligned} \tag{7}$$

Upon setting $x = 0$ in Theorem 1, the series identity (5) would follow immediately. It should be remarked in passing that the parametric constraint:

$$x \in \mathbb{R} \setminus \{-1, \cdots, -n\} \qquad (n \in \mathbb{N})\ ,$$

occurring in Theorems 1 and 2, can *trivially* be replaced by

$$x \in \mathbb{C} \setminus \{-1, \cdots, -n\} \qquad (n \in \mathbb{N})\ ,$$

$\mathbb{C}$ being the set of complex numbers.

At least two subsequent works ([6] and [10]) have been motivated by the aforementioned investigation by Díaz Barrero et al. [2]. While Sofo [10] used the method of integral representations in order to recapture Theorems 1 and 2 and to highlight closed-form evaluations of other combinatorial sums, the work of Prodinger [6] provided an alternative approach to such series identities as those given by Theorems

1 and 2, which is based upon the principles of *inverse pairs* and *partial-fraction decomposition* (see also the references cited in each of the earlier works [6] and [10]). In the present sequel, we propose to show that substantially much more general results than those asserted by Theorems 1 and 2 can be derived rather systematically by means of summation theorems for hypergeometric series (see, for other examples and illustrations of the techniques used here, [5], [7], [13] and [14]).

The following definitions and results will be required in our investigation (see, for details, [3] and [18]; see also [15]).

1. *The Psi* (*or Digamma*) *function* $\psi(z)$*:*

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(\tau)\,d\tau \;. \tag{8}$$

2. *The Polygamma function* $\psi^{(n)}(z)$ $(n \in \mathbb{N})$*:*

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}}\{\log\Gamma(z)\} = \frac{d^n}{dz^n}\{\psi(z)\} \qquad \left(n \in \mathbb{N}_0\;;\;\; z \in \mathbb{C}\setminus\mathbb{Z}_0^-\right) \tag{9}$$

or, equivalently,

$$\begin{aligned}\psi^{(n)}(z) &= (-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(k+z)^{n+1}} = \\ &= (-1)^{n+1}n!\;\zeta(n+1,z) \qquad \left(n \in \mathbb{N}_0\;;\;\; z \in \mathbb{C}\setminus\mathbb{Z}_0^-\right)\end{aligned} \tag{10}$$

in terms of the generalized (or Hurwitz) Zeta function $\zeta(s,a)$ (see [15, p. 88 et seq.]).

Some recurrence relations for the Digamma and Polygamma functions include (see, for example, [15, pp. 14 and 22])

$$\psi(z+m) = \psi(z) + \sum_{k=1}^{m}\frac{1}{z+k-1} \qquad (m \in \mathbb{N}) \tag{11}$$

and

$$\psi^{(n)}(z+m) = \psi^{(n)}(z) + (-1)^n n!\sum_{k=1}^{m}\frac{1}{(z+k-1)^{n+1}} \qquad (m \in \mathbb{N}\;;\;\; n \in \mathbb{N}_0)\;, \tag{12}$$

which, for $n = 0$, obviously yields (11).

3. *The Gauss summation theorem*:

$$\begin{aligned}{}_2F_1(a,b;c;1) := \sum_{k=0}^{\infty}\frac{(a)_k\;(b)_k}{k!\;(c)_k} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ \left(\Re(c-a-b) > 0 \quad ; \quad c \in \mathbb{C}\setminus\mathbb{Z}_0^-\right)\;.\end{aligned} \tag{13}$$

2. General combinatorial ceries identities and their consequences

In this section, we first state the main set of combinatorial series identities as Theorem 3 below (see also [17]).

Theorem 3. *In terms of the Gamma function* $\Gamma(z)$ *and the Digamma function* $\psi(z)$ *defined by* (7)*, let the two-parameter functions* $\Theta(z;\lambda,\mu)$ *and* $\Phi(z;\lambda,\mu)$ *of a* *complex variable z be given by*

$$\Theta(z;\lambda,\mu) := \frac{\Gamma(z+1)\Gamma(z-\lambda-\mu-1)}{\Gamma(z-\lambda)\Gamma(z-\mu)} \tag{14}$$

and

$$\Phi(z;\lambda,\mu) := \psi(z+1) + \psi(z-\lambda-\mu-1) - \psi(z-\lambda) - \psi(z-\mu) \ , \tag{15}$$

respectively. Suppose also that

$$\Re(z-\lambda-\mu) > 1 \quad \textit{and} \quad z \in \mathbb{C} \setminus \mathbb{Z}^- \ . \tag{16}$$

Then each of the following combinatorial series identities holds true:

$$\begin{aligned} \sum_{k=1}^{\infty} \left[\binom{\lambda+k}{k} \binom{\mu+k}{k} \left(\sum_{j=1}^{k} \frac{1}{z+j} \right) \binom{z+k}{k}^{-1} \right] = \\ = -\Theta(z;\lambda,\mu) \cdot \Phi(z;\lambda,\mu) \ , \end{aligned} \tag{17}$$

$$\begin{aligned} \sum_{k=1}^{\infty} \left[\binom{\lambda+k}{k} \binom{\mu+k}{k} \left(\sum_{1\leq i\leq j\leq k} \frac{1}{z^2+(i+j)z+ij} \right) \binom{z+k}{k}^{-1} \right] = \\ = \frac{1}{2}\, \Theta(z;\lambda,\mu) \left([\Phi(z;\lambda,\mu)]^2 + \frac{d}{dz}\{\Phi(z;\lambda,\mu)\} \right) \ , \end{aligned} \tag{18}$$

$$\begin{aligned} \sum_{k=1}^{\infty} \Bigg[\binom{\lambda+k}{k} \binom{\mu+k}{k} \Bigg(\sum_{j=1}^{k} \frac{1}{(z+j)^3} + \sum_{1\leq i<j\leq k} \frac{1}{(z+i)(z+j)(2z+i+j)} + \\ + \sum_{1\leq i<j<\ell\leq k} \frac{1}{(z+i)(z+j)(z+\ell)} \Bigg) \binom{z+k}{k}^{-1} \Bigg] = \\ = -\frac{1}{6}\, \Theta(z;\lambda,\mu) \big([\Phi(z;\lambda,\mu)]^3 + 3\Phi(z;\lambda,\mu) \cdot \\ \cdot \frac{d}{dz}\{\Phi(z;\lambda,\mu)\} + \frac{d^2}{dz^2}\{\Phi(z;\lambda,\mu)\} \Big) \ , \end{aligned} \tag{19}$$

and so on, provided that both members of each of the identities (17), (18) *and* (19) *exist.*

Proof. Since

$$\binom{\lambda+n-1}{n} = \frac{(\lambda)_n}{n!} \qquad (n \in \mathbb{N}_0)$$

in terms of the Pochhammer symbol $(\lambda)_n$ defined by (4), upon setting

$$a = \lambda+1 \quad , \quad b = \mu+1 \quad , \quad \text{and} \quad c = z+1$$

in the Gauss summation theorem (13), we obtain

$$\sum_{k=0}^{\infty} \binom{\lambda+k}{k} \binom{\mu+k}{k} \binom{z+k}{k}^{-1} = \Theta(z;\lambda,\mu) \ , \tag{20}$$

which is valid under the parametric constraints given by the hypothesis (16) of Theorem 3, with the function $\Theta(z;\lambda,\mu)$ defined as in (14).

Next, by observing that

$$\frac{d}{dz}\left\{\binom{z+k}{k}^{-1}\right\} = -\binom{z+k}{k}^{-1}\sum_{j=1}^{k}\frac{1}{z+j} \qquad (k\in\mathbb{N}_0) \tag{21}$$

and

$$\frac{d}{dz}\{\Theta(z;\lambda,\mu)\} = \Theta(z;\lambda,\mu)\cdot\Phi(z;\lambda,\mu)\ , \tag{22}$$

where the function $\Phi(z;\lambda,\mu)$ is defined as in (15), the assertion (17) of Theorem 3 follows by differentiating both sides of the identity (20) with respect to z.

Now, since

$$\begin{aligned}\frac{d^2}{dz^2}\left\{\binom{z+k}{k}^{-1}\right\} &= \binom{z+k}{k}^{-1}\left[\left(\sum_{j=1}^{k}\frac{1}{z+j}\right)^2+\sum_{j=1}^{k}\frac{1}{(z+j)^2}\right]=\\ &=\binom{z+k}{k}^{-1}\left(\sum_{1\le i\le j\le k}\frac{2}{(z+i)(z+j)}\right)=\\ &=\binom{z+k}{k}^{-1}\left(\sum_{1\le i\le j\le k}\frac{2}{z^2+(i+j)z+ij}\right)\ ,\end{aligned} \tag{23}$$

the assertion (18) of Theorem 3 can be derived by merely differentiating both sides of the identity (20) *two* times with respect to z.

Finally, in view of the following derivative formula:

$$\begin{aligned}\frac{d^3}{dz^3}\left\{\binom{z+k}{k}^{-1}\right\} &= -\binom{z+k}{k}^{-1}\left[\left(\sum_{j=1}^{k}\frac{1}{z+j}\right)^3+\right.\\ &\left.+3\left(\sum_{j=1}^{k}\frac{1}{z+j}\right)\left(\sum_{j=1}^{k}\frac{1}{(z+j)^2}\right)+\sum_{j=1}^{k}\frac{1}{(z+j)^3}\right]=\\ &=-6\left(\sum_{j=1}^{k}\frac{1}{(z+j)^3}+\sum_{1\le i<j\le k}\frac{1}{(z+i)(z+j)(2z+i+j)}+\right.\\ &\left.+\sum_{1\le i<j<\ell\le k}\frac{1}{(z+i)(z+j)(z+\ell)}\right)\ ,\end{aligned} \tag{24}$$

we prove the assertion (19) of Theorem 3 by differentiating both sides of the identity (20) *three* times with respect to z.

By *further* differentiating the identity (20) *successively* with respect to z, we can similarly derive several other combinatorial series identities belonging to the family exemplified here by the series identities (17), (18) and (19). Our proof of Theorem 3 is thus completed. □

Remark 1. By applying the relationship (3), we can easily rewrite the Gauss summation theorem (13) in the following combinatorial form (cf., e.g., [4, p. 61, Entries (7.20) and (7.21)]):

$$\sum_{k=0}^{\infty}(-1)^k\binom{-a}{k}\binom{-b}{k}\binom{-c}{k}^{-1}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{25}$$

$$\left(\Re(c-a-b)>0 \quad ; \quad c\in\mathbb{C}\setminus\mathbb{Z}_0^-\right) ,$$

which obviously is equivalent to the combinatorial series identity (20).

Remark 2. Upon setting $\mu = 0$ in the assertions (17), (18) and (19) of Theorem 3, if we make use of the recurrence relations (11) and (12) for $m = 1$ and $n = 1, 2$, we get relatively simpler combinatorial series identities given by the following corollary.

Corollary 1. *If* $\Re(z-\lambda) > 1$ *and* $z \in \mathbb{C}\setminus\mathbb{Z}^-$, *then*

$$\sum_{k=1}^{\infty}\left[\binom{\lambda+k}{k}\left(\sum_{j=1}^{k}\frac{1}{z+j}\right)\binom{z+k}{k}^{-1}\right]=\frac{\lambda+1}{(z-\lambda-1)^2} , \tag{26}$$

$$\sum_{k=1}^{\infty}\left[\binom{\lambda+k}{k}\left(\sum_{1\leq i\leq j\leq k}\frac{1}{z^2+(i+j)z+ij}\right)\binom{z+k}{k}^{-1}\right]=\frac{\lambda+1}{(z-\lambda-1)^3} , \tag{27}$$

$$\sum_{k=1}^{\infty}\left[\binom{\lambda+k}{k}\left(\sum_{j=1}^{k}\frac{1}{(z+j)^3}+\sum_{1\leq i<j\leq k}\frac{1}{(z+i)(z+j)(2z+i+j)}+\right.\right.$$

$$\left.\left.+\sum_{1\leq i<j<\ell\leq k}\frac{1}{(z+i)(z+j)(z+\ell)}\right)\binom{z+k}{k}^{-1}\right]=\frac{\lambda+1}{(z-\lambda-1)^4} , \tag{28}$$

and so on, provided that both members of each of the identities (26), (27) *and* (28) *exist.*

Remark 3. By virtue of (11) and (12), as well as of the following combinatorial identity:

$$\binom{\lambda+n-1}{n}=(-1)^n\binom{-\lambda}{n} \qquad (n\in\mathbb{N}_0) , \tag{29}$$

a special case of Theorem 3 when

$$\lambda=-n-1 \qquad (n\in\mathbb{N})$$

would readily yield Corollary 2 below.

Corollary 2. *If* $n \in \mathbb{N}$ *and* $z \in \mathbb{C}\setminus\mathbb{Z}^-$, *then*

$$\sum_{k=1}^{n}\left[(-1)^k\binom{n}{k}\binom{\mu+k}{k}\left(\sum_{j=1}^{k}\frac{1}{z+j}\right)\binom{z+k}{k}^{-1}\right]=$$

$$=\binom{z-\mu+n-1}{n}\binom{z+n}{n}^{-1}\left(\sum_{j=1}^{n}\frac{1}{z+j}-\sum_{j=1}^{n}\frac{1}{z-\mu+j-1}\right) , \tag{30}$$

$$\sum_{k=1}^{n}\left[(-1)^k\binom{n}{k}\binom{\mu+k}{k}\left(\sum_{1\le i\le j\le k}\frac{1}{z^2+(i+j)z+ij}\right)\binom{z+k}{k}^{-1}\right]=$$

$$=\frac{1}{2}\binom{z-\mu+n-1}{n}\binom{z+n}{n}^{-1}\left[\left(\sum_{j=1}^{n}\frac{1}{z+j}-\sum_{j=1}^{n}\frac{1}{z-\mu+j-1}\right)^2+\right.$$

$$\left.+\left(\sum_{j=1}^{n}\frac{1}{(z+j)^2}-\sum_{j=1}^{n}\frac{1}{(z-\mu+j-1)^2}\right)\right], \tag{31}$$

$$\sum_{k=1}^{n}\left[(-1)^k\binom{n}{k}\binom{\mu+k}{k}\left(\sum_{j=1}^{k}\frac{1}{(z+j)^3}+\sum_{1\le i<j\le k}\frac{1}{(z+i)(z+j)(2z+i+j)}+\right.\right.$$

$$\left.\left.+\sum_{1\le i<j<\ell\le k}\frac{1}{(z+i)(z+j)(z+\ell)}\right)\binom{z+k}{k}^{-1}\right]=$$

$$=\frac{1}{6}\binom{z-\mu+n-1}{n}\binom{z+n}{n}^{-1}\left[\left(\sum_{j=1}^{n}\frac{1}{z+j}-\sum_{j=1}^{n}\frac{1}{z-\mu+j-1}\right)^3+\right.$$

$$+3\left(\sum_{j=1}^{n}\frac{1}{z+j}-\sum_{j=1}^{n}\frac{1}{z-\mu+j-1}\right)\left(\sum_{j=1}^{n}\frac{1}{(z+j)^2}-\sum_{j=1}^{n}\frac{1}{(z-\mu+j-1)^2}\right)+$$

$$\left.+2\left(\sum_{j=1}^{n}\frac{1}{(z+j)^3}-\sum_{j=1}^{n}\frac{1}{(z-\mu+j-1)^3}\right)\right], \tag{32}$$

and so on, provided that both members of each of the identities (30)*,* (31) *and* (32) *exist.*

Remark 4. The first assertion (30) of Corollary 2 was recorded earlier by (for example) Gould [4, p. 60, Entry (7.15)].

Remark 5. Theorem 1 of Díaz-Barrero *et al.* [2] would follow from the assertion (27) of Corollary 1 when we set $\lambda=-n-1$ $(n\in\mathbb{N})$ and apply the combinatorial identity (29). Alternatively, by merely putting $\mu=0$ in the assertion (31) of Corollary 2, we arrive at the combinatorial series identity (6) asserted by Theorem 1.

Remark 6. With a view to deducing the combinatorial series identity (7) asserted by Theorem 2, we simply set $\lambda=-n-1$ $(n\in\mathbb{N})$, in the assertion (28) of Corollary 1 and make use the combinatorial identity (29) or (alternatively) we set $\mu=0$ in the assertion (32) of Corollary 2.

3. Further applications and generalizations

Apart from the Gauss summation theorem (13), there exist a fairly large number of summation theorems for the hypergeometric ${}_2F_1$ series as well as for other *higher-order* hypergeometric series (see, for example, [3, Chapters I and II] and [9, pp. 243-246, Appendix III]). Many of these summation theorems are expressed as combinatorial series identities by Gould [4, pp. 71-77]. In particular, the Pfaff-Saalschütz theorem [3, p. 66, Equation 2.1.5 (30)]:

$$\sum_{k=0}^{n} \frac{(-n)_k(a)_k(b)_k}{k!(c)_k(a+b-c-n+1)_k} = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \qquad (n \in \mathbb{N}_0) \ , \tag{33}$$

which yields the Gauss summation theorem (13) in the limit case when $n \to \infty$, can indeed be derived by first rewriting the familiar Euler's transformation [3, p. 64, Equation 2.1.4 (23)]

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b} \ {}_2F_1(c-a,c-b;c;z) \tag{34}$$

$$\big(|\arg(1-z)| \leqq \pi - \epsilon \quad ; \quad 0 < \epsilon < \pi \quad ; \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-\big)$$

in the form:

$$(1-z)^{a+b-c} \ {}_2F_1(a,b;c;z) = \ {}_2F_1(c-a,c-b;c;z) \tag{35}$$

$$\big(|\arg(1-z)| \leqq \pi - \epsilon \quad ; \quad 0 < \epsilon < \pi \quad ; \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-\big)$$

and then equating the coefficients of z^n from both sides of (35). More importantly, we can easily extend the results asserted by Theorem 3 (and hence also Corollaries 1 and 2) if we similarly apply the foregoing techniques to the Pfaff-Saalschütz theorem (33) as the following combinatorial series identity [4, p. 71, Entry (17.2)]:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{\lambda}{k}\binom{\mu}{k}\binom{\lambda+\mu+z+n}{k}^{-1}\binom{z+k}{k}^{-1} =$$

$$= \binom{\lambda+z+n}{n}\binom{\mu+z+n}{n}\binom{z+n}{n}^{-1}\binom{\lambda+\mu+z+n}{n}^{-1} , \tag{36}$$

where exceptional parameter values that would render either side invalid or undefined are tacitly avoided.

The details involved in the above-suggested derivations of a *further* generalization of Theorem 3 (and hence also of Corollaries 1 and 2) may be left as an exercise for the interested reader.

We conclude this presentation by remarking further that, just as exemplified by such recent investigations as (for example) [11] and [12], the subject of finding closed-form expressions for various sums involving binomial coefficients continues to attract our attention.

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