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About connection of generalized Möbius Listing's surfaces GML_2^n with sets of knots or links

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Abstract[1]. We consider the cutting process of a generalized Möbius Listing's surfaces GML_2^n along a set of lines "parallel" to its "basic line". We show connections of the resulting mathematical objects with the set of knots and links.

1. Introduction

In previous articles [2, 5, 6, 7, 8, 9, 10, 11, 12] a wide class of geometric figures - "generalized twisting and rotated" bodies (also called "surface of revolution" see [1]) - shortly GTR_m^n - was defined through their analytic representation. In particular cases, this analytic representation gives back many classical objects (torus, helicoid, helix, Möbius strip, etc.).

Aim of this article is to consider some relationship between the generalized Möbius Listing's surfaces GML_2^n and the sets of knots and links. The knots and links classifications is well known (see e.g. [3, 4, 13]). In this article we consider ribbon knots and links which appear after cutting a generalized Möbius Listing's surface GML_2^n along $\kappa \in \mathbb{N}$ different lines, which are "parallel" to its "basic line".

In previous articles [2, 12] we considered the case when GML_2^n surface had only one slit.

Notations. We use following notations:

- X, Y, Z, or x, y, z - is the ordinary notation for coordinates;

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- τ, ψ, θ - are space values (local coordinates or parameters in parallelogram):

(1)
1. $\tau \in [\tau_*, \tau^*]$, where $\tau_* \leq \tau^*$ usually are non-negative constants ;
2. $\psi \in [0, 2\pi]$;
3. $\theta \in [0, 2\pi h]$, where $h \in \mathbb{R}$ (Real) ;

But sometimes, as a special case, we suppose that

$$\tau \in [-\tau^*, \tau^*] \tag{1*}$$

- $\widetilde{x}, \widetilde{z}, \theta$ - are space values (local coordinates or parameters in the cylinder PR_∞ or in the diametral cross section of this cylinder correspondingly)

$$\begin{aligned} PR_\infty \equiv \{(\widetilde{x} = \tau \cos\psi, \widetilde{z} = \tau \sin\psi, \theta) : \tau \in [-\tau^*, \tau^*] \,;\, \psi \in [0, \pi) \,; \\ \theta \in [0, 2\pi)\} \\ T \equiv PR_2 \equiv \{(\widetilde{x} = \tau, \widetilde{z} = 0, \theta) : \tau \in [-\tau^*, \tau^*] \,;\, \psi = 0 \,; \theta \in [0, 2\pi)\} \,; \end{aligned} \tag{2}$$

2. Some geometric properties of a "regular" GML_2^n surfaces

In this part of our article we study some geometric characteristic of a "regular" generalized Möbius-Listing's surfaces GML_2^n, with circle as basic line (see e.g. Fig. 1.a, 1.b, 1.c)[2]. This means that the parametric representations of these surfaces (6)

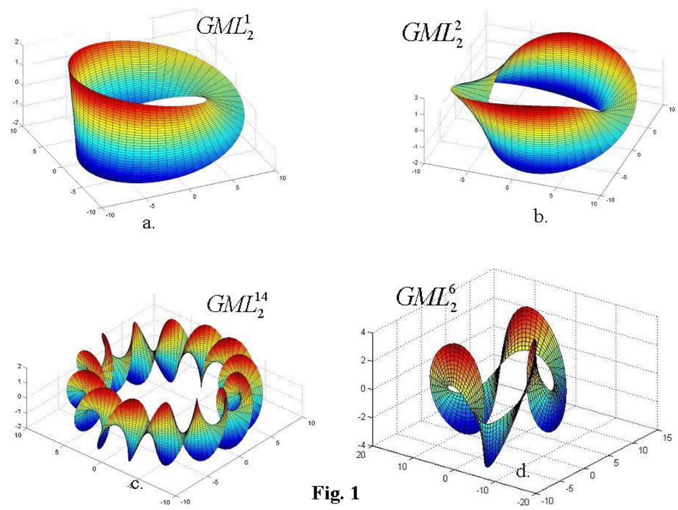

Fig. 1

[2]Note that in the present article n denotes the number of rotations and m the symmetry number of the cross section, while in [6] the meaning of these indices was reversed.

or (6*) in [6] have the following simple form

$$
(3)\qquad
\begin{aligned}
X(\tau,\theta) &= \left[R + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\cos(\theta)\\
Y(\tau,\theta) &= \left[R + \tau\cos\left(\psi + \frac{n\theta}{2}\right)\right]\sin(\theta)\\
Z(\tau,\theta) &= \tau\sin\left(\psi + \frac{n\theta}{2}\right)\ ,
\end{aligned}
$$

where, respectively:
- R - radius of basic circle - is constant;
- In the general case (see e.g. (6) in [6]) τ, defined in (1), is variable, but now, according to the notation (2), the argument τ always belongs to the interval $[-\tau^*, \tau^*]$, see (1*) where $\tau^* < R$ is some non-negative constant. So that, when $n = 1$ formula (3) is the classical (well known) form of analytic representation of the Möbius strip ([13], [1]). Actually, $2\tau^*$ is the width of the surface GML_2^n;
- the variable θ is defined in (1) and in this case $h \equiv 1$, i.e. $\theta \in [0, 2\pi]$;
- The "rule of twisting around basic line" is "regular", i.e. the function defined by equation (4) in [6] is $g(\theta) \equiv \theta$;
- n - the "number or twisting" of GML_2^n - is an arbitrary integer number, i.e. the number defined by equation (5) in [6] is $\mu \equiv n/2$. In this article we always assume, for simplicity, $n \geq 0$;
- in the present case, ψ is a constant defined in (1) (but when $n = 0$, the number ψ in equations (3) defines even the type of the corresponding surface, for example: if $\psi = 0$, then the "regular" generalized Möbius-Listing's surfaces GML_2^0, with basic line a circle, is a Ring ($R > \tau^*$) or Circle ($R = \tau^*$), and if $\psi = \pi/2$, then GML_2^0 is a cylinder, in other cases these surfaces are cones or truncated cones (see [9, 10, 5, 6, 12])).

Remark 1. When $R > \tau^*$, note that:

a) For every integer number n equation (3), when the arguments τ, ψ, θ are from (1), defines a one to one correspondence between the cylinder (2) and the body GML_∞^n ("Twisted Torus" [6, 12]);
b) In particularly, if $n = 0$, then equation (3), when the arguments τ, ψ, θ are from (1), defines a one to one correspondence between the cylinder (2) and the Torus GML_∞^0;
c) For every integer number n, equations (3), when the arguments τ, ψ, θ satisfy: $\tau \in [-\tau^+, \tau^*]$, $\theta \in [0, 2\pi)$ and y is a constant, defines a one to one correspondence between the strip T in (2) and the surface GML_2^n;
d) If n is an even number, then each function (X, Y, Z) in the representation (3) is a 2π-periodic function of the argument θ.
e) If n is a odd number, then each function (X, Y, Z) in the representation (3) is a 4π-periodic function satisfying the following properties

$$
(\mathrm{M}^*)\quad (X(\tau,\theta+2\pi);\ Y(\tau,\theta+2\pi);\ Z(\tau,\theta+2\pi)) = (X(-\tau,\theta);\ Y(-\tau,\theta);\ Z(-\tau,\theta))
$$

(in analogy to the Möbius condition see [13]).
f) For every integer number n equations (3), defines a "width preserving" correspondence (proof in ref. [8]).

Proof. By using elementary trigonometric formulas, it is easy to rewrite the representation (3) in the following form

$$
\begin{aligned}
X(\tau,\psi,\theta) &= \left[R+\widetilde{x}\cos\left(\frac{n\theta}{2}\right)-\widetilde{z}\sin\left(\frac{n\theta}{2}\right)\right]\cos(\theta)\\
Y(\tau,\psi,\theta) &= \left[R+\widetilde{x}\cos\left(\frac{n\theta}{2}\right)-\widetilde{z}\sin\left(\frac{n\theta}{2}\right)\right]\sin(\theta)\\
Z(\tau,\psi,\theta) &= \widetilde{x}\sin\left(\frac{n\theta}{2}\right)+\widetilde{z}\cos\left(\frac{n\theta}{2}\right)\ ,
\end{aligned}
\tag{4}
$$

where $\widetilde{x},\widetilde{z},\theta$ - are variables in the (2). After calculation we remark that

$$
\frac{\partial(X,Y,Z)}{\partial(\widetilde{x},\widetilde{z},\theta)} = -R-\left[\widetilde{x}\cos\left(\frac{n\theta}{2}\right)-\widetilde{z}\sin\left(\frac{n\theta}{2}\right)\right]\neq 0\ . \tag{5}
$$

i.e. the Jacobian matrix of this correspondence is different from zero ($R>\tau^*$). So that the relation (3) or (4) is a one to one correspondence. Also, parts b.) and c.) of the Remark 1 are trivial corollary of the equation (5).
According to the representation (3), the tangential vectors of the "regular" generalized Möbius-Listing's surface GML_2^n, with circle as basic line, are correspondingly

$$
\overrightarrow{\mathbf{r}}_\tau = \left\{\cos\left(\psi+\frac{n\theta}{2}\right)\cos(\theta);\ \cos\left(\psi+\frac{n\theta}{2}\right)\sin(\theta);\ \sin\left(\psi+\frac{n\theta}{2}\right)\right\} \tag{6}
$$

and

$$
\overrightarrow{\mathbf{r}}_\theta = \left\{
\begin{array}{c}
-\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\theta)-\frac{\tau n}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\cos(\theta)\ ;\\
\left[R+\tau\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos(\theta)-\frac{\tau n}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\sin(\theta)\ ;\\
\frac{\tau n}{2}\cos\left(\psi+\frac{n\theta}{2}\right)
\end{array}
\right\}\ . \tag{7}
$$

It is easy to check, that the scalar product of these two vectors (6) and (7) is

$$
(\overrightarrow{\mathbf{r}}_\tau,\overrightarrow{\mathbf{r}}_\theta)=0\ .
$$

Remark 2. For any integer number n two tangential vectors of a regular GML_2^n, with circle as basic line, are always orthogonal, i.e. the local system of coordinates (τ,θ) in this surface is an orthogonal system.

Also we may check that the unit normal vector of a "regular" generalized Möbius-Listing's surfaces GML_2^n, with circle as basic line, has the form

$$(8)\quad \overrightarrow{\nu}(\tau,\theta)=\left\{\begin{array}{l} \dfrac{-\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]\sin\left(\psi+\dfrac{n\theta}{2}\right)\cos(\theta)+\dfrac{n\tau}{2}\sin(\theta)}{\sqrt{\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]^2+\dfrac{(n\tau)^2}{4}}}; \\ \dfrac{-\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]\sin\left(\psi+\dfrac{n\theta}{2}\right)\sin(\theta)-\dfrac{n\tau}{2}\cos(\theta)}{\sqrt{\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]^2+\dfrac{(n\tau)^2}{4}}}; \\ \dfrac{\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]\cos\left(\psi+\dfrac{n\theta}{2}\right)}{\sqrt{\left[R+\tau\cos\left(\psi+\dfrac{n\theta}{2}\right)\right]^2+\dfrac{(n\tau)^2}{4}}} \end{array}\right\}.$$

Remark 3. Note that:

a) If n is an even number, then the unit normal vector $\overrightarrow{\nu}(\tau,\theta)$ is a 2π-periodic vector and consequently the GML_2^n body is a two-sided surface; i.e. to each point of the GML_2^n corresponds one (external or internal) normal vector of this surface;

b) If n is a odd number, then the unit normal vector $\overrightarrow{\nu}(\tau,\theta)$ is a 4π-periodic vector function, with the property

$$(\mathrm{M}^{**})\qquad \overrightarrow{\nu}(\tau,\theta+2\pi)=-\overrightarrow{\nu}(\tau,\theta)$$

so that the GML_2^n is a one-sided surfaces; i.e. to each point of the GML_2^n surface correspond two normal vectors to this surface and, by the geometric point of view, it is impossible to "distinguish" the external from the internal normal vector to this surface.

Additional information about similar properties for the semi-regular surfaces GML_2^n are reported in [2].

3. Relations between the set of generalized Möbius-Listing's surfaces and the sets of knots and links

We use the following definitions and notations:

• Everywhere in this article we use the terminology "Link-1" instead of "knot".

Definition 1. A closed line (similar to the basic or border's line) which is situated on a GML_2^n and is "parallel" to the basic (or border's) line of the GML_2^n - i.e. the distance between this line and basic or border's lines is constant - is called a *Slit line* or shortly a *s-line* (see e.g. Fig. 2.d).

• If the distance between an s-line and the basic line is zero, then this s-line coincides with the basic line (and sometimes is called *B-line*) (see e.g. Fif. 2.c).

Definition 2. A domain situated on the surface GML_2^n and such that its border's lines are slit lines, is called a *Slit zone* or shortly a *s-zone* (see e.g. Fig. 2.a, 2.b, 2.e, 3.a, 3.c, 3.d).

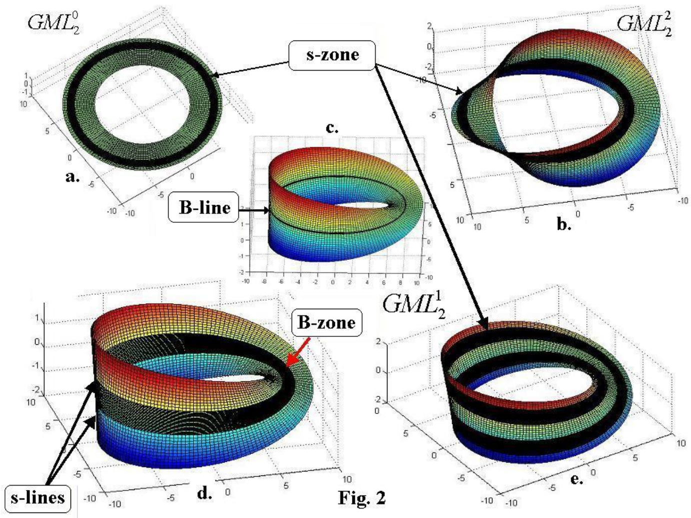

Fig. 2

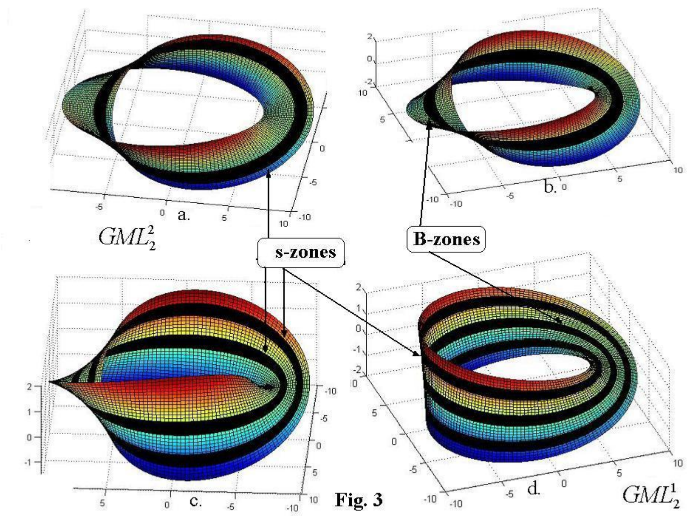

Fig. 3

- The distance between the border's lines of an s-zone is the *width* of this s-zone.
- If an s-zone's width equals to zero, then this zone reduces to an s-line.

Definition 3. If the *B-line* is properly contained inside a *Slit zone* - i.e. his distance to the border's lines is strictly positive - then this *Slit zone* will be called a *B-zone* (see e.g. Fig. 2.d, 3.b, 3.d).

Definition 4. The *process of cutting* or shortly the *cutting* is always realized along some s-lines and produces the vanishing (i.e. elimination) of the corresponding s-zone (which eventually reduces to an s-line).

• If a GML_2^n surface is cut belong an s-line (sometimes $\longrightarrow^1$), then the corresponding vanishing zone will be called a *s-slit*.

• If a GML_2^n surface is cut belong its B-line (sometimes $\longrightarrow^B$), then the corresponding vanishing zone will be called a *B-slit*.

• If the vanishing zone - after a *s-slit* (a *B-slit*) - is given by a *s-zone* (a *B-zone*), then the cutting process will be called a *s-zone-slit* (a *B-zone-slit*).

• If a GML_2^n surface is cut $(\kappa+1)$-times along $(\kappa+1)$, $\kappa = 0, 1, 2, \cdots$, different s-lines and none of them coincides with the B-line (for this process we use the symbolic notation: $\longrightarrow^{\kappa+1}$), then the resulting object is called a $(\kappa+1)$*-slitting* GML_2^n, and the corresponding vanishing zones are $(\kappa+1)$*-slits*. In this case the cutting process is called a $(\kappa+1)$*-zone-slits*.

• If a GML_2^n surface is cut $(\kappa+1)$-times along $(\kappa+1)$, $\kappa = 0, 1, 2, \cdots$, different s-lines and one of this line coincides with the B-line (for this process we use the symbolic notation: $\longrightarrow^{B+\kappa}$), then the resulting object is called a $(B+\kappa)$*-slitting* GML_2^n, and the corresponding vanishing zones are $(B+\kappa)$*-slits*. In this case the cutting process is called a $(B+\kappa)$*-zone-slits*.

• For each natural number k the segment $[0, \tau^*]$ is divided by one of the following rules:

$$\text{(9)} \qquad \begin{aligned} \widetilde{\tau}_{2j} &\equiv j \cdot \left(\frac{\tau^* - \varepsilon}{k+1} + \frac{\varepsilon}{k} \right) \quad , \quad j = 0, 1, \cdots, k \, . \\ \widetilde{\tau}_{2j+1} &\equiv (j+1) \cdot \frac{\tau^* - \varepsilon}{k+1} + j \cdot \frac{\varepsilon}{k} \quad , \quad j = 0, 1, \cdots, k \, . \end{aligned}$$

or

$$\text{(10)} \qquad \begin{aligned} \widehat{\tau}_{2j} &\equiv \frac{j \cdot \tau^*}{k} \quad , \quad j = 0, 1, \cdots, k \, . \\ \widehat{\tau}_{2j+1} &\equiv \frac{j \cdot \tau^* + \varepsilon}{k} \quad , \quad j = 0, 1, \cdots, k-1 \, . \end{aligned}$$

where $\varepsilon \in [0, \tau^*)$ is a real number.

• For each natural number k the domain of definition $T \equiv \{[-\tau^*, \tau^*] \times [0, 2\pi)\}$ (see (2)) of the representation formula (3) is divided by one of the following rules:

$$\text{(11)} \qquad \begin{aligned} T &= \widetilde{T}_k \cup \widetilde{T}_k^\varepsilon \\ \widetilde{T}_k &\equiv \left\{ [-\widetilde{\tau}_1, \widetilde{\tau}_1] \bigcup_{j=1}^{k} [\widetilde{\tau}_{2j}, \widetilde{\tau}_{2j+1}] \cup [-\widetilde{\tau}_{2j+1}, -\widetilde{\tau}_{2j}] \right\} \times [0, 2\pi) \, , \\ \widetilde{T}_k^\varepsilon &\equiv \left\{ \bigcup_{j=1}^{2k-1} [\widetilde{\tau}_{2j-1}, \widetilde{\tau}_{2j}] \cup [-\widetilde{\tau}_{2j}, -\widetilde{\tau}_{2j-1}] \right\} \times [0, 2\pi) \, , \end{aligned}$$

or

$$T = \widehat{T}_{B,k} \cup \widehat{T}^{\varepsilon}_{B,k} \ ,$$

$$(12) \qquad \widehat{T}_{B,k} \equiv \left\{ \bigcup_{j=1}^{2k-1} [\widehat{\tau}_{2j-1}, \widehat{\tau}_{2j}] \cup [-\widehat{\tau}_{2j}, -\widehat{\tau}_{2j-1}] \right\} \times [0, 2\pi) \ ,$$
$$\widehat{T}^{\varepsilon}_{B,k} \equiv \left\{ [-\widehat{\tau}_1, \widehat{\tau}_1] \bigcup_{j=1}^{k} [\widehat{\tau}_{2j}, \widehat{\tau}_{2j+1}] \cup [-\widehat{\tau}_{2j+1}, -\widehat{\tau}_{2j}] \right\} \times [0, 2\pi)$$

or

$$T = \widetilde{T}^{*}_{k} \cup \widetilde{T}^{\varepsilon,*}_{k}$$

$$(13) \qquad \widetilde{T}^{*}_{k} \equiv \left\{ [-\tau^*, \widetilde{\tau}_1] \bigcup_{j=1}^{k} [\widetilde{\tau}_{2j}, \widetilde{\tau}_{2j+1}] \right\} \times [0, 2\pi) \ ,$$
$$\widetilde{T}^{\varepsilon,*}_{k} \equiv \left\{ \bigcup_{j=1}^{2k-1} [\widetilde{\tau}_{2j-1}, \widetilde{\tau}_{2j}] \right\} \times [0, 2\pi) \ ,$$

Without loss of generality and for simplifying the process of proofing, in this article we consider the following restrictions:

• The GML^n_2 is a regular generalized Möbius-Listing's surface (see representation (3) and notation (1*);

• A *B-slit* is symmetric (see e.g. Fig. 2.c, 2.d, 3.b, 3.d), i.e. the "origin" (domain of its parametric representation) (3) of the corresponding B-zone is the domain $\widehat{T}^{\varepsilon}_{B,0}$ in (12).

• For any fixed number of cutting k, the width of the eliminated slit zones on the GML^n_2 surfaces are always identic and equal to $2\varepsilon/k$ (see e.g. Fig. 2.e, 3.c, 3.d);

• For any fixed number of cutting k, the width of the remaining slit zones on the GML^n_2 surfaces are always identic and equal to $2(\tau^* - \varepsilon)/(k+1)$.

Theorem 1. *If the GML^n_2 surface is cut $(\kappa+1)$-times along $(\kappa+1)$ different (i.e. $\kappa = 0, 1, \cdots$) s-lines, and n is an even number, then for each integer numbers n, κ, after $(B+\kappa)$-zone-slits or $(\kappa+1)$-zone-slits, an object* Link-$(\kappa+2)$ *appears, whose each component is a GML^n_2 surface (knot with structure $\{0_1\}$)[3]; i.e. if $n = 2\omega$ is an even number, then for each $\omega = 0, 1, 2, \cdots$, and κ:*

Case A.

$$(14) \qquad GML^{2\omega}_2 \longrightarrow^{B+\kappa} \textit{Link-}(\kappa+2) \ \textit{of} \ (\kappa+2) \times GML^{2\omega}_2 \ ;$$

Case B.

$$(15) \qquad GML^{2\omega}_2 \longrightarrow^{\kappa+1} \textit{Link-}(\kappa+2) \ \textit{of} \ (\kappa+2) \times GML^{2\omega}_2 \ .$$

Proof. The representation (3) is a one to one correspondence between the points of the strip T and the points of the GML^n_2 surface, and according to the remak 1, If n is an even number ($n \equiv 2\omega$), then each of the functions $X(\tau,\theta), Y(\tau,\theta), Z(\tau,\theta)$ is a 2π-periodic function of the argument θ.

[3]The topologic group of the link-$(k+2)$ in this case is at present unknown; only when $k=0$, the link-2 is of type $\{n^2_1\}$, according the standard classification (see [2]).

Let us consider the *particular case A*, when $\kappa = 0$ - This means that there exists only one B-slit-zone.

So that, according to (12), a the B-zone-slit corresponds to the elimination of the $\widehat{T}^{\varepsilon}_{B,0}$ in the domain of definition T. But in this case, the domain of definition $\widehat{T}_{B,0}$ in (12) consists of two parts and define by (3) two different objects GML_2^n. The one to one correspondence (3) guarantees that the new objects have not self-cross points.

In this case, according to the above restrictions, both new objects have identic widths.

Let us consider the *particular case B*, when $\kappa = 0$ - This means that there exists only one s-slit-zone.

So that, according to (3) and (13), a s-slit-zone corresponds to the elimination of the $\widetilde{T}_1^{\varepsilon,*}$ in the domain of definition T. But in this case, the domain of definition $\widetilde{T}_1^{*}$ in (13) consists of two parts and define by (3) two different objects GML_2^n, with different widths. The widths of components of B-zone-slits are equal, but the widths of components of s-zone-slits are different.

Let us consider now the *general case A*, when κ is an arbitrary natural number. This means that there exist $(B+\kappa)$-zone-slits.
- Recall that in the particular case A, after a B-zone-slit, an object Link-2 appears, whose both components are GML_2^n surfaces. So that the following cutting or the $(B+1)$-*zone-slits* ($\kappa = 1$) is a cutting of one of the new objects GML_2^n, which appears in the particular case A. Consequently, applying the same arguments of the particular cases A or B to each of the new GML_2^n surfaces (we do not know if the new slit-zone includes the B-line of this surface or not, but this is not important, since the number n is even and both results are the same), we find that an object Link-2 appears, whose both components are GML_2^n surfaces. Therefore, the domain of definition $\widehat{T}_{B,1}$ in (12) consists of 3 parts and define by (3) 3 different objects GML_2^n.

So that, in the general case, according to (12), to a $(B+\kappa)$-*zones-slit* or to each number κ corresponds the elimination of $\widehat{T}^{\varepsilon}_{B,\kappa+1}$ sub-domains in the domain T. But in this case, the domain of definition $\widehat{T}_{B,\kappa+1}$ in (12) consists of $(\kappa+2)$ parts and defines, by (3), $\kappa+2$ different objects GML_2^n, and the structure of these new objects is always identic to the original surface. The one to one correspondence (3) guarantees that the new objects have not self-crossing points. Some examples are given in Fig. 4.

Let us consider now the *general case B*, when κ is an arbitrary natural number. This means that there exist $(\kappa+1)$-slit-zones. In this case, by using the same arguments of the previous case A, but considering the domain of definition $\widetilde{T}^{*}_{\kappa+1}$ in (13), we find identical results. Some examples are given in Fig. 5.

Theorem 2. *If the GML_2^n surface is cut $(\kappa+1)$-times along $(\kappa+1)$ different (i.e. $\kappa = 0, 1, \cdots$) s-lines and n is an odd number, then for each integer numbers n, κ, after:*

Case A - a $(B+\kappa)$-zone-slits an object Link-$(\kappa+1)$ *appears, whose each component is a GML_2^{2n+2} surface (knot with structure $\{n_1\}$[4]); i.e. for every natural*

[4]The topologic group of the link-$(k+1)$ in this case is at present unknown; only when $k = 0$, the knot is of type $\{n_1\}$, when $n > 1$, and of type $\{0_1\}$, when $n = 1$, according the standard classification (see [2]).

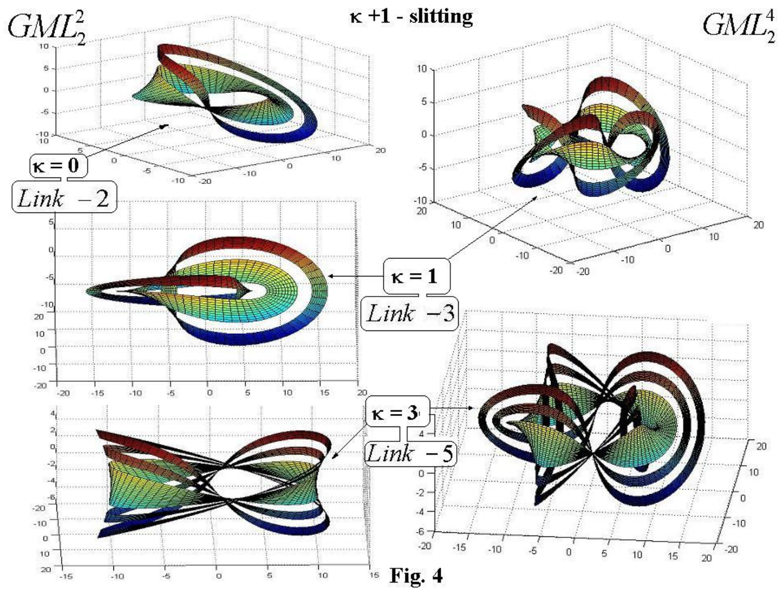

Fig. 4

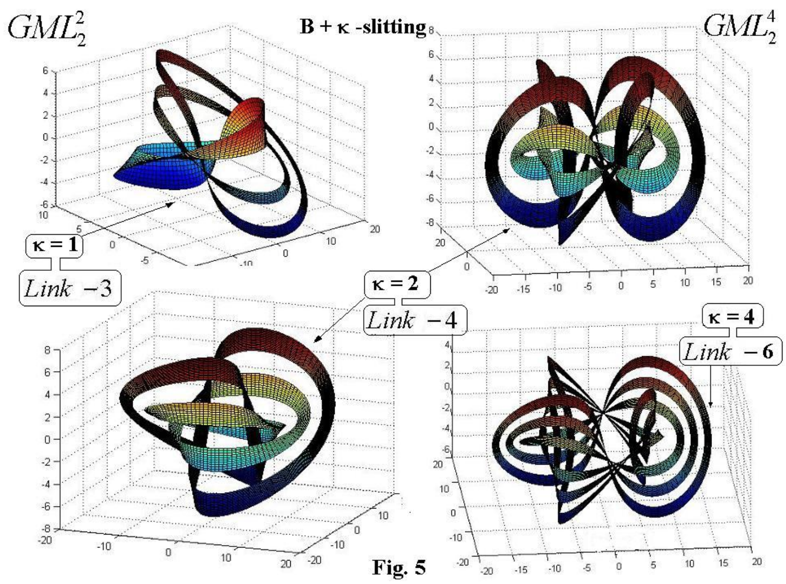

Fig. 5

numbers $\omega = 0, 1, 2, \cdots,$ *and* κ,

(16) $$GML_2^{2\omega+1} \longrightarrow^{B+\kappa} Link\text{-}(\kappa+1) \ of \ (\kappa+1) \times GML_2^{4\omega+4} \, .$$

Case B - *a* $(\kappa+1)$*-zones-slit an object* Link-$(\kappa+2)$ *appears, whose one component is a* GML_2^n *surface (knot with structure* $\{0_1\}$*), and each other component is a* GML_2^{2n+2} *surface (knot with structure* $\{n_1\}$*, except when* $n=1$*, since in this case the topological group is* $(0)_1)$[5]*; i.e. for every natural numbers* $\omega = 0,1,2,\cdots$*, and* κ*,*

(17) $GML_2^{2\omega+1} \longrightarrow^{\kappa+1}$ *Link-*$(\kappa+2)$ *of one* $GML_2^{2\omega+1}$ *and* $(\kappa+1)\times GML_2^{4\omega+4}$.

Proof. If n is an odd number ($n \equiv 2\omega+1$), then the functions $X(\tau,\theta), Y(\tau,\theta)$, $Z(\tau,\theta)$ are 4π-periodic functions of the argument θ, with property (M*); i.e. for each $\tau \in [-\tau^*,\tau^*]$

$$X(-\tau,0) = X(\tau,2\pi);\ Y(-\tau,0) = Y(\tau,2\pi);\ Z(-\tau,0) = Z(\tau,2\pi)\ . \tag{18}$$

Let us consider the *particular case A*, when $\kappa=0$ - This means that there exists only one B-slit-zone.

In this case, the one to one correspondence (3) defines a single object, in spite of the fact that the domain of definition $\widehat{T}_{B,0}$ in (12) is disconnected. This new object has a new basic line (in particular, according to equation (3) and restriction (9)) which is given by

$$\begin{aligned}
X\left(\frac{\tau^*-\varepsilon}{2},\theta\right) &= \left[R+\frac{\tau^*-\varepsilon}{2}\cos\left(\psi+\frac{n\theta}{2}\right)\right]\cos(\theta)\\
Y\left(\frac{\tau^*-\varepsilon}{2},\theta\right) &= \left[R+\frac{\tau^*-\varepsilon}{2}\cos\left(\psi+\frac{n\theta}{2}\right)\right]\sin(\theta)\\
Z\left(\frac{\tau^*-\varepsilon}{2},\theta\right) &= \frac{\tau^*-\varepsilon}{2}\sin\left(\psi+\frac{n\theta}{2}\right)\ .
\end{aligned} \tag{19}$$

This is a representation of a really closed line, but now $\theta \in [0,4\pi)$ (see Fig. 2.a. or Remark 2, when $\mu \in Q$ in [6]), and therefore the unit normal vector (8) makes $2n+2$ rotations around the new basic line (19), since it belongs to a GML_2^{2n+2} surface.

The following cutting of the GML_2^n body or $(B+1)$*-zones-slit* ($\kappa=1$) is a cutting of the new GML_2^{2n+2} surface, which appears after a *B-zones-slit.* Since the number of rotations $2n+2$ is an even number, and by using the same arguments of the previous theorem, we find that, after a $(B+1)$*-zones-slit*, an object "*Link*-$(\kappa+2)$" appears, whose both components are GML_2^{2n+2} surfaces.

In general, after a $(B+\kappa)$*-zones-slit* an object "*Link*-$(\kappa+1)$" appears, whose each component is a GML_2^{2n+2} surface. Some examples are given in Fig. 6.

Let us consider the particular *Case B*, when $\kappa=0$ - i.e. there exists only one s-slit-zone. Since n is an odd number, then according to (11) after an s-zone-slit the new domain of definition (3) is given by

$$\widetilde{T}_1 \equiv \{[-\widetilde{\tau}_1,\widetilde{\tau}_1]\times[0,2\pi)\} \cup \{[\widetilde{\tau}_2,\widetilde{\tau}_3]\times[0,2\pi)\} \cup \{[-\widetilde{\tau}_3,-\widetilde{\tau}_2]\times[0,2\pi)\}\ . \tag{20}$$

This is a disconnected domain, which consist to three parties. The domain defined from the central part of the right hand side of equation (20), according to the representation formulas (3), defines an object GML_2^n . But its width is smaller with respect to the original surface; i.e. $\tau \in [-\tau_0+\varepsilon, \tau_0-\varepsilon]$. Each of the remaining parts of the right hand side of equation (20), similarly to the case A of this theorem,

[5] The general topological group, in this case, is at present unknown.

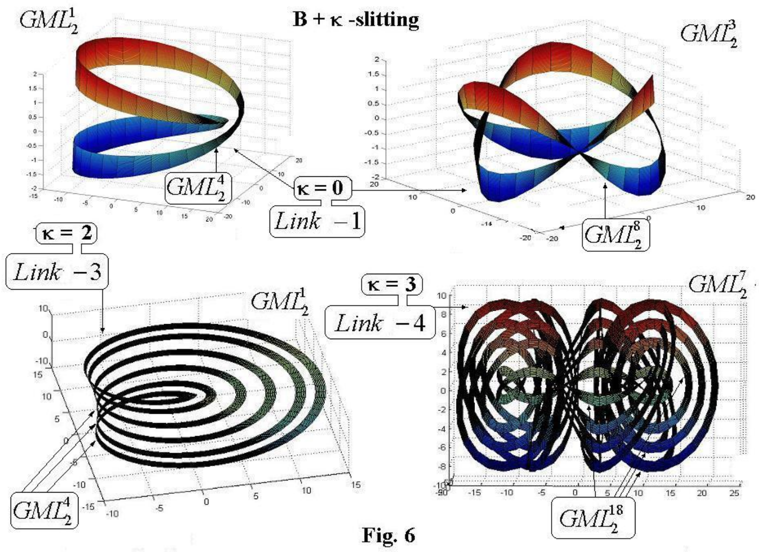

Fig. 6

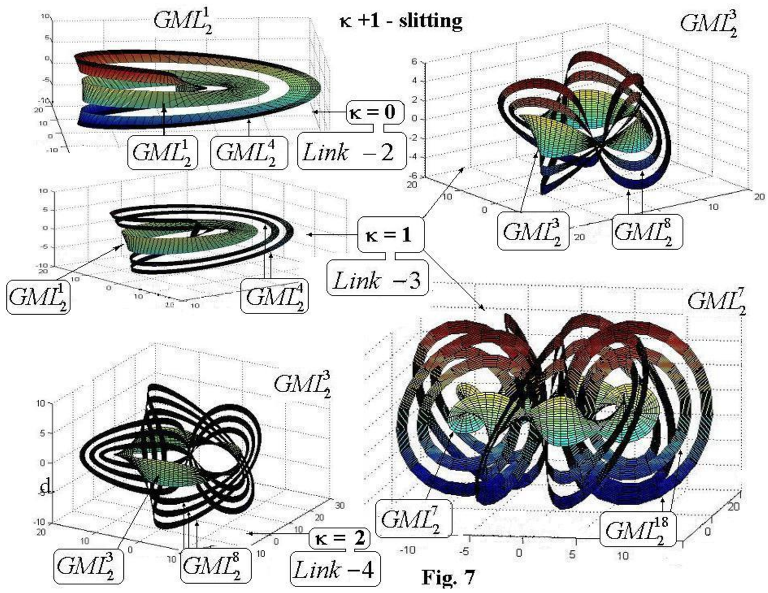

Fig. 7

define a geometrical object GML_2^{2n+2}. So that after an s-zone-slit of a GML_2^n, when n is an odd number, appears a link-2, where one of the components is a GML_2^n and the second one is a GML_2^{2n+2}.

The following cutting of the GML_2^n body or 2-*zones-slit* ($\kappa = 1$) is a cut of one of the new GML_2^{2n+2} or GML_2^n surfaces, which appear after a *B-zones-slit*

If the second cut is a cutting of the GML_2^{2n+2} surface, then since $2n+2$ is a even number, by using the same arguments of previous theorem, we find that after a 2-zones-slit an object "*Link*-3" appears, whose one component is a GML_2^n surface, and each other component is a GML_2^{2n+2} surface.

But if second cut is a cutting of the GML_2^n surface, then, since n is an odd number, by using the same argument of the particular case B of this theorem, we find that after a 2-zones-slit an object "*Link*-3" appears, whose one component is a GML_2^n surface, and all other components are GML_2^{2n+2} surfaces.

Let us consider the *general case B*, when κ is an arbitrary natural number. This means that there exist $(\kappa + 1)$-slit-zones. In this case we can use the previous arguments (when $\kappa = 1$) and remark that the domain of definition $\widetilde{T}_{\kappa+1}$ in (11), consists of $(2\kappa + 3)$ sub-strips and defines an object "*Link*-$(\kappa + 2)$". Therefore, we can conclude that one object is a GML_2^n surface, and each other object is a GML_2^{2n+2} surface. Some examples are given in Fig. 7.

Lastly, from the above Theorems and Remarks 1, 2, 3 (also Remarks 1, 2, 3, 4 of [2]) we can deduce the following facts

Remark 4. Both the previous Theorems still hold when the basic line is a closed space line.

After cutting each regular Möbius-Listing surface GML_2^n, whose basic line is a circle, appear objects (object) with the following properties:

- the tangential vectors of the new objects (object) $\overrightarrow{r}_\tau$ and $\overrightarrow{r}_\theta$ (8),(9) are orthogonal;
- each point of these objects are Hyperbolic (saddle) points if $n \neq 0$;
- each point of these objects are Parabolic points if $n = 0$.

References

[1] E. Abbena, A. Gray A. & S.M. Salamon, *Modern differential geometry of curves and surfaces with Mathematica*, Studies in Advanced Mathematics, (Third Edition), Chapman & Hall/CRC, Boca Raton, Florida, 2006.

[2] C. Cassisa & I. Tavkhelidze, *About some geometric characteristic of the generalized Möbius Listing's surfaces GML_2^n*, Georgian Electronic Scientific Journal: Computer Science and Telecommunications, 4(21)(2009), 54–84.

[3] H. Doll & J. Hoste, *A tabulation of oriented links*, Math. Comput., 57(1991), 747–761.

[4] G. Kuperberg, *Quadrisecants of knots and links*, J. Knot Theory Ramifications, 3(1994), 41–50.

[5] P.E. Ricci & I. Tavkhelidze, *About the analytic representation of a class of geometrical figures, surfaces and lines*, *Reports of enlarged session of the seminar of "I. Vekua" Institute of Applied Mathematics*, 20(2005), 12–15.

[6] P.E. Ricci & I. Tavkhelidze, *Classification of a wide set of geometric figures, surfaces and lines (trajectories)*, Rend. Accad. Naz. Sci. XL, Mem. Mat. Appl. (5), 30(2006), 191–212.

[7] I. Tavkhelidze, *On the some properties of one class of geometrical figures*, Bulletin of TICMI, Tbilisi, 4(2000), 51–55.

[8] I. Tavkhelidze, *On the some properties of one class of geometrical figures and lines*, *Reports of enlarged sessions of the seminar of "I. Vekua" Institute of Applied Mathematics*, 16(1)(2001), 35–38.

[9] I. Tavkhelidze, *Some remarks on a class of geometric figures*, Bulletin of TICMI, Tbilisi, 6(2002), 41–45.

[10] I. Tavkhelidze, *On the analytic representation of one class of geometric figures, surfaces and lines*, Reports of the seminar of "I. Vekua" Institute of Applied Mathematics, 29(2003), 26–32.

[11] I. Tavkhelidze, *Some properties of a class of geometrical bodies, surfaces and lines*, Proceedings of Odlar Urdu University, Baku, Azerbaijan 2003, 60–65 (in Russian).

[12] I. Tavkhelidze, *Classification of a wide set of geometric figures*, Lecture Notes of TICMI, Tbilisi, 8(2007), 53–61.

[13] E.W. Weisstein, *CRC concise encyclopedia of mathematics*, Chapman & Hall/CRC, (Second edition), CRC Press, Boca Raton, Florida, 2003.