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Finite difference methods for Hamilton-Jacobi equations on the Heisenberg group

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Abstract[1]. We propose and analyze numerical schemes for viscosity solutions of Hamilton-Jacobi equations on the Heisenberg group. The main idea is to construct a grid compatible with the noncommutative group geometry. Under suitable assumptions on the data, the Hamiltonian and the parameters for the discrete first order scheme, we prove that the error produced by the finite difference scheme behaves like $\sqrt{h}$ where is h is the grid step. Such an estimate is similar to those available in the Euclidean geometrical setting. We present numerical simulations for both steady and unsteady problems.

1. Basic facts and notations

Let us start by recalling well known properties of the Heisenberg group $H = (\mathbb{R}^3, \oplus)$, where

$$y \oplus x = \big(x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(x_1 y_2 - x_2 y_1)\big) .$$

It is obvious that in general, $x \oplus y \neq y \oplus x$. Note that $x \oplus y = y \oplus x$ if and only if $x_1 y_2 - x_2 y_1 = 0$. Let α be a nonnegative parameter, the dilation of x by α is defined by

$$\alpha \cdot x = (\alpha x_1, \alpha x_2, \alpha^2 x_3) . \tag{1}$$

One can verify that $\alpha \cdot (x \oplus y) = \alpha \cdot x \oplus \alpha \cdot y$.
Observe that for all $x \in \mathbb{R}^3$ and $y = (y_1, y_2, 0)$, one has

$$x \oplus ty = x(t) , \tag{2}$$

where $x(t)$ is the solution of the ordinary differential equation

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x_2(t) & -2x_1(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} ,$$

with the initial value $x(0) = x$.

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We introduce the horizontal gradient

$$D_H u = \begin{pmatrix} \partial_{x_1} u + 2x_2 \partial_{x_3} u \\ \\ \partial_{x_2} u - 2x_1 \partial_{x_3} u \end{pmatrix} , \tag{3}$$

The operator D_H commutes with left translations, i.e. for all $y \in \mathbb{R}^3$, calling $\tau_y^L u$ the function $x \mapsto u(y \oplus x)$,

$$D_H(\tau_y^L u) = \tau_y^L (D_H u) \ . \tag{4}$$

On the contrary, calling $\tau_y^R u$ the function $x \mapsto u(x \oplus y)$,

$$\left(D_H(\tau_y^R u)\right)(x) = \left(\tau_y^R (D_H u)\right)(x) + 4((\partial_{x_3} u)(x \oplus y)) \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \ .$$

The horizontal gradient has the following behavior with respect to dilatations: calling $u \circ \alpha$ the function $x \mapsto u(\alpha \cdot x)$, we have

$$D_H(u \circ \alpha) = \alpha \ (D_H u) \circ \alpha \ . \tag{5}$$

For any fixed $y \in \mathbb{R}^3$, the eikonal problem

$$|D_H w_y(x)| = 1 \quad \text{in } \mathbb{R}^3 \setminus \{y\} \quad , \quad w_y(y) = 0 \tag{6}$$

has a unique viscosity solution satisfying

$$w_y(x) \geq 0 \quad , \quad \forall\, x, y \in \mathbb{R}^3 \ ,$$

$$\lim_{|x-y| \to \infty} w_y(x) = +\infty \ ,$$

$$w_y(x) + w_z(y) \geq w_z(x) \quad , \quad \forall\, x, y, z \in \mathbb{R}^3 \ ,$$

see [3, 4], where $| \cdot |$ is the standard Euclidean norm in $\mathbb{R}^2$.
We use the notation $d(x; y) = w_y(x)$ for the so-called Carnot-Carathéodory distance. It follows easily from the left invariance and homogeneity of D_H, see (4) and (5), that

$$d(z \oplus x; z \oplus y) = d(x; y) \quad , \quad \text{and} \quad d(\alpha \cdot x; \alpha \cdot y) = \alpha d(x; y).; \tag{7}$$

It is also well-known, see [6], that for any $R > 0$ there exists a constant $K(R) > 0$ such that

$$d(x; y) \leq K(R)\, |x - y|^{1/2} \quad \text{for all} \quad x, y \in \mathbb{R}^3 \ , \ \ |x - y| \leq R \ . \tag{8}$$

We denote by $| \cdot |_K$ the Korànyi homogeneous norm in $\mathbb{R}^3$, which is naturally associated with the Heisenberg group:

$$|x|_K = \left((x_1^2 + x_2^2)^2 + x_3^2\right)^{1/4} \ . \tag{9}$$

It is clear that

$$|x|_K = \sqrt{x_1^2 + x_2^2} = |x|$$

for any horizontal vector $x = (x_1, x_2, 0)$. Note also that for each $\alpha \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^3$, $|\alpha \cdot x|_K = \alpha |x|_K$ and $| - y \oplus x|_K = | - x \oplus y|_K$. It is proved in [12] that $(x, y) \mapsto | - y \oplus x|_K$ defines a metric in $\mathbb{R}^3$. It can be seen that $x \mapsto | - y \oplus x|_K$ is a viscosity subsolution of (6).

We also recall that there exist two positive constants $c_1 < c_2$ such that

$$c_1 |-x \oplus y|_K \le d(x; y) \le c_2 |-x \oplus y|_K \ , \tag{10}$$

see [6]. For what follows, we define the Carnot-Carathéodory balls

$$B_C(r) = \{x \in \mathbb{R}^3,\ d(x; 0) \le r\} \ ,$$

and the Korànyi balls

$$B_K(r) = \{x \in \mathbb{R}^3,\ |x|_K \le r\} \ .$$

We shall say that u is Lipschitz continuous with respect to the left translations with a constant L if, for all $y \in \mathbb{R}^3$,

$$\sup_{z\in\mathbb{R}^3} |u(y \oplus z) - u(z)| \le L|y|_K \ .$$

Similarly, u is Lipschitz continuous with respect to the right translations with a constant L if, for all $y \in \mathbb{R}^3$,

$$\sup_{z\in\mathbb{R}^3} |u(z \oplus y) - u(z)| \le L|y|_K \ .$$

For example, for any real valued Lipschitz continuous function χ on $\mathbb{R}_+$, $x \mapsto \chi(|x|_K)$ is Lipschitz continuous w.r.t. right translations. Also, any bounded subsolution of $|D_H w| \le 1$ in $\mathbb{R}^3$ is Lipschitz continuous with respect to right translations, see [7].

2. Hamilton-Jacobi equations on the Heisenberg group

We consider the approximation of solutions of Cauchy problems for some first order degenerate Hamilton-Jacobi partial differential equation, of the form

$$\begin{aligned} \frac{\partial u}{\partial t} + \Phi(|D_H u|) &= 0\ , && \text{in } \mathbb{R}^3 \times (0,\infty)\ , \\ u(x,0) &= u_0(x)\ , && \text{in } \mathbb{R}^3\ , \end{aligned} \tag{11}$$

where Φ is a nondecreasing and convex function on $\mathbb{R}_+$ such that $\Phi(0) = 0$ and $\lim_{q\to+\infty} \Phi(q) = +\infty$. If the initial datum u_0 is bounded and continuous and under the previously mentioned assumptions on Φ, the Hopf-Lax formula

$$u(x,t) = \inf_{y\in\mathbb{R}^3} \left(u_0(y) + t\Phi^* \left(\frac{d(x;y)}{t} \right) \right) , \tag{12}$$

see [13, 8, 9, 10], where Φ^* is the conjugate function, $\Phi^*(q) = \sup_{p\ge 0} \big(p\, q - \Phi(p)\big)$. Moreover, if we also assume that the Φ^* is such that

$$\lim_{q\to+\infty} \frac{\Phi^*(q)}{q} = +\infty \ , \tag{13}$$

then

$$u(x,t) = \min_{y\in\mathbb{R}^3} \left(u_0(y) + t\Phi^* \left(\frac{d(x;y)}{t} \right) \right) . \tag{14}$$

For example, assumption (13) is satisfied by $\Phi(p) = (1/\alpha)\, p^\alpha$ with $\alpha \ge 1$.

Properties of the viscosity solutions. For each $t \geq 0$, let $S(t)$ be the time t map associated with (11), i.e. $S(t)u_0(x) = u(x,t)$ where u is the viscosity solution of (11). In the following proposition we summarize several useful properties of $S(t)$.

Proposition 1. *Let Φ be a nondecreasing and convex function on $\mathbb{R}_+$ such that $\Phi(0) = 0$, $\lim_{q\to+\infty}\Phi(q) = +\infty$ and (13). Then, for continuous functions u_0 and v_0,*

(1) $\|(S(t)u_0 - S(t)v_0)^+\|_\infty \leq \|(u_0 - v_0)^+\|_\infty$.

(2) $\|S(t)u_0 - S(t)v_0\|_\infty \leq \|u_0 - v_0\|_\infty$.

(3) $\inf_{\mathbb{R}^3} u_0 \leq S(t)u_0 \leq \sup_{\mathbb{R}^3} u_0$.

(4) $\|\tau_y^L(S(t)u_0) - S(t)u_0\|_\infty \leq \|\tau_y^L(u_0) - u_0\|_\infty$.

(5) *If u_0 is Lipschitz continuous with respect to left translations with a constant L_1, then so is $S(t)u_0$.*

(6) $S(t+\tau)u_0 \leq S(t)u_0\,, \forall \tau > 0$.

(7) *If u_0 is Lipschitz continuous with respect to right translations with a constant L, then for $K = \Phi(L/c_1)$, where c_1 appears in (10),*

$$\|S(t)u_0 - S(t')u_0\| \leq K|t-t'| \quad , \quad \forall\, t,t' \geq 0\,. \tag{15}$$

(8) *If $\operatorname{supp}(u_0) \subset B_C(R_0)$, then $S(t)u_0$ is compactly supported and there exists a function $R : \mathbb{R}_+ \to \mathbb{R}_+$, nondecreasing, which only depends on Φ^* and on $\|u_0^-\|_\infty$, such that*

$$\operatorname{supp}(S(t)u_0) \subset B_C(R_0 + R(t)) \subset B_K(\frac{1}{c_1}\,(R_0 + R(t)))\,. \tag{16}$$

(9) *If*
- *u_0 is supported in the Carnot-Carathéodory ball $B_C(R_0)$,*
- *u_0 is Lipschitz continuous with respect to left translations with a constant L_1,*
- *there exists L_2 such that $\|u_0(\cdot \oplus \delta e_3) - u_0(\cdot)\| \leq L_2|\delta|$, for all $\delta > 0$,*

then $S(t)u_0$ is Lipschitz continuous with respect to right translations with a constant

$$L(t) = L_1 + \frac{4L_2(R_0 + R(t))}{c_1}\,. \tag{17}$$

Proof. See [1].

□

Remark 1. In Proposition 1, the assumptions of point 9 imply the fact that u_0 is Lipschitz continuous with respect to the right translations with a constant $L = L_1 + 4(L_2R/c_1)$. Therefore the assumptions of point 9 imply point 7 with

$$K = \Phi\left(L_1 + 4\,\frac{L_2R}{c_1}\right)\,. \tag{18}$$

3. Finite difference methods

3.1. **Finite difference schemes on discrete groups.** Let T be a positive time. We are interested in approximating u for times $t \leq T$. Let P be a positive integer and $\Delta t = T/P$. Let h be a positive real number. Hereafter, we assume that there exists a constant C such that

$$\Delta t \leq Ch\ . \tag{19}$$

For three integers i, j, k we define the nodes $\xi_{i,j,k} = (ih, jh, (4k+2ij)h^2)$, and for a nonnegative integer n, we define $t_n = n\Delta t$. This lattice was introduced in [2] as the key ingredient for a second order finite difference scheme for the Kohn Laplacian on the Heisenberg group. Calling (e_1, e_2, e_3) the canonical basis of $\mathbb{R}^3$, we have

$$\begin{aligned} \xi_{i,j,k} \oplus \pm he_1 &= \xi_{i\pm 1,j,k}\ , \\ \xi_{i,j,k} \oplus \pm he_2 &= \xi_{i,j\pm 1,k\mp i}\ . \end{aligned} \tag{20}$$

More generally,

$$\xi_{\ell,m,n} \oplus \xi_{i,j,k} = \xi_{\ell+i,m+j,k+n-j\ell}\ . \tag{21}$$

Formulas (20) and (21) clearly show the relationship between the grid and the group operations $\oplus$ and $\cdot$. Since $\xi_{i,j,k} \oplus \xi_{\ell,m,n} = \xi_{\ell+i,m+j,k+n-im}$, we see that $\xi_{i,j,k} \oplus \xi_{\ell,m,n}$ and $\xi_{\ell,m,n} \oplus \xi_{i,j,k}$ coincide if and only if $im = j\ell$.
Capital letters U, V,... will stand for discrete functions defined on the lattice $\{\xi_{i,j,k}, i, j, k \in \mathbb{Z}\}$ and their values at $\xi_{i,j,k}$ will be written $U_{i,j,k}$, $V_{i,j,k}$,... . The notations $\Delta^1_+ U$ and $\Delta^2_+ U$ will be used for the discrete functions:

$$(\Delta^1_+ U)_{i,j,k} = U_{i+1,j,k} - U_{i,j,k} \quad , \quad (\Delta^2_+ U)_{i,j,k} = U_{i,j+1,k-i} - U_{i,j,k}\ .$$

The value of the numerical approximation of $u(\xi_{i,j,k}, t_n)$ will be written $U^n_{i,j,k}$.
We shall consider numerical schemes

$$U^{n+1}_{i,j,k} = G(U^n_{i,j,k}, U^n_{i+1,j,k}, U^n_{i-1,j,k}, U^n_{i,j+1,k-i}, U^n_{i,j-1,k+i}), \tag{22}$$

such that there exists a continuous function $g : \mathbb{R}^4 \to \mathbb{R}$, called the *numerical Hamiltonian*, with

$$\begin{aligned} G(U_{i,j,k}, U_{i+1,j,k}, U_{i-1,j,k}, U_{i,j+1,k-i}, U_{i,j-1,k+i}) = U_{i,j,k} - \\ -\Delta t g\left(\frac{1}{h}(\Delta^1_+ U)_{i,j,k}, \frac{1}{h}(\Delta^1_+ U)_{i-1,j,k}, \frac{1}{h}(\Delta^2_+ U)_{i,j,k}, \frac{1}{h}(\Delta^2_+ U)_{i,j-1,k+i}\right)\ . \end{aligned} \tag{23}$$

For the scheme (22) to be consistent with the Hamilton-Jacobi equation, we must have

$$g(a, a, b, b) = \Phi\left(\left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\|\right)\ . \tag{24}$$

We will say that (22) is monotone if G is a nondecreasing function of each of its five arguments. We will say that (22) is monotone on $[-R, R]$ if $G(U_{i,j,k}, U_{i+1,j,k}, U_{i-1,j,k}, U_{i,j+1,k-i}, U_{i,j-1,k+i})$ is a nondecreasing function of each of its five arguments as long as $(\Delta^1_+ U)_{i,j,k}$, $(\Delta^1_+ U)_{i-1,j,k}$, $(\Delta^2_+ U)_{i,j,k}$ and $(\Delta^2_+ U)_{i,j-1,k+i}$ are contained in $[-R, R]$.
For brevity, we will use the notation $\vec{G}(U) = (G(U)_{i,j,k})_{i,j,k\in\mathbb{Z}}$. We will also use the notation $\|U\|_\infty = \sup_{i,j,k\in\mathbb{Z}} |U_{i,j,k}|$. We will say that $U \in \ell^\infty(\mathbb{Z}^3)$ if $\|U\|_\infty < +\infty$.
For $\Lambda > 0$, we call $\mathcal{C}_\Lambda$ the set

$$\mathcal{C}_\Lambda = \{U \in \ell^\infty(\mathbb{Z}^3)\ ,\ |(\Delta^1_+ U)_{i,j,k}| < \Lambda h\ ,\ |(\Delta^2_+ U)_{i,j,k}| < \Lambda h\ ,\ \forall\, i, j, k \in \mathbb{Z}\}\ . \tag{25}$$

Finally, for $(\ell,m,n)\in\mathbb{Z}^3$, we note $\tau^L_{(\ell,m,n)}U$ the discrete function defined by

$$(\tau^L_{(\ell,m,n)}U)_{i,j,k} = U_{\ell+i,m+j,k+n-4j\ell}\ .$$

Proposition 2. *Assume that the scheme (22) is consistent and monotone on $[-\Lambda,\Lambda]$. Then*

(1) *Identifying $\lambda\in\mathbb{R}$ with the constant function λ on $\mathbb{Z}^3$, we have $\vec{G}(U+\lambda) = \vec{G}(U)+\lambda$, for all discrete function U.*
(2) *For U and V in $\mathcal{C}_\Lambda$,*

$$\|(\vec{G}(U)-\vec{G}(V))^+\|_\infty \le \|(U-V)^+\|_\infty\ . \tag{26}$$

(3) *For U and V in $\mathcal{C}_\Lambda$ such that $U\le V$, $\vec{G}(U)\le\vec{G}(V)$.*
(4) *For U and V in $\mathcal{C}_\Lambda$,*

$$\|\vec{G}(U)-\vec{G}(V)\|_\infty \le \|U-V\|_\infty\ . \tag{27}$$

(5) *The operator $\vec{G}$ commutes with the left lattice translations: for $(\ell,m,n)\in\mathbb{Z}^3$,*

$$\vec{G}(\tau^L_{(\ell,m,n)}U) = \tau^L_{(\ell,m,n)}\vec{G}(U)\ . \tag{28}$$

(6) *If $U^0\in\mathcal{C}_\Lambda$ and if there exists a positive number L_1 such that for all $(\ell,m,n)\in\mathbb{Z}^3$, $\|\tau^L_{(\ell,m,n)}U^0-U^0\|_\infty\le L_1|\xi_{\ell,m,n}|_K$, then for all $p\ge 0$, $U^p=\vec{G}^p(U^0)$ has the same property.*
(7) *If the discrete function U^0 satisfies: there exist two positive integers I_0 and J_0 and two positive real numbers L_1 and L_2 such that*
   - *$U^0_{i,j,k}=0$ if $|i|>I_0$ and $|j|>J_0$,*
   - *for all $(\ell,m,n)\in\mathbb{Z}^3$, $\|\tau^L_{(\ell,m,n)}U^0-U^0\|_\infty\le L_1|\xi_{\ell,m,n}|_K$,*
   - *for all $k\in\mathbb{Z}$, $\|\tau^L_{(0,0,k)}U^0-U^0\|_\infty\le 4L_2|k|h^2$,*
   - *$L_1+4L_2(P+\max(I_0,J_0))h<\Lambda$,*

   then for all $p\ge 0$, $U^p=\vec{G}^p(U^0)$ is such that

$$\begin{aligned}\|\Delta^1_+U^p\|_\infty &\le (L_1+4L_2(p+J_0)h)h\ ,\\ \|\Delta^2_+U^p\|_\infty &\le (L_1+4L_2(p+I_0)h)h\ .\end{aligned} \tag{29}$$

(8) *Under the assumptions of point 7 on U^0, there exists a constant K' depending on L_1, L_2, Ph, $(I_0+J_0)h$ such that, for all $p<P$,*

$$\|U^{p+1}-U^p\|_\infty \le K'\Delta t\ . \tag{30}$$

Proof. See [1].

□

3.2. **Examples.**

Godunov like schemes. Take equation (11) with Φ satisfying all the assumptions above. The upwind scheme proposed by Osher and Sethian, [14] and also [15], reads (22), with (23) and

$$g(u_1,u_2,v_1,v_2) = \tag{31}$$

$$= \Phi\left(\left(\min(u_1,0)^2 + \max(u_2,0)^2 + \min(v_1,0)^2 + \max(v_2,0)^2 \right)^{1/2} \right) .$$

From the hypothesis on Φ, we see that the scheme is monotone on $[-\Lambda,\Lambda]$ if $1 - (2\Delta t/h)\,\Phi'(2\Lambda) \geq 0$.

The Lax-Friedrichs scheme. The Lax-Friedrichs scheme for equation (11) is (22), with (23) and

$$g(u_1,u_2,v_1,v_2) = \tag{32}$$

$$= \Phi\left(\left(\left(\frac{u_1+u_2}{2}\right)^2 + \left(\frac{v_1+v_2}{2}\right)^2 \right)^{1/2} \right) - \theta \frac{h}{\Delta t}\,(u_1 - u_2 + v_1 - v_2) \ ,$$

where θ is a positive constant. It can be verified that the scheme is monotone on $[-\Lambda,\Lambda]$ provided $0 < \theta < 1/4$ and $\theta - (\Delta t/2h)\,\Phi'(\sqrt{2}\Lambda) \geq 0$.

3.3. **Numerical analysis.** We now give the main theorem:

Theorem 1. *Under the following assumptions:*

(1) Φ *satisfies the assumptions of Proposition 1,*
(2) *the difference scheme* (22) *is in the form* (23)*, monotone on* $[-\Lambda,\Lambda]$ *and consistent with* (11)*,*
(3) *the function* u_0 *satisfies the assumptions in point 9 of Proposition 1, and the interpolation* U^0 *of* u_0 *on the lattice* $(ih, jh, (4k+2ij)h^2)$*,* $i,j,k \in \mathbb{Z}$*, satisfies the assumptions in point 7 of Proposition 2,*
(4) $L(T)$ *defined by* (17) *satisfies* $L(T) < \Lambda$*,*
(5) *the numerical Hamiltonian* g *is locally Lipschitz continuous,*
(6) *for a positive constant* C*,* $\Delta t \leq Ch$*,*

there exist two positive constants H *and* c *(independent of* h*) such that for* $h < H$*,*

$$|U^p_{i,j,k} - u(\xi_{i,j,k}, t_p)| \leq ch^{1/2} \ , \tag{33}$$

for all $0 \leq p \leq P$ *and* $i,j,k \in \mathbb{Z}$*.*

Sketch of the Proof. The full proof is contained in [1]. The strategy is similar to that of [11]. We seek to estimate

$$\sup_{\substack{i,j,k\in\mathbb{Z}\\ 0\leq p\leq P}} |U^p_{i,j,k} - u(\xi_{i,j,k}, p\Delta t)| \ .$$

For that purpose, we will assume

$$\sup_{\substack{i,j,k\in\mathbb{Z}\\ 0\leq p\leq P}} \left(u(\xi_{i,j,k}, p\Delta t) - U^p_{i,j,k} \right) = \sigma > 0 \ , \tag{34}$$

and look for an upper bound on σ. Were

$$\inf_{\substack{i,j,k\in\mathbb{Z}\\ 0\le p\le P}} \Big(u(\xi_{i,j,k}, p\Delta t) - U^p_{i,j,k} \Big) = -\sigma < 0 \ ,$$

we could estimate σ exactly in the same way, so we have bounds from below and from above. For that, we define

$$M = \|u_0\|_{L^\infty(\mathbb{R}^3)} + 1 \ . \tag{35}$$

Note that Propositions 1 and 2 above imply that

$$|u| \le M \quad \text{on } Q \quad , \quad \text{and} \quad \|U^p\|_\infty \le M \quad 0 \le p \le P \ . \tag{36}$$

For simplifying the notations, we call $Q = \mathbb{R}^3 \times [0,T]$ and $Q^d = \{(\xi_{i,j,k}, p\Delta t), i,j,k \in \mathbb{Z}, 0 \le p \le P\}$. The main ingredient for obtaining the desired estimate will be to consider the function $\Psi : Q \times Q^d \to \mathbb{R}$,

$$\Psi(\eta, t, \xi, s) = u(\eta, t) - U^p_{i,j,k} + \Big(5M + \frac{\sigma}{2}\Big)\beta_\epsilon(-\xi \oplus \eta, t-s) - \frac{\sigma(t+s)}{4T} \tag{37}$$

where $\xi = \xi_{i,j,k}$, $s = p\Delta t$ and $\beta_\epsilon(x,t) = \beta(|(1/\epsilon)\cdot x|_K, t/\epsilon)$, with ϵ is a positive real number and β a smooth function on $\mathbb{R}\times\mathbb{R}$, satisfying

$$\beta(0,0) = 1 \ \ , \ \ 0 \le \beta \le 1 \ \ , \ \ \beta(r,t) = 0 \ \text{ if } r^4 + t^4 > 1 \ . \tag{38}$$

We choose

$$\epsilon = h^{3/8} \ , \tag{39}$$

and the function β such that there exists a smooth function $b : \mathbb{R}_+ \to [0,1]$, with

$$\beta(x,t) = b(|x|_K^4 + t^4) \ , \tag{40}$$

$$\begin{aligned} b(z) &= 1 - z \ , && \text{if } z \le \frac{1}{2} \ , \\ b(z) &= 0 \ , && \text{if } z \ge 1 \ , \\ b(z) &\le \frac{1}{2} \ , && \text{if } z \ge \frac{1}{2} \ . \end{aligned}$$

First step. We first use a lemma due to Crandall and Lions[11]: there exists $(\eta_0, t_0, \xi_0, s_0) \in Q \times Q^d$ such that $\Psi(\eta_0, t_0, \xi_0, s_0) = \max_{Q\times Q^d} \Psi$ and we have $\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) \ge 3/5$, thus $|-\xi_0 \oplus \eta_0|_K^4 + |t_0 - s_0|^4 \le 2\epsilon^4/5$.
Second step. We improve the previous estimates on $-\xi_0 \oplus \eta_0$ and $|t_0 - s_0|$. Indeed, the Lipschitz regularity of u w.r.t. right translations yields lower bounds on $|D_H\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0)|$, $|\partial_3\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0)|$ and $|D_t\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0)|$, which imply

$$\begin{aligned} &(\xi_{0,1} - \eta_{0,1})^2 + (\xi_{0,2} - \eta_{0,2})^2 \lesssim \epsilon^{8/3} \sim h \ , \\ &|\xi_{0,3} - \eta_{0,3} + 2(\eta_{0,2}\xi_{0,1} - \eta_{0,1}\xi_{0,2})| \lesssim \epsilon^4 \ , \\ &|t_0 - s_0| \lesssim \epsilon^{4/3} \sim h^{1/2} \ , \ \text{ if } 0 < t_0 < T \ . \end{aligned}$$

Third step. Assuming for simplicity $t_0 > 0$ and $s_0 > 0$, we have

$$\begin{aligned} \frac{\sigma}{4T} \le \Big(5M + \frac{\sigma}{2}\Big) D_t\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) - \\ -\Phi\left(\Big(5M + \frac{\sigma}{2}\Big) \ |(D_H\beta_\epsilon)(-\xi_0 \oplus \eta_0, t_0 - s_0)|\right) \ , \end{aligned}$$

from the facts that u is a viscosity solution of (11) and that (η_0, t_0) is a maximum in Q of

$$(\eta, t) \mapsto u(\eta, t) + \left(5M + \frac{\sigma}{2}\right) \beta_\epsilon(-\xi_0 \oplus \eta, t - s_0) - \frac{\sigma t}{4T} .$$

Fourth step. We obtain a similar estimate on the discrete side, mainly using the fact that the scheme is monotone, the definition of $(\eta_0, \xi_0, t_0, s_0)$ and the estimates obtained at the second step of the proof:

$$\frac{\sigma}{4T} \leq -\left(5M + \frac{\sigma}{2}\right) D_t \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) +$$
$$+ \Phi\left(\left(5M + \frac{\sigma}{2}\right) |(D_H \beta_\epsilon)(-\xi_0 \oplus \eta_0, t_0 - s_0)|\right) + Ch^{1/2} .$$

Fifth step. Summing the last two estimates yields

$$\sigma \lesssim h^{1/2} .$$

Sixth step. Deal with the case $t_0 = 0$ or $s_0 = 0$ as in [11].

□

Remark 2. The error produced by the first order scheme is of the order of $\sqrt{h}$, which is precisely the estimate obtained by Crandall and Lions [11] for monotonous finite difference schemes in the nondegenerate case.

4. Numerical results

4.1. **The eikonal equation.** To test the methods against semi-analytical results, we first consider the static eikonal equation (6) for which a complete theory is available. More generally, we consider the static problems

$$\begin{aligned} &\Phi(|D_H u|) = f \quad , \quad \text{in } \mathbb{R}^3 \setminus \overline{\omega} , \\ &u(x) = u_0(x) \quad , \quad \text{in } \overline{\omega} , \end{aligned} \tag{41}$$

where ω is a given subset of $\mathbb{R}^3$. The numerical schemes are essentially the same as for the time dependent problems. The first order scheme that we use is

$$\Phi\left(\left(\begin{array}{c} \max(-\frac{1}{h}(\Delta_+^1 U)_{i,j,k}, \frac{1}{h}(\Delta_+^1 U)_{i-1,j,k}, 0)^2 \\ \\ + \max(-\frac{1}{h}(\Delta_+^2 U)_{i,j,k}, \frac{1}{h}(\Delta_+^2 U)_{i,j-1,k+i}, 0)^2 \end{array}\right)^{\frac{1}{2}}\right) = f_{i,j,k} \ , \xi_{i,j,k} \notin \overline{\omega} , \tag{42}$$

$$U_{i,j,k} = 0 \quad \xi_{i,j,k} \in \overline{\omega} .$$

Assuming Φ is a one to one mapping from $\mathbb{R}_+$ onto $\mathbb{R}_+$, $\Phi^{-1}(f_{i,j,k})$ can be computed by a Newton method and the equation in (42) is equivalent to the quadratic equation

$$\max(-(\Delta_+^1 U)_{i,j,k}, (\Delta_+^1 U)_{i-1,j,k}, 0)^2 + \tag{43}$$
$$+ \max(-(\Delta_+^2 U)_{i,j,k}, (\Delta_+^2 U)_{i,j-1,k+i}, 0)^2 = \left(h\Phi^{-1}\left(f_{i,j,k}\right)\right)^2 .$$

The numerical schemes produce a system of nonlinear equations,which is solved by the fast marching method of Sethian, see [15].

It is possible to obtain a more accurate fast marching method by using a higher order scheme where it is possible to use already computed value. For brevity, we do not describe this scheme here and we refer to [15]. Roughly speaking, this new scheme is second order in the regions where the solution is smooth and first order near singularities.

We first aim at numerically computing the Carnot-Carathéodory distance to the origin, that is the solution u of problem (41) with $\Phi(s) = s$, $f = 1$, $\overline{\omega} = \{(0,0,0)\}$ and $u_0 = 0$. As shown in Beals, Gaveau and Greiner [5], the geodesics or Hamiltonian paths relative to the origin and a point $x = (x_1, x_2, x_3)$ such that $x_1^2 + x_2^2 > 0$, (which satisfy $x(0) = 0$, $x(t) = x$, for some $t > 0$), are given by

$$\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} = \frac{\sin(2s\theta)}{\sin(2t\theta)} e^{(s-t)\theta\Xi} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{with } \Xi = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \tag{44}$$

$$x_3 - x_3(s) = \frac{4(t-s)\theta - \sin(2t\theta) + \sin(2s\theta)}{2\sin^2(2t\theta)} (x_1^2 + x_2^2),$$

where θ is a solution to

$$\mu(2t\theta) = \frac{x_3}{x_1^2 + x_2^2}, \tag{45}$$

and where we have set

$$\mu(\phi) = \frac{\phi}{\sin^2 \phi} - \cot \phi. \tag{46}$$

It is proved that (45) has a unique solution $2t\theta$ in the interval $[0, \pi)$, and that the square of the Carnot-Carathéodory distance $d^2(x; 0)$ is the action integral corresponding to the Hamiltonian curve:

$$d^2(x;0) = \frac{4t^2\theta^2}{2t\theta + \sin^2(2t\theta) - \sin(2t\theta)\cos(2t\theta)} (|x_3| + x_1^2 + x_2^2) \quad \text{if } \theta \neq 0, \tag{47}$$

$$d^2(x;0) = \left(x_1^2 + x_2^2\right) \quad \text{if } \theta = 0.$$

Thus if $x_1^2 + x_2^2 > 0$, computing $d(x; 0)$ requires solving the one dimensional nonlinear equation (45) in $[0, \pi)$, which can be done numerically with Newton's method for example. If, on the contrary $x_1^2 + x_2^2 = 0$, the Carnot-Carathéodory distance is given by $d(x; 0) = \sqrt{\pi |x_3|}$.
Let u be the solution to the eikonal equation $|D_H u(x)| = 1$ for $x \neq 0$ and $u(0) = 0$, then the geodesic curve joining x to the origin is computed as follows:

- set $t = u(x)$.
- Compute $x(s)$, $s \in [0, t]$, by solving the Cauchy problem:

$$\begin{aligned} \frac{dx}{dt}(s) &= -\frac{1}{|D_H u(x(s))|^2} (\sigma(x(s)))^T D_H u(x(s)) \quad 0 < s < t, \\ x(0) &= x. \end{aligned} \tag{48}$$

We have tested the fast marching method with the two schemes mentioned above. Table 4.1 contains the error $\max_{\xi_{i,j,k} \in [-1/2, 1/2]^3} |U_{i,j,k} - d_(\xi_{i,j,k}; 0)|$ where U has been computed with the fast marching method and either the first order scheme (42) or the first/second order scheme. The first line of the table contains the number of unknowns, i.e. $1/4h^4$. In Figure 1, we have plotted the error versus h in logarithmic scale. We see that the error produced by scheme (42) behaves like $O(\sqrt{h})$, in agreement with the theory above. The error produced by the first/second order scheme is smaller, and the slope (in logarithmic scale) of the curve lies between 1/2 and 1.

In Figure 2, we have plotted some Carnot-Carathéodory spheres centered at 0, intersected with the planar region $\{0\} \times [-0.5, 0.5]^2$: these spheres are obtained as the level sets of U computed by the first/second order scheme with $h = 1/100$. We very well see that the spheres have a conical singularity near the axis $x_1 = x_2 = 0$, with an angle that gets sharper as $|x_3|$ grows. Note that, for obvious reasons, the grid used for representing the Carnot Carathéodory spheres is coarser than the one used for computation, and corresponds to $h = 1/60$.
In Figure 3, we have plotted the Carnot-Carathéodory geodesic curve between the point $(0.15, 0.15, 0.3)$ and the origin, computed by the semi-analytic formula (44) or by a discrete solution to (48):

- the parameter h is $1/120$.
- in (48) $D_H u$ is first approximated at the grid nodes by a second order difference formula applied to U, where U has been computed with one of the two finite difference methods described above.
- for a point x not on the grid, $D_H u(x)$ is computed by a bilinear interpolation of the values previously computed at the grid nodes.
- A second order midpoint scheme is used for integrating (48).

In Figure 3, we see that the geodesic curve is well approximated by the discrete method.
In Figure 4, we have computed the Carnot-Carathéodory distance to some compact sets $\overline{\omega}$, by solving the boundary value problem (41) with the first/second order finite difference scheme and $h = 1/120$. On the left of figure, we choose $\overline{\omega}$ as the convex set $\{x;\ |x_1| + |x_2| + |x_3| \le 0.2\}$. On the right of the figure, $\overline{\omega}$ is nonconvex, and has the shape of a three-dimensional cross.

$1/h$	20	40	60	80	100	120
size	4.10^4	$6.4\ 10^5$	$3.24\ 10^6$	$1.024\ 10^7$	$2.5\ 10^7$	$5.184\ 10^7$
scheme (42)	0.121287	0.0769367	0.060584	0.0497205	0.0446911	0.0405027
2^{nd} scheme	0.0996706	0.0499173	0.0361482	0.0286559	0.0244234	0.0218842

TABLE 1. L^∞ Error between the theoretical and computed values of $d(x;0)$ for $x \in [-\frac{1}{2}, \frac{1}{2}]^3$ vs. h.

4.2. **The initial value problem.** We consider the following boundary value problem (11), with $\Phi(d) = d$ and

$$u_0(x) = -0.5 \qquad \text{if } |x|_K \le 0.1\ ,$$
$$u_0(x) = 0.5 - \exp(-10^3 |x|_K^4 + 0.1) \quad \text{if } |x|_K > 0.1\ . \tag{49}$$

We have discretized this equation for $x \in (-1, 1)^2 \times (-1/2, 1/2)$, $t \in (0, 1)$, with the scheme (22) (23), (31). We have taken $h = 1/100$; the lattice in the x variable has $200^2 \times 100^2/4 = 10^8$ nodes. We have chosen $\Delta t = 1/200$, so the first order scheme in (22), (23), (31) is monotone.
In Figure 5, we have plotted five level sets of u at time $t = 0.25$, around the front $u = 0$, corresponding to $u = -0.2, -0.1, 0, 0.1, 0.2$, computed by the finite difference scheme We see clearly the singular behavior of u around the axis $x_1 = x_2 = 0$.

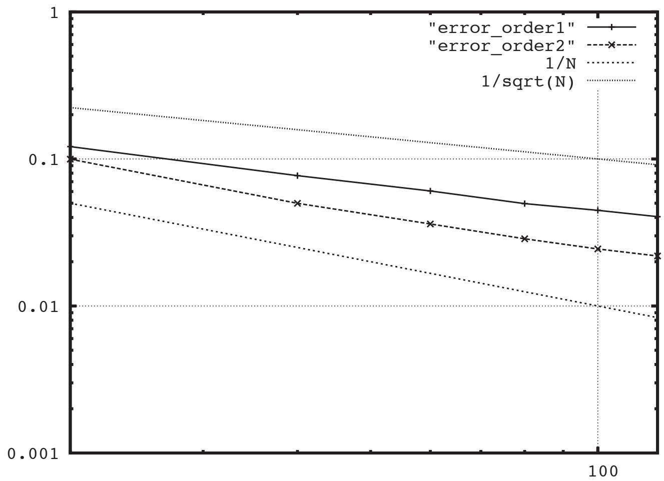

FIGURE 1. L^∞ Error between the theoretical and computed values of $d(x;0)$ for $x \in [-\frac{1}{2}, \frac{1}{2}]^3$ vs. $N = \frac{1}{h}$.

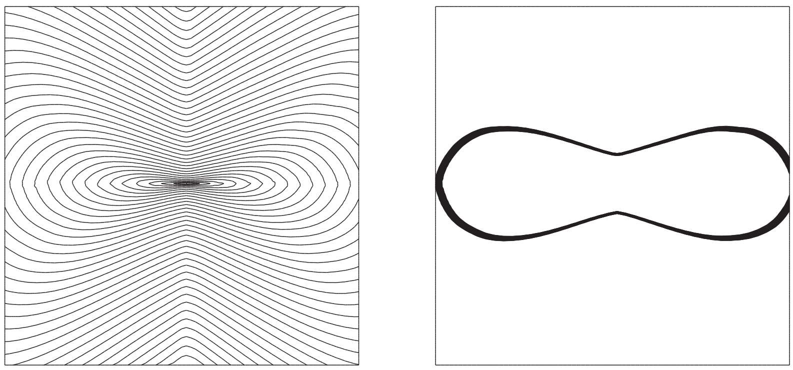

FIGURE 2. Left: Carnot-Carathéodory spheres ∂B_C intersected with the plane $x_1 = 0$, found as the level sets of U computed with the first/second order finite difference scheme $h = 1/100$. Right: some Carnot-Carathéodory spheres with radius close to 0.5.

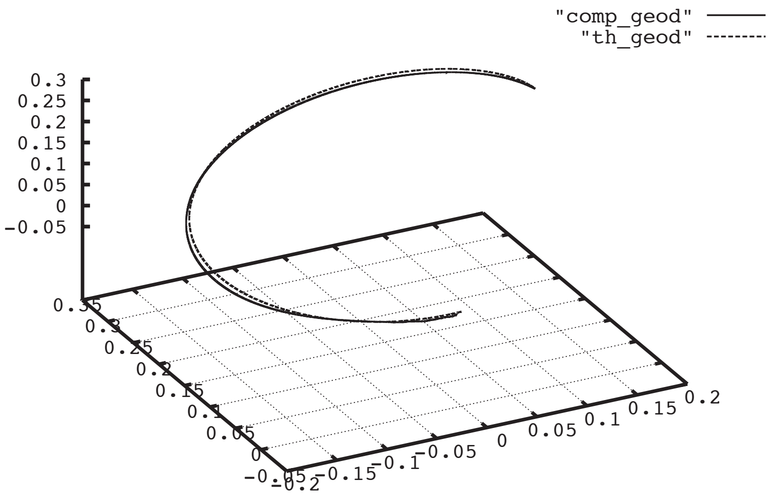

FIGURE 3. Comparison between the Carnot-Carathéodory geodesic joining $(0.15, 0.15, 0.3)$ and the origin, computed either by (44) (45) or by (48), with u computed by the finite difference scheme on a grid with $120 \times 120 \times \frac{(120^2)}{4}$ nodes.

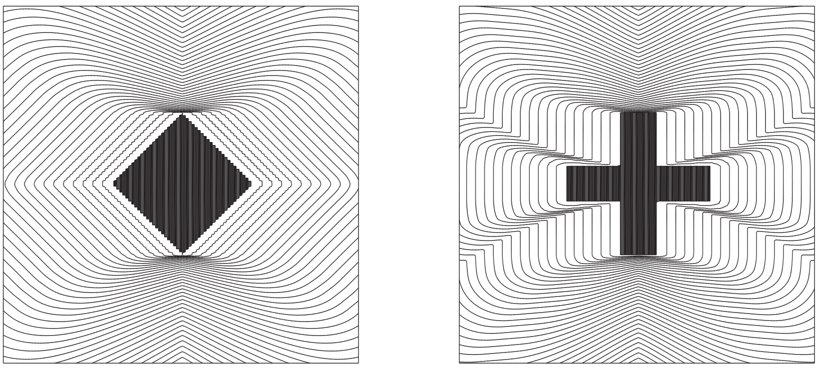

FIGURE 4. Level sets (intersected with the plane $x_1 = 0$) of the Carnot-Carathéodory distance to a convex set (the set $|x_1|+|x_2|+|x_3| \leq 0.2$) and to a nonconvex set.

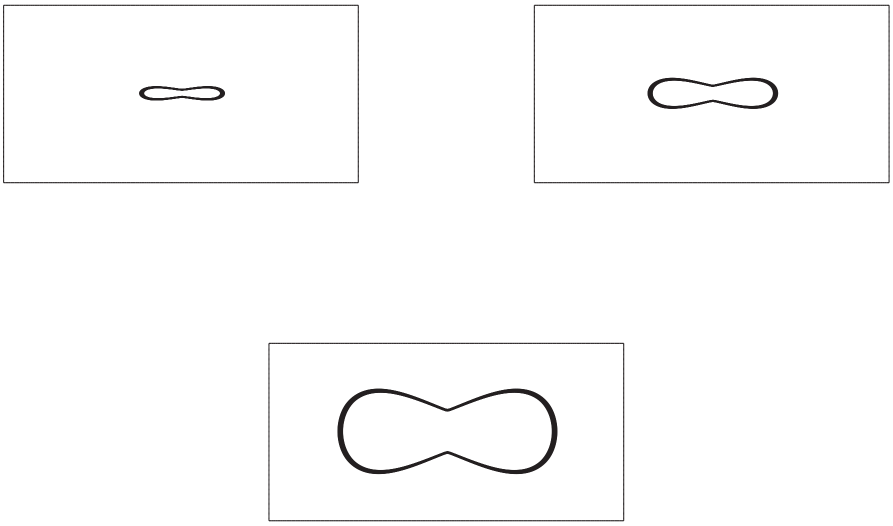

FIGURE 5. The front of the solution to (49) intersected with the plane $x_1 = 0$ at times $t = 0.125$, $t = 0.25$ and $t = 0.5$.

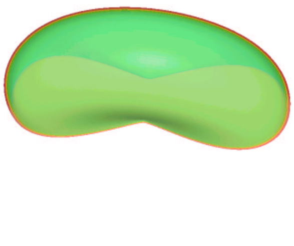

FIGURE 6. 3D view of the front at $t = 0.5$, in the half space $x_1 < 0$.

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