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Compensated compactness, div-curl theorem and H-convergence in general Heisenberg groups

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Abstract[1]. In this paper we present some recent results concerning existence and estimates of the fundamental solution for hypoelliptic differential operators in the contact complex of Heisenberg groups, and their application to the compensated compactess for intrinc Heisenberg differential forms. As a consequence, we prove a general div-rot theorem for horizontal vector fields and a compactness theorem with respect to the H-convergence of differential operators in general Heisenberg groups $\mathbb{H}^n$ with $n \geq 1$.

1. Introduction

In this note, we present some recent results proved in [1], [2], [3] concerning existence and estimates of the fundamental solution for homogeneous left invariant hypoelliptic differential operators in the contact complex of Heisenberg groups, together with their application to the compensated compactness for intrinsic differential forms. Finally we apply our results to prove a div-curl theorem for horizontal vector fields and a compactness result for the H-convergence of differential operators in the Heisenberg group $\mathbb{H}^n$ with $n \geq 1$, extending to $\mathbb{H}^n$ similar results proved for Euclidean spaces in [15], [16], [13], and for $\mathbb{H}^1$ in [9].

The compensated compactness (or div-curl) theorem of Murat and Tartar in the Euclidean space $\mathbb{R}^n$ states basically that the scalar product of two weakly convergent sequences of vector fields $(D_k)_{k\in\mathbb{N}}$ and $(E_k)_{k\in\mathbb{N}}$ in $(L^2(\mathbb{R}^n))^n$ still converges in the sense of distributions, provided $\{\operatorname{div} D_k \ : \ k \in \mathbb{N}\}$ and $\{\operatorname{curl} E_k \ : \ k \in \mathbb{N}\}$ are compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $(H^{-1}_{\mathrm{loc}}(\mathbb{R}^n))^{n(n-1)/2}$, respectively.

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The compensated compactness theorem provides a key tool in homogenization theory for elliptic pde's, in the study of H-convergence, and in the analysis of continuity properties for functionals of the calculus of variations.

This theorem has a natural formulation in the setting of De Rham complex of differential forms (see [13] and [19]). If we want to develop a complete theory of compensated compactness for differential forms in higher dimensional Heisenberg groups $\mathbb{H}^n$ (identified through exponential coordinates with $\mathbb{R}^{2n+1}$) with $n \geq 1$, the natural setting is provided by the contact complex introduced by Rumin ([20], [17], and [10], [1] for a slightly different presentation). This notion will be made precise later on in Section 2. A key feature of the contact complex is the presence of a *second order* differential operator D replacing the exterior differential on (intrinsic) n-forms. The presence of this second order operator reflects the non-Euclidean character of the contact complex, and is somehow related to the gap between the topological dimension ($= 2n+1$) of $\mathbb{H}^n$, and its intrinsic Hausdorff dimension ($Q := 2n+2$), see [10] for an exhaustive theory on the dimension of intrinsic submanifolds of $\mathbb{H}^n$. The structure of the complex, together with the presence of the second order operator D, affects the statement of our compensated compactness theorem, as well as that of the div-curl theorem. In fact, the compactness of the differentials of the weakly convergent forms has to be assumed in (intrinsic) Sobolev spaces of order either -1 or -2 according to the order of the differentials involved and hence to the degree of the forms. Consequently, also the integrability exponents in the div-curl theorem must be changed coherently.

2. Notations

We denote by $\mathbb{H}^n$ the n-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (p_1, \dots, p_{2n}, p_{2n+1}) = (p', p_{2n+1})$, with $p' \in \mathbb{R}^{2n}$ and $p_{2n+1} \in \mathbb{R}$, or by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If p and $q \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot q = (p' + q', p_{2n+1} + q_{2n+1} + 2\sum_{j=1}^{n}(p_j q_{j+n} - p_{j+n} q_j)) .$$

We denote as $p^{-1} := (-p', -p_{2n+1})$ the inverse of p (remember 0 is the identity of $\mathbb{H}^n$).

For a general review on Heisenberg groups and their properties, we refer to [22], [12] and to [23]. We limit ourselves to fix some notations, following [10].

For fixed $q \in \mathbb{H}^n$ and for $r > 0$, left translations $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ and non isotropic dilations $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ are automorphisms of the group defined as

$$\tau_q(p) := q \cdot p \quad \text{and as} \quad \delta_r(p) := (rp', r^2 p_{2n+1}) . \tag{1}$$

The Heisenberg group $\mathbb{H}^n$ can be endowed with the homogeneous norm (Koranyi norm)

$$\varrho(p) = \left(|p'|^4 + p_{2n+1}^2\right)^{1/4} , \tag{2}$$

and we define the gauge distance as

$$d(p,q) := \varrho(p^{-1} \cdot q) . \tag{3}$$

Finally, set $B_\rho(p,r) = \{q \in \mathbb{H}^n;\ d(p,q) < r\}$. We denote by $Q := 2n+2$ the Hausdorff dimension of $(\mathbb{H}^n, d)$.

Let $\mathfrak{h}$ be the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_{x_i} + 2y_i\partial_t \ , \ Y_i := \partial_{y_i} - 2x_i\partial_t \ , \ T := \partial_t \ .$$

The only non-trivial commutation relations are $[X_j, Y_j] = -4T$, for $j = 1, \dots, n$.

The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$. Coherently, from now on, we refer to $X_1, \dots, X_n, Y_1, \dots, Y_n$ (identified with first order differential operators) as to the *horizontal derivatives*. Sometimes, to avoid cumbersome notations, we put

$$W_i := X_i \ , \ W_{i+n} := Y_i \ , \ W_{2n+1} := T \ , \ \text{for } i = 1, \cdots, n \ ,$$

Following [8], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_{2n+1})$ is a multi-index, we set $W^I = W_1^{i_1} \cdots W_{2n}^{i_{2n}} T^{i_{2n+1}}$. By the Poincaré-Birkhoff-Witt theorem (see, e.g. [4], I.2.7), the differential operators W^I form a basis for the algebra of left invariant differential operators in $\mathbb{H}^n$. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \ .$$

The vector spaces $\mathfrak{h}$ and $\mathfrak{h}_1$ can be endowed with an inner product, indicated as $\langle \cdot, \cdot \rangle$, making $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ and T orthonormal. Following e.g. [8], we can define a group convolution in $\mathbb{H}^n$: if, for instance, $f \in \mathcal{D}(\mathbb{H}^n)$ and $g \in L^1_{\text{loc}}(\mathbb{H}^n)$, we set

$$f * g(p) := \int f(q) g(q^{-1}p) \, dq \quad \text{for } q \in \mathbb{H}^n \ . \tag{4}$$

We remind also the notion of *kernel of order* α. Following [7], a kernel of order α is a homogeneous distribution of degree $\alpha - Q$ (with respect to group dilations δ_r as in (1)), that is smooth outside of the origin.

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $X_1, \dots, Y_n, T$ is the family of covectors $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$ where $\theta := dt + 2\sum_{j=1}^n (x_j dy_j - y_j dx_j)$ is called the *contact form* in $\mathbb{H}^n$. We indicate as $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $dx_1, \dots, dy_n, \theta$ an orthonormal basis.

We put $\bigwedge_0 \mathfrak{h} := \bigwedge^0 \mathfrak{h} = \mathbb{R}$ and, for $1 \le k \le 2n+1$,

$$\begin{aligned} \textstyle\bigwedge_k \mathfrak{h} \quad &:= \text{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\} \ , \\ \textstyle\bigwedge^k \mathfrak{h} \quad &:= \text{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\} \ , \end{aligned}$$

where we set

$$\theta_i := dx_i \ , \ \theta_{i+n} := dy_i \ , \ \theta_{2n+1} := \theta \ , \quad \text{for } i = 1, \cdots, n \ .$$

As customary, the volume $(2n+1)$-form $\theta_1 \wedge \cdots \wedge \theta_{2n+1}$ will be also written as dV.

The action of a k-covector φ on a k-vector v is denoted by $\langle \varphi, v \rangle$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_k \mathfrak{h}$ and to $\bigwedge^k \mathfrak{h}$ making both bases $W_{i_1} \wedge \cdots \wedge W_{i_k}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ orthonormal.

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$, obtaining the *horizontal k-vectors* and *horizontal k-covectors*

$$\begin{aligned} \bigwedge\nolimits_k \mathfrak{h}_1 &:= \mathrm{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} \ : \ 1 \le i_1 < \cdots < i_k \le 2n\} \\ \bigwedge\nolimits^k \mathfrak{h}_1 &:= \mathrm{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} \ : \ 1 \le i_1 < \cdots < i_k \le 2n\}\ . \end{aligned}$$

The *symplectic* 2*-form* $d\theta \in \bigwedge^2 \mathfrak{h}_1$ is $d\theta = 4\sum_{i=1}^n dx_i \wedge dy_i$.

Proposition 2.1. *For* $1 \le k \le 2n+1$*, we can define canonical linear isomorphisms* (*see* [6] *1.7.8,* [10]*, Definition 2.3*)

$$* : \bigwedge\nolimits_k \mathfrak{h} \longleftrightarrow \bigwedge\nolimits_{2n+1-k} \mathfrak{h} \quad \textit{and} \quad * : \bigwedge\nolimits^k \mathfrak{h} \longleftrightarrow \bigwedge\nolimits^{2n+1-k} \mathfrak{h}\ ,$$

such that

If $v \in \bigwedge_k \mathfrak{h}$ we define $v^\natural \in \bigwedge^k \mathfrak{h}$ by the identity $\langle v^\natural, w\rangle := \langle v, w\rangle$, and analogously we define $\varphi^\natural \in \bigwedge_k \mathfrak{h}$ for $\varphi \in \bigwedge^k \mathfrak{h}$.

We remind now the definitions of the vector spaces ${}_H\bigwedge_k$ and ${}_H\bigwedge^k$ of *integrable k-vectors and k-covectors* ([10], Definition 2.5).

Definition 2.2. We set ${}_H\bigwedge_0 = \mathbb{R}$ and, for $1 \le k \le n$,

$$\begin{aligned} &{}_H\bigwedge\nolimits_k := \mathrm{span}\,\{v \in \bigwedge\nolimits_k \mathfrak{h}_1 \ : \ v \text{is simple and integrable}\}\ , \\ &{}_H\bigwedge\nolimits_{2n+1-k} := *\,({}_H\bigwedge\nolimits_k)\ , \end{aligned}$$

where the simple k-vector $v = v_1 \wedge \cdots \wedge v_k \in \bigwedge_k \mathfrak{h}_1$ is said to be *integrable* if the distribution of k-planes span$\{v_1, \dots, v_k\}$ is integrable. Integrable covectors are defined for $0 \le k \le 2n+1$ by

$${}_H\bigwedge\nolimits^k \stackrel{\text{def}}{=} \left\{\varphi \in \bigwedge\nolimits^k \mathfrak{h} \ : \ \varphi^\natural \in {}_H\bigwedge\nolimits_k\right\}\ ,$$

and ${}_H\bigwedge^k$ turns out to be isomorphic to $\bigwedge^1({}_H\bigwedge_k)$.

Notice that ${}_H\bigwedge_1 = \bigwedge_1 \mathfrak{h}_1 = \mathfrak{h}_1$. On the contrary, for $1 < k \le n$, $\{0\} \ne {}_H\bigwedge_k \subsetneq \bigwedge_k \mathfrak{h}_1$. In addition, we point out that any $v \in {}_H\bigwedge_k$ with $k > n$ can be written in the form

$$v = w \wedge T \ , \ \text{ with } w \in \bigwedge\nolimits_{k-1} \mathfrak{h}_1\ . \tag{5}$$

By Proposition 2.1, if $1 \le k \le n$ we have also

$${}_H\bigwedge\nolimits_k = *\left({}_H\bigwedge\nolimits_{2n+1-k}\right)\ .$$

We set $N_k := \dim {}_H\bigwedge_k = \dim {}_H\bigwedge^k$ for $k = 1, \dots, 2n+1$. The spaces of integrable covectors are canonically isomorphic to the spaces defined by Rumin in [20]. Following [20], we define $\mathcal{I}^*$ and $\mathcal{J}^* \subset \bigwedge^* \mathfrak{h}$, where $\mathcal{I}^*$ is the graded ideal generated by θ, that is $\mathcal{I}^* := \{\beta \wedge \theta + \gamma \wedge d\theta : \beta, \gamma \in \bigwedge^* \mathfrak{h}\}$ and $\mathcal{J}^*$ is the annihilator of $\mathcal{I}^*$, that is $\mathcal{J}^* := \{\alpha \in \bigwedge^* \mathfrak{h} \ : \ \alpha \wedge \theta = 0 \text{ and } \alpha \wedge d\theta = 0\}$. Both $\mathcal{I}^*$ and $\mathcal{J}^*$ are graded, indeed $\mathcal{I}^* = \oplus_{k=1}^{2n+1} \mathcal{I}^k$ and $\mathcal{J}^* = \oplus_{k=1}^{2n+1} \mathcal{J}^k$, where $\mathcal{I}^k, \mathcal{J}^k \subset \bigwedge^k \mathfrak{h}$ and

$$\begin{aligned} \mathcal{I}^k &= \{\beta \wedge \theta + \gamma \wedge d\theta \ : \ \beta \in \bigwedge\nolimits^{k-1} \mathfrak{h}, \gamma \in \bigwedge\nolimits^{k-2} \mathfrak{h}\} \\ \mathcal{J}^k &= \{\alpha \in \bigwedge\nolimits^k \mathfrak{h} \ : \ \alpha \wedge \theta = 0 \text{ and } \alpha \wedge d\theta = 0\}\ . \end{aligned}$$

The following identities, or natural isomorphisms, hold ([10], Theorem 2.9).

Theorem 2.3. *For* $1 \le k \le n$*,*

$$ {}_H\bigwedge\nolimits_k = \ker \mathcal{I}^k \quad \textit{and} \quad {}_H\bigwedge\nolimits_{2n+1-k} \simeq \frac{\bigwedge_{2n+1-k}\mathfrak{h}}{\ker \mathcal{J}^{2n+1-k}} \,, \tag{6} $$

$$ {}_H\bigwedge\nolimits^k \simeq \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k} \quad \textit{and} \quad {}_H\bigwedge\nolimits^{2n+1-k} = \mathcal{J}^{2n+1-k} \,, \tag{7} $$

where $\ker \mathcal{I}^k = \{v \in \bigwedge_k \mathfrak{h} : \langle \varphi, v\rangle = 0 \quad \forall \varphi \in \mathcal{I}^k\}$ *and* $\ker \mathcal{J}^{2n+1-k}$ *is analogously defined.*

Remark 2.4. For our purposes, it will be convenient to write explicitly a canonical isomorphism realizing (7). To this end, if $1 \le k \le n$, let denote by

$$ R \,:\, {}_H\bigwedge\nolimits^k \to \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k} $$

the map defined by $R\alpha := [\alpha]$, where $[\alpha]$ is the equivalence class of α. Then us denote by

$$ P \,:\, \frac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k} \to {}_H\bigwedge\nolimits^k $$

the map $[\alpha] \to P[\alpha] := \pi(\alpha)$ that associates with a class $[\alpha]$ the orthogonal projection $\pi(\alpha)$ in $\bigwedge^k \mathfrak{h}$ of a representative of $[\alpha]$ on the orthogonal complement $\mathcal{I}_k^\perp$ of the linear space $\mathcal{I}_k := \{\beta \wedge \theta + \gamma \wedge d\theta \,, \text{ with } \beta, \gamma \in \bigwedge^* \mathfrak{h}\}$. Clearly, this definition does not depend on the representative chosen.

We have $PR\alpha = \alpha$ for any $\alpha \in {}_H\bigwedge^k$, and $RP[\alpha] = [\alpha]$ for any $[\alpha] \in \bigwedge^k \mathfrak{h}/\mathcal{I}^k$.

We observe that our previous algebraic construction yields, by left translation, canonically several bundles over $\mathbb{H}^n$. These are the bundles of k-vectors and k-covectors, that, thanks to the left invariance of the structure, we can still indicate as $\bigwedge_k \mathfrak{h}$ and $\bigwedge^k \mathfrak{h}$. Analogously, we can define the bundles $\bigwedge_k \mathfrak{h}_1$ and $\bigwedge^k \mathfrak{h}_1$ of the horizontal k-vectors and k-covectors and the bundles ${}_H\bigwedge_k$ and ${}_H\bigwedge^k$ of the integrable k-vectors and k-covectors. The fiber of $\bigwedge_k \mathfrak{h}$ over $p \in \mathbb{H}^n$ is denoted by $\bigwedge_{k,p} \mathfrak{h}$ and analogously the other ones. For $q, q' \in \mathbb{H}^n$ and for any linear map $f : T\mathbb{H}^n_q \to T\mathbb{H}^n_{q'}$,

$$ \Lambda_k f : \bigwedge\nolimits_k T\mathbb{H}^n_q \to \bigwedge\nolimits_k T\mathbb{H}^n_{q'} $$

is the linear map defined by

$$ (\Lambda_k f)(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k) \,. $$

The inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge_k \mathfrak{h}$ induces an inner product $\langle \cdot, \cdot \rangle_p$ on each fiber ${}_H\bigwedge_{k,p}$ by the identity

$$ \langle \Lambda_k d\tau_p(v), \Lambda_k d\tau_p(w) \rangle_p := \langle v, w \rangle \,. $$

Analogously,

$$ \Lambda^k f \,:\, \bigwedge\nolimits^k T\mathbb{H}^n_{q'} \to \bigwedge\nolimits^k T\mathbb{H}^n_q $$

is the linear map defined by

$$ \langle (\Lambda^k f)(\alpha), v_1 \wedge \cdots \wedge v_k \rangle = \langle \alpha, (\Lambda_k f)(v_1 \wedge \cdots \wedge v_k) \rangle $$

for any $\alpha \in \bigwedge^k T\mathbb{H}^n_{q'}$ and any simple k-vector $v_1 \wedge \cdots \wedge v_k \in \bigwedge_k T\mathbb{H}^n_q$. If $p, q \in \mathbb{H}^n$, then

$$ \Lambda_k d\tau_q \,:\, {}_H\bigwedge\nolimits_{k,p} \to {}_H\bigwedge\nolimits_{k,qp} \quad \text{and} \quad \Lambda^k d\tau_{q^{-1}} \,:\, {}_H\bigwedge\nolimits^k_p \to {}_H\bigwedge\nolimits^k_{qp} $$

are isomorphisms and isometries.

3. Forms, currents and function spaces

If $\Omega \subset \mathbb{H}^n$ is a bounded open set, $1 \le p \le \infty$, and $k \in \mathbb{N}$, then we denote by $W^{k,p}_{\mathbb{H}}(\Omega)$ the space of all $u \in L^p(\Omega)$ such that $W^I u \in L^p(\Omega)$ for any multi-index I with $d(I) \le k$, endowed with the natural norm. If $1 \le p < \infty$, we denote also by $\mathring{W}_{\mathbb{H}}{}^{k,p}(\Omega)$ the completion of $\mathcal{D}(\Omega)$ in $W^{k,p}_{\mathbb{H}}(\Omega)$.

If $\mathcal{U} \subset \mathbb{H}^n$ is an open subset, then we denote by $\mathcal{D}^m_{\mathbb{H}}(\mathcal{U})$ the space of all smooth sections of ${}_H\bigwedge^m$ over $\mathcal{U}$ with compact support, endowed with the usual topology making it a Fréchet space. We refer to the elements of $\mathcal{D}^m_{\mathbb{H}}(\mathcal{U})$ as to the *Heisenberg (smooth) m-differential forms in $\mathcal{U}$* with compact support. The definition of $\mathcal{E}^m_{\mathbb{H}}$ is given in the same way.

Let $\{\xi_1, \dots, \xi_{N_m}\}$ be an orthonormal basis of ${}_H\bigwedge^m_e$. Then, by left translation, we can define N_m smooth sections of ${}_H\bigwedge^m$, that we still denote by $\xi_1, \dots, \xi_{N_m}$. Obviously, $\{\xi_{1,p}, \dots, \xi_{N_m,p}\}$ is an orthonormal basis of ${}_H\bigwedge^m_p$. In the sequel, we shall refer to $\{\xi_1, \dots, \xi_{N_m}\}$ as an (orthonormal) left invariant moving frame in ${}_H\bigwedge^m$.

A left invariant orthonormal moving frame of ${}_H\bigwedge^m$ yields a special trivialization of the fiber bundle ${}_H\bigwedge^m$ that makes computations remarkably simpler. Indeed, the map that associates with α belonging to the fiber over p the local coordinates

$$\alpha \to (p, \langle \alpha, \xi_{1,p}\rangle_p, \dots \langle \alpha, \xi_{N_m,p}\rangle_p)$$

defines a trivialization of ${}_H\bigwedge^m$.

From now on, function spaces of intrinsic forms are defined by means of their coordinates with respect to a left invariant orthonormal moving frame. For instance, if $1 \le p \le \infty$ and $A \subset \mathbb{H}^n$ is an open set, we can define the spaces $L^{m;p}(A) := L^p(A, {}_H\bigwedge^m)$ and $L^p_m(A) := L^p(A, {}_H\bigwedge_m)$ of all sections of ${}_H\bigwedge^m$ (of ${}_H\bigwedge_m$, respectively) with components in $L^p(A)$, endowed with the natural norm. Finally, if $1 \le p \le \infty$ and $s \ge 0$, we define in the same way the Sobolev space $W^{m;s,p}_{\mathbb{H}}(A)$ (the Sobolev space $W^{s,p}_{\mathbb{H};m}(A)$) of all sections of ${}_H\bigwedge^m$ (of ${}_H\bigwedge_m$, respectively) with components in $W^{s,p}_{\mathbb{H}}(A)$.

We proved in [1], Definition 6 and Remark 4, that it is possible to define in a simple but intrinsic way the convolution $\Phi * \alpha$, for any $\Phi \in \mathcal{D}(\mathbb{H}^n)$ and any $\alpha \in L^{m;1}_{\mathrm{loc}}(\mathbb{H}^n)$. In coordinates, if $\xi_1, \dots, \xi_{N_m}$ is a left invariant orthonormal moving frame, and $\alpha = \sum_j \alpha_j \xi_j$, we have

$$\Phi * \alpha = \sum_j (\Phi * \alpha_j)\xi_j \; .$$

Following [20], [9], we can obtain from ${}_H\bigwedge^*$ a complex of intrinsic differential forms that fits the structure of Heisenberg groups in the same way as De Rham complex does for usual differential forms in Euclidean spaces.

Theorem 3.1. *Let $\mathcal{U} \subset \mathbb{H}^n$ be an open set. If $0 \le k < n$, we denote by $d_c : {}_H\bigwedge^k \to {}_H\bigwedge^{k+1}$ the map induced on the quotient spaces by the exterior differential on forms. If $n+1 \le k \le 2n+1$, d_c will be the usual exterior differential. Then there exists a left invariant homogeneous differential operator of order two $D : {}_H\bigwedge^n \to {}_H\bigwedge^{n+1}$*

such that the complex (the contact complex)

$$0 \to \mathcal{D}^0_{\mathbb{H}}(\mathcal{U}) \xrightarrow{d_c} \mathcal{D}^1_{\mathbb{H}}(\mathcal{U}) \xrightarrow{d_c} \cdots \xrightarrow{d_c} \mathcal{D}^n_{\mathbb{H}}(\mathcal{U}) \xrightarrow{D}$$

$$\xrightarrow{D} \mathcal{D}^{n+1}_{\mathbb{H}}(\mathcal{U}) \xrightarrow{d_c} \cdots \xrightarrow{d_c} \mathcal{D}^{2n+1}_{\mathbb{H}}(\mathcal{U}) \to 0$$

has the same cohomology as De Rham complex.

Remark 3.2. The above definition of intrinsic differential d_c relies on Rumin's notion of intrinsic k-covectors as quotient classes. Indeed, if $1 \le k \le n-1$ and $[\alpha] \in \bigwedge^k \mathfrak{h}/\mathcal{I}^k$ we have set $d_c[\alpha] := [d\alpha]$. Clearly, there is a corresponding notion (that we shall not distinguish if this does not lead to any ambiguity) relying on our original definition of ${}_H\bigwedge^k$, if we take $d'_c := P d_c R$, or, in other words, making the following diagram commutative:

$$\begin{array}{ccc} {}_H\bigwedge^k & \xrightarrow{d'_c} & {}_H\bigwedge^{k+1} \\ R\big\downarrow & & \big\uparrow P \\ \dfrac{\bigwedge^k \mathfrak{h}}{\mathcal{I}^k} & \xrightarrow{d_c} & \dfrac{\bigwedge^{k+1} \mathfrak{h}}{\mathcal{I}^{k+1}} \end{array}$$

Remark 3.3. In the sequel, it will be useful to have an explicit representation of the operator D. To this end, let $\beta \in \mathcal{E}^n_{\mathbb{H}}$. To this aim, we recall ([24], [20]) that - denoting by L the algebraic operator on horizontal forms given by $L(\alpha) := d\theta \wedge \alpha$ - we have $L : \mathcal{D}^s_{\mathrm{H}\mathbb{H}} \to \mathcal{D}^{s+2}_{\mathrm{H}\mathbb{H}}$, and, in addition, L is injective if $s \le n-1$, and L is surjective if $s \ge n-1$ (hence for $s = n-1$ it is an isomorphism). In the class β there exists always a purely horizontal element (since any form β can be written as $\beta = \beta_H + \theta \wedge \beta_T$, with $\beta_H, \beta_T \in \mathcal{D}^*_{\mathrm{H}\mathbb{H}}$ purely horizontal), that we can still denote by β. Then the operator D is defined ([20], [21], and also [9]) by

$$D\beta = \theta \wedge \left(\mathcal{L}_T \beta + d_H L^{-1}(d_H \beta)\right) , \tag{8}$$

where $\mathcal{L}_T$ is the Lie derivative along T and the horizontal differential d_H on horizontal forms is defined by

$$d_H(f\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}) := \sum_{j=1}^{2n} (W_j f)\theta_j \wedge \theta_{i_1} \wedge \cdots \wedge \theta_{i_k} ,$$

with $i_1, \dots, i_k \le 2n$.

The following notion of current in Heisenberg groups appears in [10]. For general results in the Euclidean setting, we refer to [6], [11] and [5]. The main result we are interested here is precisely Proposition 3.7 below, i.e. the representation of a current as a distributional coefficients form proved in [1],

Definition 3.4. If $\mathcal{U}$ is an open set, we say that T is a *Heisenberg m-current* if T is a continuous linear functional on $\mathcal{D}^m_{\mathbb{H}}(\mathcal{U})$ endowed with the usual topology. We write $T \in \mathcal{D}'_{\mathbb{H},m}(\mathcal{U})$.

The definition of $\mathcal{E}'_{\mathbb{H},m}(\mathcal{U})$ is given analogously.

Proposition 3.5. *If $\mathcal{U} \subset \mathbb{H}^n$ is an open set, and $T \in \mathcal{D}'(\mathcal{U})$ is a (usual) distribution, then T can be identified canonically with a $(2n+1)$-current $\tilde{T} \in D'_{\mathbb{H},2n+1}(\mathcal{U})$ through the formula*

$$\langle \tilde{T}, \alpha \rangle := \langle T, *\alpha \rangle \tag{9}$$

for any $\alpha \in \mathcal{D}_{\mathbb{H}}^{2n+1}(\mathcal{U})$. *Reciprocally, by* (9), *any* $(2n+1)$-*current* $\tilde{T}$ *can be identified with an usual distribution* $T \in D'(\mathcal{U})$.

We want to define here the wedge product of two (say) smooth forms $\alpha \in \mathcal{E}_{\mathbb{H}}^k(\mathcal{U})$ and $\beta \in \mathcal{E}_{\mathbb{H}}^h(\mathcal{U})$. This is not always possible if we want to preserve all "good properties" with respect to the exterior differential of the usual wedge product between forms. Nevertheless, the definition is possible when $k+h \le n$ or $k \le n < h$. Our definition turns out to be much simpler if we think in terms of the covectors belonging to the classes (7).

Definition 3.6. If $\alpha \in \mathcal{E}_{\mathbb{H}}^k(\mathcal{U})$ and $\beta \in \mathcal{E}_{\mathbb{H}}^h(\mathcal{U})$ with $k+h \le n$, then for any $p \in \mathcal{U}$, both $\alpha(p)$ and $\beta(p)$ are identified with quotient classes. Thus, if α_0 and β_0 are representatives of $\alpha(p)$ and $\beta(p)$, respectively, we denote by $(\alpha \overset{\circ}{\wedge} \beta)(p)$ the equivalence class of $\alpha_0 \wedge \beta_0$.

If $\alpha \in \mathcal{E}_{\mathbb{H}}^k(\mathcal{U})$ and $\beta \in \mathcal{E}_{\mathbb{H}}^h(\mathcal{U})$ with $k \le n$, $h > n$ then for any $p \in \mathcal{U}$, $\alpha(p)$ is identified with a quotient class. Thus, if α_0 is a representative of $\alpha(p)$, we define $(\alpha \overset{\circ}{\wedge} \beta)(p)$ to be the form $\alpha_0 \wedge \beta(p)$. These definitions do not depend on the representatives we choose for quotient classes. A pair (k,h) satisfying one of the above assumptions will be called a wedge-admissible pair.

Following [6], 4.1.7, if $T \in D'_{\mathbb{H},m}(\mathcal{U})$, and $\phi \in \mathcal{E}_{\mathbb{H}}^k(\mathcal{U})$, with $k \le m$, we define $T \llcorner \phi \in \mathcal{D}'_{\mathbb{H},m-k}(\mathcal{U})$ by the identity

$$\langle T \llcorner \phi, \alpha \rangle := \langle T, \phi \overset{\circ}{\wedge} \alpha \rangle$$

for any $\alpha \in \mathcal{D}_{\mathbb{H}}^{m-k}(\mathcal{U})$, provided k and m are such that $\phi \overset{\circ}{\wedge} \alpha$ is well defined in $D_{\mathbb{H}}^m(\mathcal{U})$.

The following result is taken from [1], Propositions 5 and 6, and Definition 10, but we refer also to [5], Sections 17.3, 17.4 and 17.5.

Proposition 3.7. *Let* $\mathcal{U} \subset \mathbb{H}^n$ *be an open set. If* $1 \le m \le 2n+1$, $\xi_1, \dots \xi_{N_m}$ *is a left invariant orthonormal moving frame of* ${}_H\bigwedge^m$ *and* $T \in \mathcal{D}'_{\mathbb{H},m}(\mathcal{U})$, *then there exist* (*uniquely determined*) $T_1, \dots, T_{N_m} \in \mathcal{D}'(\mathcal{U})$ *such that we can write*

$$T = \sum_j \tilde{T}_j \llcorner (*\xi_j) \ .$$

Notice $*\xi_j \overset{\circ}{\wedge} \alpha$ *is well defined for any* $\alpha \in \mathcal{D}_{\mathbb{H}}^m(\mathcal{U})$, *for* $*\xi_j \in \mathcal{E}_{\mathbb{H}}^{2n+1-m}(\mathcal{U})$, *and the pair* $(2n+1-m, m)$ *is always wedge-admissible, since* $2n+1-m$ *and* m *can not be both less than* n.

If $\mathcal{U}, \mathcal{V}$ are open subsets of $\mathbb{H}^n$, and $f : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism, then for any $\alpha \in \mathcal{E}^m(\mathcal{V})$, we denote by $f^\sharp\alpha$ the pull-back form in $\mathcal{E}^m(\mathcal{U})$ defined by

$$f^\sharp\alpha(p) := (\Lambda^m df(p))\alpha(f(p)) \ ,$$

for any $p \in \mathcal{U}$. Notice that, if $\Lambda^m df(p)$ maps ${}_H\bigwedge^m_{f(p)}$ in ${}_H\bigwedge^m_p$, then clearly the pull-back of any Heisenberg m-differential form in $\mathcal{V}$ is still a Heisenberg m-differential form in $\mathcal{U}$. By duality, $(\Lambda^m df(p))({}_H\bigwedge^m_{f(p)}) \subset {}_H\bigwedge^m_p$ if and only if $(\Lambda_m df(p))({}_H\bigwedge_{m,p}) \subset {}_H\bigwedge_{m,f(p)}$, since $(\Lambda^m df(p))(v^\natural) = \big((\Lambda_m df(p))(v)\big)^\natural$.

Proposition 3.8. *Let $\{\xi_1,\dots,\xi_{N_m}\}$ be a left invariant orthonormal moving frame of ${}_H\bigwedge^m$, where $1\le m\le 2n+1$. If $\alpha=\sum_j\alpha_j\xi_j\in\mathcal{E}^m_{\mathbb{H}}(\mathbb{H}^n)$ ($\alpha\in\mathcal{D}^m_{\mathbb{H}}(\mathbb{H}^n)$) and $t>0$, then $\delta_t^\sharp\alpha$ is well defined and belongs to $\mathcal{E}^m_{\mathbb{H}}(\mathbb{H}^m)$ ($\alpha\in\mathcal{D}^m_{\mathbb{H}}(\mathbb{H}^n)$, respectively).*

In addition

$$\delta_t^\sharp\alpha=t^{d(m)}\sum_j(\alpha_j\circ\delta_t)\xi_j\ , \tag{10}$$

where $d(m)=m$ if $1\le m\le n$, and $d(m)=m+1$ if $n+1\le m\le 2n+1$.

Identity (10) suggests the following notation, that by itself is not correct, but nevertheless allows simpler statements. If $\alpha=\sum_j\alpha_j\xi_j\in\mathcal{E}^m_{\mathbb{H}}(\mathbb{H}^n)$, $t>0$, and $p\in\mathbb{H}^n$, we write

$$(\alpha\circ\delta_t)(p):=t^{-d(m)}(\delta_t^\sharp\alpha)(p)=\sum_j(\alpha_j\circ\delta_t)(p)\xi_{j,p}\ . \tag{11}$$

In fact, the notation is not formally correct, since the map $p\to\alpha(\delta_t(p))$ is not a section of ${}_H\bigwedge^m$, for $\alpha(\delta_t(p))$ belongs to ${}_H\bigwedge^m_{\delta_t(p)}$, and not to ${}_H\bigwedge^m_p$. Notice that (10) yields that the definition of $\alpha\circ\delta_t$ is independent of the left invariant orthonormal moving frame chosen.

Definition 3.9. We say that a left invariant differential operator

$$\mathcal{L}:D^m_{\mathbb{H}}(\mathbb{H}^n)\to D^m_{\mathbb{H}}(\mathbb{H}^n)$$

is *homogeneous of degree* $a\in\mathbb{N}$ if, with our notation (11), for any $t>0$

$$t^a(\mathcal{L}\alpha)\circ\delta_t=\mathcal{L}(\alpha\circ\delta_t)\ .$$

for any $\alpha\in D^m_{\mathbb{H}}(\mathbb{H}^n)$. We say that $\mathcal{L}$ is *left invariant* if for any $q\in\mathbb{H}^n$

$$\tau_q^\sharp\big(\mathcal{L}(\alpha)\big)=\mathcal{L}(\tau_q^\sharp\alpha)$$

for any $\alpha\in D^m_{\mathbb{H}}(\mathbb{H}^n)$.

Proposition 3.10. *If $1<p<\infty$, $1/p+1/p'=1$, $k\in\mathbb{N}$, and $\Omega\subset\mathbb{H}^n$ is a bounded open set, then the dual space $\big(\overset{\circ}{W}{}^{m;k,p'}_{\mathbb{H}}(\Omega)\big)^*$ coincides with the set of all currents $T\in D'_{\mathbb{H},m}(\Omega)$ of the form*

$$T=\sum_j\tilde T_j\llcorner(*\xi_j)\ , \tag{12}$$

with $T_1,\dots,T_{N_m}\in W^{-k,p}_{\mathbb{H}}(\Omega)$. In particular, it is natural to write

$$W^{m;-k,p}_{\mathbb{H}}(\Omega):=\big(\overset{\circ}{W}{}^{m;k,p'}_{\mathbb{H}}(\Omega)\big)^*\ .$$

The following result is well known.

Theorem 3.11. *Let $\Omega\subset\mathbb{H}^n$ be a bounded open set. If $k,h\in\mathbb{N}$, $h\le k$, and $1<p,q<\infty$, then $\overset{\circ}{W}{}^{m;k,p}_{\mathbb{H}}(\Omega)$ is compactly embedded in $\overset{\circ}{W}{}^{m;k-h,q}_{\mathbb{H}}(\Omega)$ for $1<q<pQ/(Q-hp)$ if $p<Q/h$ and for any $q>1$ if $p\ge Q/h$. By duality, if $1<r,s<\infty$, then $W^{m;-k+h,r}_{\mathbb{H}}(\Omega)$ is compactly embedded in $W^{m;-k,s}_{\mathbb{H}}(\Omega)$ for $1<r'<s'Q/(Q-hs')$ if $s'<Q/h$ and for any $r>1$ if $s'\ge Q/h$ where (r,r') and (s,s') are Hölder conjugate pairs.*

4. Horizontal Laplace operators on forms

The following result - together with the hypoellipticity of the intrinsic Laplace operator for the contact complex proved in [20] - provides the crucial tool for the Hodge decomposition of intrinsic forms (see Proposition 5.2 below). This result was proved under more restrictive assumptions in [1], [2] (i.e. assuming that the operator $\mathcal{L}$ is *maximal hypoelliptic*), and in the present form in [3]. Its proof is inspired by Folland's paper [7] for the scalar case, but relies also on the proof of a Liouville's type theorem in homogeneous groups given in [14]. Recently, some related results have been proved independently, by means of pseudodifferential techniques, in [18].

Theorem 4.1. *Suppose*

$$\mathcal{L} : D^m_{\mathbb{H}}(\mathbb{H}^n) \to D^m_{\mathbb{H}}(\mathbb{H}^n)$$

is a hypoelliptic differential operator such that ${}^t\mathcal{L} = \mathcal{L}$. *Suppose also that* $\mathcal{L}$ *is homogeneous of degree* $a \le Q$. *If* $\xi_1, \dots \xi_{N_m}$ *is a left invariant orthonormal moving frame of* ${}_H\bigwedge^m$, *then for* $j = 1, \dots, N_m$ *there exists*

$$K_j = \sum_i \tilde{K}_{ij} \llcorner (*\xi_i) \in D'_{\mathbb{H},m}(\mathbb{H}^n) \cap \mathcal{E}^m_{\mathbb{H}}(\mathbb{H}^n \setminus \{0\}) \,, \tag{13}$$

with $K_{ij} \in \mathcal{D}'(\mathbb{H}^n)$, $i, j = 1, \dots, N_m$ *such that*

i) $\mathcal{L}K_j = \tilde{\delta} \llcorner (*\xi_j)$;

ii) *if* $a < Q$, *then the* K_{ij}*'s are kernels of type* a *in the sense of* [7], *for* $i, j = 1, \dots, N_m$ *(i.e. they are smooth functions outside of the origin, homogeneous of degree* $a - Q$, *and hence belonging to* $L^1_{\mathrm{loc}}(\mathbb{H}^n)$, *by Corollary 1.7 of* [7]*). If* $a = Q$, *then the* K_{ij}*'s satisfy the logarithmic estimate* $|K_{ij}(p)| \le C(1 + |\ln \rho(p)|)$ *and hence belong to* $L^1_{\mathrm{loc}}(\mathbb{H}^n)$. *Moreover, their horizontal derivatives (i.e.* $W_\ell K_{ij}$ *for* $\ell = 1, \dots, 2n$*) are kernels of type* $Q - 1$ *in the sense of* [7];

iii) *when* $\alpha \in D^m_{\mathbb{H}}(\mathbb{H}^n)$, *if we set*

$$\mathcal{K}\alpha := \sum_{ij} (\alpha_j * \tilde{K}_{ij}) \llcorner (*\xi_i) \,, \tag{14}$$

then $\mathcal{L}\mathcal{K}\alpha = \alpha$. *Moreover, if* $a < Q$, *also* $\mathcal{K}\mathcal{L}\alpha = \alpha$.

iv) *if* $a = Q$, *then for any* $\alpha \in D^m_{\mathbb{H}}(\mathbb{H}^n)$ *there exists a "constant form"* $\beta_\alpha = \sum_j \beta_j \xi_j$, *where* β_j *is a real constant for* $j = 1, \dots, N_m$, *such that*

$$\mathcal{K}\mathcal{L}\alpha - \alpha = \beta_\alpha \,.$$

5. Compensated Compactness Theorem

Our compensated compactness theorem in the contact complex of Heisenberg groups read as follows (for an analogous result in De Rham complex, see for instance [19], [13]).

Theorem 5.1. *Assume that* $\alpha_1^\varepsilon, \cdots, \alpha_\ell^\varepsilon$ *are smooth Heisenberg differential forms on the bounded open set* $\Omega \subset \mathbb{H}^n$, *of degree* $s_1, \cdots, s_\ell$, *respectively. Assume that, for any open set* $\Omega_0 \subset\subset \Omega$,

$$\alpha_i^\varepsilon \to \alpha_i \ \ \text{weakly in} \ \ L^{s_i;p_i}(\Omega_0) \,, \tag{15}$$

with $1 < p_i < \infty$ *for* $i = 1, \dots, \ell$, *and* $1/p_1 + \cdots + 1/p_\ell = 1$, *and that*

$$\{d_c \alpha_i^\varepsilon\} \ \ \text{is a compact set in} \ \ W^{s_i+1;-1,p_i}_{\mathbb{H}}(\Omega_0) \ \ , \ \ \text{if}\ s_i \ne n \,, \tag{16}$$

and

$$\{D\alpha_i^\varepsilon\} \text{ is a compact set in } W_{\mathbb{H}}^{n+1;-2,p_i}(\Omega_0) \ , \quad \text{if } s_i = n \ . \tag{17}$$

If $s := s_1 + \cdots + s_\ell \le 2n+1$ *and*

$$\sum_{s_i \le n} s_i \le n \ , \tag{18}$$

then we have

$$\int \alpha_1^\varepsilon \mathbin{\mathring{\wedge}} \cdots \mathbin{\mathring{\wedge}} \alpha_\ell^\varepsilon \mathbin{\mathring{\wedge}} \phi \to \int \alpha_1 \mathbin{\mathring{\wedge}} \cdots \mathbin{\mathring{\wedge}} \alpha_\ell \mathbin{\mathring{\wedge}} \phi \tag{19}$$

for any $\phi \in D_{\mathbb{H}}^{2n+1-s}(\Omega)$.

The proof relies on the following Hodge decomposition for intrinsic forms.

Proposition 5.2. *We can write*

$$\alpha_i^\varepsilon = \omega_i^\varepsilon + d_c\psi_i^\varepsilon \quad \text{if } s_i \neq n+1; \, , \tag{20}$$

and

$$\alpha_i^\varepsilon = \omega_i^\varepsilon + D\psi_i^\varepsilon \quad \text{if } s_i = n+1 \ , \tag{21}$$

with

$$\omega_i^\varepsilon \to \omega_i \quad \text{strongly in } L^{s_i;p_i}(\Omega'') \tag{22}$$

$$\psi_i^\varepsilon \to \psi_i \quad \text{strongly in } L^{s_i-1;p_i}(\Omega'') \quad \text{if } s_i \neq n+1 \ , \tag{23}$$

$$\psi_i^\varepsilon \to \psi_i \quad \text{strongly in } W_{\mathbb{H}}^{n;1,p_i}(\Omega'') \quad \text{if } s_i = n+1 \ , \tag{24}$$

$$d_c\psi_i^\varepsilon \to d_c\psi_i \text{ and } D\psi_i^\varepsilon \to D\psi_i \quad \text{weakly in } L^{s_i;p_i}(\Omega'') \ , \tag{25}$$

respectively if $s_i \neq n$ *and if* $s_i = n$.

6. A general div-curl theorem and H-convergence

If F is an horizontal vector field, i.e. if F is a (say) smooth section of ${}_H\bigwedge_1$ (usually denoted also by $H\mathbb{H}^n$), as customary we set

$$\operatorname{div}_{\mathbb{H}} F := (*d_c(*F^\natural))^\natural \ ,$$

and

$$\operatorname{curl}_{\mathbb{H}} F \ := (*DF^\natural)^\natural \qquad \text{if } n = 1 \ ,$$

$$\operatorname{curl}_{\mathbb{H}} F \ := (*d_cF^\natural)^\natural \qquad \text{if } n > 1 \ .$$

Moreover, if f is a (say) smooth function, we denote by $\nabla_{\mathbb{H}} f$ the horizontal vector field

$$\nabla_{\mathbb{H}} f := (d_c f)^\natural \ .$$

We remind that a left invariant orthonormal moving frame of ${}_H\bigwedge_1$ is given by $\{W_1, \ldots, W_{2n}\}$, and hence the horizontal vector field F can be written in the form $F := \sum_j F_j W_j$ and therefore identified with the vector-valued function $(F_1, \ldots, F_{2n})$. Thus

$$\operatorname{div}_{\mathbb{H}} F = \sum_j W_j F_j \ .$$

On the other hand, if $n = 1$, then an orthonormal left invariant moving frame of ${}_H\bigwedge_2$ is given by $\{W_2 \wedge W_3, -W_1 \wedge W_3\}$. Using (8), a straightforward computation shows that $\mathrm{curl}_{\mathbb{H}} F$ can be identified with the second order vector-valued operator

$$(F_1, F_2) \to$$
$$\to \frac{1}{4}(W_1W_2F_2 + W_2^2F_1 - 2W_2W_1F_2, W_1^2F_2 - 2W_1W_2F_1 + W_2W_1F_1) \ ,$$

that is precisely the formula given in the proof of Proposition 2.6 of [9]. Indeed, by Remark 3.3, $d_H F^\natural = (W_2F_1)\theta_2 \wedge \theta_1 + (W_1F_2)\theta_1 \wedge \theta_2 = (W_1F_2 - W_2F_1)\theta_1 \wedge \theta_2$; on the other hand, $d\theta := d\theta_3 = 4\theta_1 \wedge \theta_2$, so that $L^{-1}(d_H F^\natural) = (1/4)(W_1F_2 - W_2F_1)$. On the other hand $\mathcal{L}_T F^\natural = -(1/4)\big((W_1W_2 - W_2W_1)F_1\theta_1 + (W_1W_2 - W_2W_1)F_2\theta_2\big)$, so that

$$\begin{aligned} DF^\natural = \theta_3 \wedge \big\{\big(-(W_1W_2 - W_2W_1)F_1 + W_1(W_1F_2 - W_2F_1)\big)\theta_1 + \\ +\big(-(W_1W_2 - W_2W_1)F_2 + W_2(W_1F_2 - W_2F_1)\big)\theta_2\big\} = \\ = \big(-(W_1W_2 - W_2W_1)F_1 + W_1(W_1F_2 - W_2F_1)\big)(-\theta_1 \wedge \theta_3) + \\ +\big((W_1W_2 - W_2W_1)F_2 - W_2(W_1F_2 - W_2F_1)\big)\theta_2 \wedge \theta_3 \ , \end{aligned}$$

and the assertion follows by duality.

If $n > 1$, we denote by π the orthogonal projection in $\bigwedge^2 \mathfrak{h}$ along the linear subspace $\mathcal{I}^2$, and by π_0 the orthogonal projection in $\bigwedge^2 \mathfrak{h}_1$ along the linear space Θ spanned by $d\theta_{2n+1}$. Then, by Remark 3.2,

$$d_c F^\natural = \pi(dF^\natural) = \pi_0(d_H F^\natural) \ .$$

Suppose for instance $n = 2$. In this case

$$d_H F^\natural = d_H\Big(\sum_{j=1}^4 F_j\theta_j\Big) =$$
$$= \sum_{1\le i<j\le 4} (W_iF_j - W_jF_i)\theta_i \wedge \theta_j := \sum_{1\le i<j\le 4} F_{ij}\,\theta_i \wedge \theta_j \ .$$

On the other hand, $d\theta_5 = 4(\theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4)$, so that an orthonormal basis of $\bigwedge^2 \mathfrak{h}_1 \cap \Theta^\perp$ is given by

$$\big\{\theta_1 \wedge \theta_2,\ \theta_1 \wedge \theta_4,\ \theta_2 \wedge \theta_3,\ \theta_3 \wedge \theta_4,\ \frac{1}{\sqrt{2}}(\theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4)\big\} \ ,$$

and

$$\begin{aligned} \pi_0(d_H F^\natural) = F_{12}\,\theta_1 \wedge \theta_2 + F_{14}\,\theta_1 \wedge \theta_4 + F_{23}\,\theta_2 \wedge \theta_3 + F_{34}\,\theta_3 \wedge \theta_4 + \\ +\frac{1}{2}(F_{13} + F_{24})(\theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4) \ . \end{aligned}$$

Thus, by duality, $\mathrm{curl}_{\mathbb{H}} F$ can be identified with the first order vector-valued operator

$$(F_1, F_2, F_3, F_4) \to \big(F_{12},\ F_{14},\ F_{23},\ F_{34},\ \frac{1}{\sqrt{2}}(F_{13} + F_{24})\big) \ .$$

Theorem 5.1 yields the following result that generalizes to the case $n \ge 1$ and $p \ge 1$ Theorem 3.3 of [9], extending to the setting of Heisenberg groups Theorem 5.3 and its Corollary 5.4 of [13].

Theorem 6.1. *Let $\Omega \subset \mathbb{H}^n$ be an open set, let $p, q > 1$ a Hölder conjugate pair, and let $a, b > 1$ be such that*

i) *if* $n = 1$, *then* $a > Qp/(Q + 2p)$, $b > Qq/(Q + 2q)$;
ii) *if* $n > 1$, *then* $a > Qp/(Q + p)$, $b > Qq/(Q + q)$.

Let now $E^n \in L^p_{\mathrm{loc}}(\Omega, H\mathbb{H}^n)$ *and* $D^n \in L^q_{\mathrm{loc}}(\Omega, H\mathbb{H}^n)$ *be horizontal vector fields for* $n \in \mathbb{N}$, *weakly convergent to* E *and* D *in* $L^p_{\mathrm{loc}}(\Omega, H\mathbb{H}^n)$ *and in* $L^q_{\mathrm{loc}}(\Omega, H\mathbb{H}^n)$, *respectively.*

If $\{\mathrm{curl}_{\mathbb{H}} E^n\}$ *is bounded in* $L^a_{\mathrm{loc}}(\Omega, H\mathbb{H}^n)$ *and* $\{\mathrm{div}_{\mathbb{H}} D^n\}$ *is bounded in* $L^b_{\mathrm{loc}}(\Omega, H\mathbb{H}^n)$, *then*

$$\langle E^n, D^n\rangle \to \langle E, D\rangle \quad \text{in } \mathcal{D}'(\Omega)\ ,$$

i.e.

$$\int_\Omega \langle E^n(p), D^n(p)\rangle_p \phi(p)\, dp \to \int_\Omega \langle E(p), D(p)\rangle_p \phi(p)\, dp$$

for any $\phi \in \mathcal{D}(\Omega)$.

Proof. We want to apply Theorem 5.1 (with its notations) to the forms

$$\alpha_1^n := (E^n)^\natural \quad \text{and} \quad \alpha_2^n := *(D^n)^\natural\ ,$$

taking $s_1 = 1$, $s_2 = 2n$, $p_1 = p$, $p_2 = q$. Indeed, $\alpha_1^n \overset{\circ}{\wedge} \alpha_2^n = \alpha_1^n \wedge \alpha_2^n = \langle (E^n)^\natural, (D^n)^\natural\rangle\, dV = \langle E^n, D^n\rangle\, dV$, and thus the assertion will follow by showing that $\{\mathrm{div}_{\mathbb{H}} D^n\}$ is compact in $W^{-1,q}_{\mathbb{H},\mathrm{loc}}(\Omega)$ and $\{\mathrm{curl}_{\mathbb{H}} E^n\}$ is compact in $W^{-2,p}_{\mathbb{H};2,\mathrm{loc}}(\Omega)$ if $n = 1$, or in $W^{-1,p}_{\mathbb{H};2,\mathrm{loc}}(\Omega)$ if $n > 1$. But this follows by a simple computation from Theorem 3.11, since

i) $L^b_{\mathrm{loc}}(\Omega, H\mathbb{H}^n)$ is compactly embedded in $W^{-1,q}_{\mathbb{H},\mathrm{loc}}(\Omega)$ for $n \ge 1$;
ii) $L^a_{\mathrm{loc}}(\Omega, H\mathbb{H}^n)$ is compactly embedded in $W^{-2,p}_{\mathbb{H},\mathrm{loc}}(\Omega)$ for $n = 1$;
iii) $L^a_{\mathrm{loc}}(\Omega, H\mathbb{H}^n)$ is compactly embedded in $W^{-1,p}_{\mathbb{H},\mathrm{loc}}(\Omega)$ for $n > 1$.

Indeed, consider first the case i): we have to show that the choice $r = b$ and $s = q$ satisfies the assumptions of Theorem 3.11. If now $Qq/(Q + q) \le 1$, then our assumptions reduce to $b > 1$; but $Qq/(Q+q) \le 1$ yields $s' := q' \ge Q$, and hence the assumptions of Theorem 3.11 are satisfied. On the other hand, if $Qq/(Q + q) > 1$, then $s' < Q$, and $r' := b' < qQ/(qQ - Q - Q) = q'Q/(Q - q') := s'Q/(Q - s')$, and the assumptions of Theorem 3.11 are still satisfied. Finally, cases ii) and iii) can be handled in the same way.

□

In particular, Theorem 6.1 makes possible to extend the notion of Murat-Tartar H-convergence (see e.g. [16]), given in [9] for $\mathbb{H}^1$, to $\mathbb{H}^n$ for $n \ge 1$. In fact, the definitions given in [9] are naturally stated in general Heisenberg groups as follows.

Definition 6.2. If $0 < \alpha \le \beta < \infty$ and Ω is an open subset of $\mathbb{H}^n$, $n \ge 1$, we denote by $M(\alpha, \beta; \Omega)$ the set of $(2n \times 2n)$-matrix-valued measurable functions in Ω such that

$$\langle A(p)\xi, \xi\rangle_{\mathbb{R}^{2n}} \ge \frac{1}{\beta}\, |A(p)\xi|^2_{\mathbb{R}^{2n}} \quad \text{and} \quad \langle A(p)\xi, \xi\rangle_{\mathbb{R}^{2n}} \ge \alpha|\xi|^2_{\mathbb{R}^{2n}}$$

for all $\xi \in \mathbb{R}^{2n}$ and for a.e. $p \in \Omega$.

Definition 6.3. We say that a sequence of matrices $A^n \in M(\alpha, \beta; \Omega)$ *H-converges* to the matrix $A^{\mathit{eff}} \in M(\alpha', \beta'; \Omega)$ for some $0 < \alpha' \le \beta' < \infty$, if for every $f \in$

$W^{-1,2}_{\mathbb{H}}(\Omega)$, called u_n the solutions in $\overset{\circ}{W}{}^{1,2}_{\mathbb{H}}(\Omega)$ of the problems $-\mathrm{div}_{\mathbb{H}}\,(A^n\nabla_H u_n) = f$, the following convergences hold:

$$u_n \to u_\infty \quad \text{in} \quad \overset{\circ}{W}{}^{1,2}_{\mathbb{H}}(\Omega)\text{-weak}$$

$$A^n\nabla_H u_n \to A^{\mathit{eff}}\nabla_H u_\infty \quad \text{in} \quad L^2(\Omega; H\mathbb{H}^n)\text{-weak}\,.$$

Therefore u_∞ is solution of the problem $-\mathrm{div}_{\mathbb{H}}\,(A^{\mathit{eff}}\nabla_H u_\infty) = f$ in Ω.

Theorem 6.1 above enables us now to extend to higher order Heisenberg groups Theorem 4.4 of [9], showing that the sets $M(\alpha,\beta;\Omega)$ are compact in the topology of the H-convergence.

Theorem 6.4. *If $0 < \alpha \le \beta < \infty$ and Ω is a bounded open subset of $\mathbb{H}^n$, $n \ge 1$, then for any sequence of matrices $A^n \in M(\alpha,\beta;\Omega)$ there exists a subsequence A^{n_k} and a matrix $A^{\mathit{eff}} \in M(\alpha,\beta;\Omega)$ such that A^{n_k} H-converges to A^{eff}.*

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