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Existence theorems for fully nonlinear equations in the Heisenberg group

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Abstract[1]. We prove the existence of a viscosity solution for a class of fully nonlinear subelliptic equations in the Heisenberg group.

1. Introduction

The principal aim of this work is to prove the existence of solution of Dirichlet problems for a class of fully non linear equations, acting in the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^n, \circ)$. The contest is that of viscosity solutions, since the operators we consider are of a non variational nature. Viscosity solutions in the Heisenberg setting has been lately studied in different papers, see e.g. the work of Beatrous, Bieske and Manfredi, [4], Bardi and Mannucci [3], Manfredi and Stroffolini [15], see also [14]. For existence results in the Euclidean setting see e.g. [5].

It is well known how important is the role of the distance function for elliptic PDE in general. In the Heisenberg group, we have the availability of two distances: the smooth distance and the Carnot-Carathéodory one.

The smooth distance of ξ from a compact set K is defined by

$$\delta_K(\xi) = \inf\{|\xi \circ \eta^{-1}|_{\mathbb{H}^n} \ ; \ \eta \in K\} \ ,$$

the Koranyi norm $|.|_{\mathbb{H}^n}$ will be defined in the next section.

The Carnot-Carathéodory distance $d_K(\xi)$ is given by the minimum time to reach K from ξ with "horizontal" curves of speed one. The horizontal curves are curves which are tangent to the space generating the Heisenberg algebra.

So far, first order regularity for the Carnot-Carathéodory distance has been studied: Monti and Serra Cassano proved that it is a weak solution of the eikonal equation [16] and Dragoni proved that it is also a viscosity solution of the same equation [9]. In a recent preprint [7] Cannarsa and Rifford proved that it is semiconcave using a very abstract proof. We shall use their result. To our knowledge it is not know if the smooth distance is semiconcave, though Arcozzi and Ferrari study the Hessian of the smooth distance in [1].

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We wish to mention an interesting preprint of A. Cutri and N. Tchou [8] which was completed while this note was processed where existence is proved for a class of fully non linear equations. Their technique differs from ours and their work is somehow more complete, but we think that our proof is simpler.

We expect, in an up-coming work, to give a new and somehow simpler proof of the semiconcavity of the Carnot-Carathéodory distance.

2. Notation

We will set $n \in \mathbb{N}$, $n \geq 1$ and we consider the "space" variables

$$x = (x_1, \cdots, x_n) \in \mathbb{R}^n \quad \text{and} \quad y = (y_1, \cdots, y_n) \in \mathbb{R}^n .$$

Also, we will often write $z = (x, y) \in \mathbb{R}^{2n}$ and we identify it with the vector $z \in \mathbb{C}^n$ with components $z_k = x_k + \sqrt{-1}\, y_k$, for $k = 1, \cdots, n$. The vector $\overline{z} \in \mathbb{C}^n$ has components $\overline{z}_k = x_k - \sqrt{-1}\, y_k$, for $k = 1, \cdots, n$.

Given z, $w \in \mathbb{C}^n$, the notation zw stands for the product in $\mathbb{C}^n$, that is

$$zw = \sum_{j=1}^{n} z_j w_j .$$

Given $t \in \mathbb{R}$, we will use the notation $\xi = (z, t) = (x, y, t) \in \mathbb{R}^{2n+1}$. We will also consider the radial variables $\rho' = |z|$ and

$$\rho = |\xi|_{\mathbb{H}^n} = \left(|z|^4 + t^2\right)^{1/4} . \tag{2.1}$$

As usual, $\mathbb{H}^n$ denotes the Heisenberg group, endowed with the action

$$\xi \circ \xi_0 = (z + z_0,\, t + t_0 + 2\,\mathrm{Im}\,(\overline{z} z_0)) .$$

The Hörmander vector fields that generate the Heisenberg algebra are:

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \quad , \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}$$

which are left-invariant with respect to $\circ$. The horizontal gradient will be denoted by $\nabla_{\mathbb{H}^n}$ and we will write:

$$\nabla_{\mathbb{H}^n} u(\xi) = (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u)(\xi) = \sigma(\xi) \nabla u(\xi) .$$

Here $\sigma(\xi)$ is the $2n \times (2n+1)$ matrix

$$\sigma(\xi) := \begin{pmatrix} I & 0 & 2y^T \\ 0 & I & -2x^T \end{pmatrix} .$$

I denote the $n \times n$ identity matrix and $\xi = (x, y, t)$.

We will define the smooth distance $\delta(\xi, \eta)$ using the smooth gauge:

$$\delta(\xi, \eta) = |\eta^{-1} \circ \xi|_{\mathbb{H}^n} .$$

We will consider the so-called "Koranyi ball" centered at ξ_0 of radius r, defined by

$$\mathfrak{B}_r(\xi_0) = \{\xi \ \text{ s.t. } \ \delta(\xi, \xi_0) \leq r\} .$$

Remark. Since $\xi^{-1} = -\xi$ in the sequel for simplicity we shall use $-\xi$.

To give the definition of viscosity solution first we need to introduce the semi-jets:

Definition 2.1. Let S^{2n} denote the symmetric $2n \times 2n$ matrices. For any continuous function g we define the *intrinsic semi-jets* by:

$$J^{2,+}g(\xi) = \{(p,X) \in \mathbb{R}^{2n} \times S^{2n} \text{ such that } g(\xi \circ (h,0)) \leq g(\xi) + p \cdot h + \frac{1}{2}\langle Xh, h\rangle\} ,$$

$$J^{2,-}g(\xi) = \{(p,X) \in \mathbb{R}^{2n} \times S^{2n} \text{ such that } g(\xi \circ (h,0)) \geq g(\xi) + p \cdot h + \frac{1}{2}\langle Xh, h\rangle\} .$$

Consider a second order subelliptic equation

$$F(\xi, u(\xi), \nabla_{\mathbb{H}^n} u(\xi), D^2_h u(\xi)) = 0 \tag{2.2}$$

with F continuous, increasing in the p variable and decreasing in the X variable (degenerate ellipticity).

We shall say that u is a viscosity subsolution of (2.2) in Ω if and only if for any $\xi_0 \in \Omega$ and any $(p,X) \in J^{2,+}u(\xi_0)$ we get $F(\xi_0, u(\xi_0), p, X) \geq 0$. Analogously, we give the definition of viscosity supersolution replacing $J^{2,+}u(\xi_0)$ with $J^{2,-}u(\xi_0)$ and by requiring that the opposite inequality holds.

u is a viscosity solution if it is both a sub and a super viscosity solution. From now on we shall drop the "viscosity" term since we only consider viscosity solutions.

2.1. **Properties of the distance function.** We now introduce the Carnot-Caratheodory distance, control distance. Let $V = \{v : \mathbb{R} \to \mathbb{R}^{2n},\ v(s) \in C(R),\ |v(s)| = 1\}$. We shall say that $\phi = \phi^{\xi,v}$ is a **horizontal curve** starting a ξ if it is a solution of the Cauchy problem

$$\begin{cases} \dot{\phi} = \sigma(\phi)^T v \\ \phi(0) = \xi \ . \end{cases}$$

Definition 2.2. Let K be a closed set of $\mathbb{H}^n$. For any $\xi \notin K$, the *Carnot Caratheodory distance* to K is defined by

$$d_K(\xi) := \inf_{v \in V} \inf\{s \text{ such that } \phi^{\xi,v}(s) \in K\} .$$

So d_K is just the minimal time to reach K through horizontal curves. It is a solution of the eikonal equation, in the a.e. sense, see [16] and in the viscosity sense, see [9].

Definition 2.3. We shall say that a function g is *h-semiconcave* in Ω if there exists a constant C such that , for any $\xi \in \Omega$ and for any $h = (h,0) \in \mathbb{H}^n$ such that $\xi \circ \delta_s h \in \Omega$ for all $s \in [-1,1]$,

$$g(\xi \circ h) + g(\xi \circ -h) \leq 2g(\xi) + \frac{C}{2}|h|^2 .$$

The main reason we want to prove the h-semiconcavity is that h-semiconcave functions satisfy the following

Proposition 2.4. *If g is h-semiconcave in Ω with constant of semiconcavity equal to C then for any $\xi \in \Omega$ if $(p,X) \in J^{2,-}g(\xi)$ then $X \leq CI$.*

Proof. Indeed, identifying $h = (h,0)$:

$$\begin{aligned} 2g(\xi) + \frac{C}{2}\,|h|^2 \geq g(\xi \circ h) + g(\xi \circ -h) \geq \\ \geq 2g(\xi) + \langle Xh, h\rangle \ . \end{aligned}$$

□

Theorem 2.5 (Cannarsa-Rifford)**.** *If the closed set K is bounded and it satisfies the interior sphere condition then d_K is h-semiconcave in $\mathbb{H}^n \setminus K$.*

This is a consequence of the result of Cannarsa and Rifford which states that the distance to a point is semiconcave "away" from the point.

Proof. **First step.** For $K = \overline{\mathfrak{B}}_R(0) = \{\xi \text{ such that } d_{\{0\}}(\xi) \leq R\}$, d_K is h-semiconcave with constant of semiconcavity bounded by a constant C which depends on R.

Indeed from the dynamical programming, for $\xi \notin K$, $d_K(\xi) = d_{\{0\}}(\xi) - R$ and $d_{\{0\}}(\xi|) > R$ and then the semi-concavity is just given by the result of Cannarsa and Rifford.

Second step. If u_α is a family of h-semiconcave functions then $u(\xi) = \inf_\alpha u_\alpha(\xi)$ is h-semiconcave.

Indeed we want to prove that

$$u(\xi \circ h) + u(\xi \circ -h) \leq 2u(\xi) + C|h|^2 .$$

Knowing that this holds for each u_α.

For any $\varepsilon > 0$ there exists α such that $u(\xi) \geq u_\alpha(\xi) - (\varepsilon/2)$. Hence

$$\begin{aligned} &\lambda u(\xi \circ h) + u(\xi \circ -h) - 2u(\xi) \leq \\ &\leq \lambda u_\alpha(\xi \circ h) + u_\alpha(\xi \circ -h) - 2u_\alpha(\xi) + \varepsilon \leq \\ &\leq C|h|^2 + \varepsilon . \end{aligned}$$

This holds for any ε and the second step is proved.

Third step. Conclusion.

If K satisfies the interior sphere condition, then $K = \cup_\alpha B_{R_\alpha}(\xi_\alpha)$, since K is bounded, $\inf R_\alpha = R > 0$. Hence

$$d_K(\xi) = \inf_\alpha d_{B_{R_\alpha}(\xi_\alpha)}(\xi)$$

by the first step $d_{B_{R_\alpha}(\xi_\alpha)}(\xi)$ are all h-semiconcave with constant of semiconcavity bounded by $1/R_\alpha$. Hence by the second step d_K is also h-semiconcave.

□

Remark 2.6. Observe that Cannarsa and Rifford in fact prove that $d_{\{0\}}$ is Euclidean semiconcave

$$d_{\{0\}}(\xi + h) + d_{\{0\}}(\xi - h) \leq 2d_{\{0\}}(\xi) + \frac{C}{2}(|h|^2) ,$$

but this is a stronger condition then the h-semiconcavity.

Remark 2.7. Observe that, for $\xi \notin K$, this distance d_K is controlled by the smooth distance δ_K.

3. Existence Theorem

We will consider fully nonlinear subelliptic equations of the form:

$$F(\xi, D^2_{\mathbb{H}^n} u(\xi)) = f(\xi) \tag{3.1}$$

where $F : \mathbb{H}^n \times S^{2n} \to \mathbb{R}$ is a continuous function satisfying:

(F1) $F(\xi, 0) = 0$

(F2) for some $0 < \lambda < \Lambda$, $\forall\ M, N \in S^{2n}$ with $N \geq 0$ and for any $\xi \in \overline{\Omega}$:

$$\lambda \mathrm{Tr}(N) \leq F(\xi, M+N) - F(\xi, N) \leq \Lambda \mathrm{Tr}(N)\ ,$$

(F3) there exists a continuous function $\omega, \omega(0) = 0$ such that for every $\xi, \eta \in \mathbb{H}^n$ and every $X \in S^n$,

$$|F(\xi, X) - F(\eta, X)| \leq \omega(d(\xi, \eta))|X|\ .$$

Theorem 3.1. *Let Ω be a bounded open set satisfying the exterior ball condition. Suppose that F satisfies $F1, F2, F3$ and let f be a bounded continuous function in Ω, then there exists u solution of*

$$\begin{cases} F(\xi, D^2_{\mathbb{H}^n} u(\xi)) = f(\xi) & in\ \ \Omega\,, \\ u = 0 & on\ \ \partial\Omega\ . \end{cases} \tag{3.2}$$

To prove Theorem 3.1 it is well known, using Ishii's adaptation of Perron's method, that it is enough to construct a sub and a supersolution of (3.2). This is what we do in the next Proposition 3.2.

Proposition 3.2. *Let Ω be a bounded open set satisfying the exterior ball condition. Suppose that F satisfies $F1, F2, F3$ and $d(\xi)$ is the distance to the complement of Ω. Then , for any $\beta > 0$ and any $\gamma \in (0, 1)$ there exists k such that*

$$u_\beta(\xi) = 1 - \frac{1}{(1 + d(\xi)^\gamma))^k}$$

is a viscosity supersolution of

$$\begin{cases} F(\xi, D^2_{\mathbb{H}^n} u(\xi)) = -\beta & in\ \ \Omega\ , \\ u = 0 & on\ \ \partial\Omega\ , \end{cases}$$

and there exists k such that $v_\beta(\xi) = \left[1/(1 + d(\xi)^\gamma)^k\right] - 1$ is a subsolution of

$$\begin{cases} F(\xi, D^2_{\mathbb{H}^n} u(\xi)) = \beta & in\ \ \Omega\ , \\ u = 0 & on\ \ \partial\Omega\ . \end{cases}$$

Remark. Observe that the conditions on F don't imply that F is odd in the Hessian hence the power k will be different for the two functions.

Proof. We shall first remark that if Ω is bounded and it satisfies the exterior ball condition, then the complement of Ω satisfies the interior ball condition.

We define the following function

$$u(\xi) = 1 - \frac{1}{(1 + d(\xi)^\gamma)^k}$$

with $\gamma \in (0, 1)$ and k to be chosen later. Clearly $u = 0$ on the boundary of Ω.

We need to show that u is a supersolution. Suppose that ϕ is a test function at a point ξ_0 for u from below, i.e. $(D_{\mathbb{H}^n}\phi(\xi_0), D^2_{\mathbb{H}^n}\phi(\xi_0)) \in J^{2,-}u(\xi_0)$, then there exists a function ψ defined by

$$\phi(\xi) = 1 - \frac{1}{(1 + \psi(\xi)^\gamma)^k}$$

which is a test function for d at the same point from below. Hence for $(p, X) \in J^{2,-}(d(\xi_0))$,

$$D_{\mathbb{H}^n}\phi(\xi_0) = \frac{k\gamma d^{\gamma-1}}{(1+d^\gamma)^{k+1}}\, p$$

and

$$D^2_{\mathbb{H}^n}\phi(\xi_0) = \frac{k\,\gamma\, d^{\gamma-2}}{(1+d^\gamma)^{k+2}}\left[(\gamma - 1 - (k+2-\gamma)d^\gamma)\, p\otimes p + d(1+d^\gamma)X\right].$$

Let us recall that $|p| = 1$. We have already proved the h-semiconcavity of d that guarantees a uniform bound on the eigenvalues of X i.e. there exists $C \in \mathbb{R}$ such that $\operatorname{Tr} X \leq C$. Also, we have that $P = p \otimes p$ satisfies: $0 \leq P \leq I|p|^2 = I$. Hence we get:

$$F(\xi_0, D^2_{\mathbb{H}^n}\phi(\xi_0)) \leq \frac{k\,\gamma\, d^{\gamma-2}}{(1+d^\gamma)^{k+2}}\left[\lambda(\gamma - 1 - (k+2-\gamma)d^\gamma) + \Lambda C d(1+d^\gamma)\right].$$

We can choose k large enough so that

$$\left[\lambda(\gamma - 1 - (k+2-\gamma)d^\gamma) + \Lambda C d(1+d^\gamma)\right] < 0\,.$$

Hence for $\beta > 0$ fixed, since $\gamma \in (0,1)$ we can choose k large in order that $F(\xi_o, D^2_{\mathbb{H}^n}\phi) \leq -\beta$.

The proof of the other case proceeds in the same way, just observing that if ϕ is a test function from above for v then ψ defined by

$$\phi(\xi) = \frac{1}{(1+\psi(\xi)^\gamma)^k} - 1$$

is a test function for d from below, and hence we can similarly use Proposition 2.4.

□

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