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Γ-convergence for strongly local Dirichlet forms in perforated domains with homogeneous Neumann boundary conditions

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Abstract[1]. We consider the homogenization problems with holes for strongly local Dirichlet forms in the case of Neumann homogeneous conditions on the boundaries of holes. The main difficulties arise from the absence of a group structure on the underlying space, then from the non-periodic distribution of the holes and from the absence of classical extension results in the holes. The complete proofs of the results will appear in the paper [4].

1. Introduction

In this paper we will give some results on the Γ-convergence of problems with holes and homogeneous Neumann boundary conditions on the boundaries of holes relative to a Riemannian type Dirichlet form. The main difficulties arise from the fact that we operate on a locally compact metrizable Hausdorff space, then no structure of group is available, so there is no notion of periodicity on our space. Moreover we consider holes with irregular boundaries; then no classical extension theorems in the holes are available.

For the case of Neumann boundary condition the first paper in which for a particular Euclidean elliptic problem with periodically distributed holes the convergence to a limit problem is rigorously proved in [11], more general Euclidean cases are considered in [14]. L. Tartar made the remark that all the proofs depends on the existence of suitable extension operators (in the holes); the problem in Euclidean setting without periodicity but with the assumption of the existence of suitable extension operators (in the holes) has been solved in [8]. We remark that the question of existence of extension operators with the required properties is solved in Euclidean setting with some assumption on the regularity of the boundary of the holes but becomes very delicate in the subelliptic setting so it is important to investigate methods, which can operates in absence of classical extension theory; for paper in this direction see [14], [1], [7], [12], [19], [20], [21] for the Euclidean case and [5] for the periodic case in the Heisenberg (subelliptic) group.

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We will now introduce the notion of Poincaré type or Riemannian Dirichlet forms.
Let X be a metrizable locally compact Hausdorff space with a positive Radon measure m such that $supp[m] = X$; we assume that we are given a *strongly local diffusion type, regular Dirichlet form* in the Hilbert space $L^2(X, m)$, in the sense of M. Fukushima, [13], whose domain is denoted by $D[a]$. Such a form a admits the following integral representation $a(u, v) = \int_X \alpha(u, v)(dx)$ for every $u, v \in D[a]$ where $\alpha(u, v)$ is a signed Radon measure on X, uniquely associated with the functions u, v. Moreover for any open subset Ω of X the restriction of $\alpha(u, v)$ to Ω depends only on the restrictions of u and v to Ω and $\alpha(u, v) = 0$ whenever $u = cst$ on a neighborhood of the support of v. Let Ω be an open set; by $D_0[a, \Omega]$ we denote the closure of $C_0(\Omega) \cap D[a]$ in $D[a]$. By $D_{loc}[a, \Omega]$ we denote the space of all m-measurable functions u, v in X, that coincide m-a.e. on every compact subset of Ω with some function of $D[a]$. The measure $\alpha(u, v)$ is defined unambiguously in Ω for all $u, v \in D_{loc}[a, \Omega]$. We refer to [2], [13], [17], [18] for the properties of $\alpha(u, v)$ with respect to Leibniz, chain and truncation rules. We assume now that the space X is endowed with a pseudo distance d and is complete with respect to d (we assume also that d define a topology on X equivalent to the initial one). We denote $B(x, r) = \{y; d(x, y) < r\}$, $B(r)$ will be the ball $B(x, r)$ with a fixed center x. We say that the Dirichlet form is of *Poincaré type* if the following two assumptions hold:
(H_1) For every compact set K there exists constants $0 < R_0 < +\infty$, $\nu > 0$ and $c_0 > 0$, such that

$$0 < c_0 \left(\frac{r}{R}\right)^{\nu} m(B(x, R)) \leq m(B(x, r))$$

for every $x \in K$ and every $0 < r < R_0$.
(H_2) For every compact set K there exists constants c_1, $k \geq 1$, R_1 such that the following *scaled Poincaré inequality* holds

$$\int_{B(x,r)} |f - f_{x,r}|^2 m(dx) \leq c_1 r^2 \int_{B(x,kr)} \alpha(u, u)(dx)$$

where $f \in D_{loc}[\Omega]$, $x \in K$ and c_1, k are independent of x, r (we can assume $R_1 = R_0$).

Remark 1.1. We observe that (H_1) is verified if *a duplication property* holds for the balls $B(x, r)$, $0 < r < R_0$, $x \in K$ that is

$$m(B(x, 2r)) \leq c_0^\star \, m(B(x, r)) \ ,$$

where $c_0^\star$ is a positive constant independent of x, r (but possibly depending on K). In this case we have $\nu \geq lg_2 c_0^\star$.

Remark 1.2. We recall,[13], that there a Choquet capacity associated to a given Dirichlet form of Poincaré type.

If d is a distance on X and $d \in D_{loc}[a]$ with $\alpha(d, d) \leq m$ we say that our Dirichlet form is a *Riemannian* Dirichlet form.

Remark 1.3. We observe that if

$$d_a(x, y) = \sup \{\phi(x) - \phi(y) \ ; \ \forall \phi \in D_{loc}[a] \text{ with } \alpha(\phi, \phi) \leq m\}$$

and $d_a(x, y)$ is finite and separating (i.e. if $x \neq y$ then $d_a(x, y) \neq 0$) then d_a is a distance and $d_a \in D_{loc}[a]$ with $\alpha(d_a, d_a) \leq m$.

Remark 1.4. If a is Riemannian there exists cut-off functions between balls (i.e. given two balls $B(x,r)$ and $B(x,R)$, $r<R$, there exists a function ϕ in $D_{loc}[a]$ with $\phi=1$ on $B(x,r)$, $\phi=0$ out of $B(x,R)$ and $\alpha(\phi,\phi)\le 1/(R-r)^2$).

In section 2, we give the results obtained with a sketch of the proofs In section 3. we give two examples, which (to our knowledge) are not investigated in other papers concerning: (a) weighted uniformly elliptic operators (b) degenerate elliptic operators generated by vector fields satisfying an Hörmander condition.

2. Results and sketch of the proofs.

In this section we outline the proofs of the results, [4], since [4] is not yet available as publication. We consider a denumerable covering $\mathcal{B}_\epsilon$ of the space obtained by balls of radius $(1+\delta)\epsilon$ ($\epsilon>0$, $\delta>0$ fixed), such that the homotetic balls of radius ϵ do not intersect.We assume also that $\mathcal{B}_\epsilon$ has a property of finite intersection uniform with respect to ϵ (i.e. every point of X belongs to at most Q balls in the covering, where Q does not depend on ϵ).

We denote by $B_{i,(1+\delta)\epsilon}$, $i=1,2,..$, the balls in $\mathcal{B}_\epsilon$ and by $B_{i,\epsilon}$, $i=1,2,..$, the homotetic balls of radius ϵ. Consider now a relatively compact open set Ω with boundary Γ and denote by $B_{j,\epsilon}$, $j=1,2,..,q$, a subfamily of the balls $B_{i,\epsilon}$, such that $B_{i,\epsilon(1+\delta)}$ is contained in Ω (the number of the balls $B_{j,\epsilon}$ is finite due to the homogeneous structure of X). In every ball $B_{j,\epsilon(1-\delta)}$ we consider a compact set $T_{j,\epsilon}$. We denote $\Omega_\epsilon=\Omega-\cup_j T_{j,\epsilon}$. Let θ_ϵ be the characteristic function of Ω_ϵ We assume that θ_ϵ converges in the weak* topology of $L^\infty(\Omega,m)$ to a function θ (this property always hold at least after extraction of subsequences) with $0<\sigma<\theta\le 1$. We consider on X a Riemannian type form $a(u,v)=\int\alpha(u,v)(dx)$ with domain $D[a]$; we denote by $D_0[a,\Omega]$ the domain of the restriction of the form to Ω and we denote by $V_\epsilon(\Omega)$ the closure of the space of the functions v in $C(\Omega_\epsilon)\cap D_{loc}[a,\Omega_\epsilon]$ with $supp(v)\cap\Gamma=\emptyset$ and $\int_{\Omega_\epsilon}\alpha(v,v)(dx)<+\infty$ for the norm

$$\|v\|_\epsilon=[\int_{\Omega_\epsilon}\alpha(v,v)(dx)+\int_{\Omega_\epsilon}|v|^2m(dx)]^{1/2}\ .$$

We assume that the following scaled Poincaré inequality holds

$$(P_1)\qquad \int_{B_{j,\epsilon}-T_{j,\epsilon}}|v-\bar v_{j,\epsilon}|^2m(dx)\le C_2\epsilon^2\int_{B_{j,\epsilon(1+\delta)}-T_{j,\epsilon}}\alpha(v,v)(dx)$$

for every $v\in V_\epsilon(\Omega)$ where C_2 is a constant independent of j, ϵ and $\bar v_{j,\epsilon}$ is the average of v for the measure $m(dx)$ on $B_{j,\epsilon}-B_{j,\epsilon(1-\delta)}$.

Remark 2.1.
(a) The Poincaré inequality (P_1) has an equivalent formulation replacing $\bar v_{j,\varepsilon}$, the average of v for the measure $m(dx)$ on $B_{j,\varepsilon}-B_{j,\varepsilon(1-\delta)}$, by the average of v for the measure $m(dx)$ on $B_{j,\varepsilon}-T_{j,\varepsilon}$.
(b) The Poincaré inequality (P_1) is in some sense a weak regularity assumption on the holes. We remark that (P_1) holds in the case studied in [1] (periodic distribution of holes in Euclidean geometry with $m(dx)=dx$).

Using (P_1) we can prove the following coercivity inequality

Proposition 2.1. *There exists $\epsilon_0>0$ such that for $\epsilon\le\epsilon_0$ we have*

$$\int_{\Omega_\epsilon}\alpha(v,v)(dx)\ge\lambda\|v\|^2_{L^2(\Omega_\epsilon)}$$

where $v \in V_\epsilon(\Omega)$ and λ is a positive constant.

Let now $\eta_{j,\epsilon}$ be the cut-off function of $B_{j,\epsilon(1-\delta)}$ with respect to $B_{j,\epsilon}$ and define the linear operator $P^\epsilon : V_\epsilon \to D_0[a,\Omega]$ as

$$P^\epsilon v = (1 - \sum_j \eta_{j,\epsilon})v + \sum_j \eta_{j,\epsilon}\bar{v}_{j,\epsilon} \quad , \quad \text{on } \Omega_\epsilon$$

$$P^\epsilon v = \sum_j \eta_{j,\epsilon}\bar{v}_{j,\epsilon} \quad , \quad \text{on } \Omega - \Omega_\epsilon \ .$$

We have

Proposition 2.2. *The linear operator P^ϵ is bounded. Moreover denoted by $\tilde{v}_\epsilon$ the extension of $v \in V_\epsilon(\Omega)$ by $\bar{v}_{j,\epsilon}$ to $T_{j,\epsilon}$ ($\tilde{v}$ is so defined on Ω) we have that $\lim_{\epsilon\to 0}(P^\epsilon v - \tilde{v}_\epsilon) = 0$ in $L^2(\Omega, m)$. Consider now a sequence v_ϵ in $V_\epsilon(\Omega)$ such that $\|v_\epsilon\|_\epsilon$ is bounded; then there exists a subsequence, denoted again by v_ϵ, such that $\lim_{\epsilon\to 0} P^\epsilon v_\epsilon = w$ weakly in $D_0[a,\Omega]$, strongly in $L^2(\Omega, m)$ and $\lim_{\epsilon\to 0}(P^\epsilon v_\epsilon - \widetilde{(v_\epsilon)}_\epsilon) = 0$ in $L^2(\Omega, m)$ (equivalently $\|P^\epsilon v_\epsilon - v_\epsilon\|_{L^2(\Omega_\epsilon,m)}$ converges to 0), more exactly*

$$\|(P^\epsilon v_\epsilon - \widetilde{(v_\epsilon)}_\epsilon)\|^2_{L^2(\Omega)} \le C\epsilon^2 \int_{\Omega_\epsilon} \alpha(v_\epsilon, v_\epsilon)(dx)$$

$$\|(P^\epsilon v_\epsilon - v_\epsilon)\|^2_{L^2(\Omega_\epsilon)} \le C\epsilon^2 \int_{\Omega_\epsilon} \alpha(v_\epsilon, v_\epsilon)(dx)$$

so $\widetilde{(v_\epsilon)}_\epsilon$ converges to w in $L^2(\Omega)$ and $\|v_\epsilon - w\|_{L^2(\Omega_\epsilon,m)}$ converges to 0. Moreover if v is a function in $D_0[a,\Omega]$ then $(P^\epsilon v - v)$ converges to 0 in $L^2(\Omega, m)$ (we denote by $P^\epsilon v = P^\epsilon(v|_\Omega)$).

The proof is based on the Poincaré inequality (P_1) and the compact embedding of $D_0[a,\Omega]$ into $L^2(X,m)$ [6].

In the following we will use P^ϵ as an approximate extension operator.

We consider now the following problems

$$(2.1_\epsilon) \qquad \int_{\Omega_\epsilon} \alpha(u_\epsilon, v)(dx) = \langle f, P^\epsilon v\rangle_{(D_0[a,\Omega])', D_0[a,\Omega]}$$

$$u_\epsilon \in V_\epsilon(\Omega) \qquad \forall\, v \in V_\epsilon(\Omega) \ .$$

The above problems have a unique solution u_ϵ. Choosing $v = u_\epsilon$ we obtain that

$$\int_{\Omega_\epsilon} \alpha(u_\epsilon, u_\epsilon)(dx) \le C$$

where C is a constant independent of ϵ. We define the linear operator $B_\epsilon : (D_0[a,\Omega])' \to D_0[a,\Omega]$ as

$$B_\epsilon f = P^\epsilon u_\epsilon \ .$$

It is easy to prove that the operators B_ϵ are uniformly bounded then, at least after extraction of a subsequence denoted again by B_ϵ , weakly converge at every $f \in (D_0[a,\Omega])'$ to a linear operator $B_0 : (D_0[a,\Omega])' \to D_0[a,\Omega]$ (i.e. $\lim_{\epsilon\to 0} B_\epsilon u = B_0 f$ weakly in $D_0[a,\Omega]$ for every fixed $f \in (D_0[a,\Omega])'$). From Proposition 2.2 we have that $P^\epsilon u_\epsilon$ is bounded in $D_0[a,\Omega]$ then (taking into account that $B_\epsilon f$ weakly converges to $B_0 f$ in $D_0[a,\Omega]$ for every fixed $f \in (D_0[a,\Omega])'$) we have that

$$\lim_{\epsilon\to 0} P^\epsilon u_\epsilon = u_0 = B_0 f$$

weakly in $D_0[a,\Omega]$ and strongly in $L^2(\Omega,m)$. Then the operator $B_0 : (D_0[a,\Omega])' \to D_0[a,\Omega]$ is defined as

$$B_0 f = u_0 \ .$$

At first we observe that for $f, g \in (D_0[a,\Omega])'$

$$\langle g, B_0 f\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} = \langle f, B_0 g\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]}$$

and

$$\langle f, B_0 f\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} = \langle f, u_0\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} =$$

$$= \lim_{\epsilon\to 0} \langle f, P^\epsilon u_\epsilon\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} = \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \alpha(u_\epsilon,u_\epsilon)(dx) \geq$$

$$\geq C \ \limsup_{\epsilon\to 0} \|P^\epsilon u_\epsilon\|^2_{D_0[a,\Omega]} \geq C\|u_0\|^2_{D_0[a,\Omega]} \geq C\|B_0 f\|^2_{D_0[a,\Omega]} \ .$$

We prove now that B_0 is coercive.

We have

$$\langle f, B_0 f\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} = \langle f, u_0\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} = \tag{2.2}$$

$$= \lim_{\epsilon\to 0} \langle f, P^\epsilon u_\epsilon\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} = \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \alpha(u_\epsilon,u_\epsilon)(dx) \ .$$

We have also

$$\int_{\Omega_\epsilon} \alpha(u_\epsilon, v)(dx) = \int_\Omega \alpha(A^{-1}f, P^\epsilon v)(dx)$$

where A is the operator defined by the form a. Choose now v as the restriction to Ω_ϵ of $A^{-1}f$, we obtain

$$\int_{\Omega_\epsilon} \alpha(u_\epsilon, A^{-1}f)(dx) = \int_\Omega \alpha(A^{-1}f, P^\epsilon A^{-1}f)(dx)$$

then

$$\frac{1}{2}\int_{\Omega_\epsilon} \alpha(u_\epsilon,u_\epsilon)(dx) + \frac{1}{2}\int_{\Omega_\epsilon} \alpha(A^{-1}f, A^{-1}f)(dx) \geq$$

$$\geq \int_\Omega \alpha(A^{-1}f, P^\epsilon A^{-1}f)(dx) \ .$$

We obtain in the limit as $\epsilon \to 0$

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \alpha(u_\epsilon,u_\epsilon)(dx) \geq \int_\Omega \alpha(A^{-1}f, A^{-1}f)(dx) = \|f\|_{(D_0[a,\Omega])'}$$

then using (2.2)

$$\langle f, B_0 f\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} \geq \|f\|_{(D_0[a,\Omega])'} \ .$$

Taking into account the previous properties we have that the operator B_0 is invertible and we denote by $A_0 : D_0[a,\Omega] \to (D_0[a,\Omega])'$ the operator B_0^{-1}. We observe that we can prove that A_0 is defined on all $D_0[a,\Omega]$ and is bounded, coercive and selfadjoint from $D_0[a,\Omega]$ into $(D_0[a,\Omega])'$; moreover

$$\lambda \ \langle Au, u\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} \leq$$

$$\leq \ \langle A_0 u, u\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]} \ \leq \Lambda \ \langle Au, u\rangle_{(D_0[a,\Omega])',D_0[a,\Omega]}$$

(where A is the operator from $D_0[a,\Omega]$ into $(D_0[a,\Omega])'$ relative to the form a) so A_0 defines a bilinear symmetric form $a_0(u,v)$ on $D_0[a,\Omega] \times D_0[a,\Omega]$, which is positive and such that

$$\lambda a(u,u) \leq a_0(u,u) \leq \Lambda a(u,u)$$

for every $u \in D_0[a,\Omega]$. The above domination inequality implies that the bilinear form a_0 is regular ($D_0[a_0,\Omega] = D_0[a,\Omega]$ and the norms of the graph are equivalent in the two cases). Finally we observe that $u_0 = B_0 f$ is the unique solution of the problem

$$a_0(u_0, v) = \langle f, v\rangle_{(D_0[a,\Omega])', D_0[a,\Omega]} \quad , \quad \text{for every } v \in D_0[a,\Omega] = D_0[a_0,\Omega] \; .$$

We will now prove that a_0 is a Dirichlet form. To prove that a_0 is a Dirichlet form it is enough to prove that for a_0 a Markov type property holds. The proof of Markov property uses essentially the following semicontinuity property:

Proposition 2.3. *Let $v_\epsilon \in V_\epsilon$ be a sequence such that*

$$\int_{\Omega_\epsilon} \alpha(v_\epsilon, v_\epsilon)(dx) \le C$$

and $P^\epsilon v_\epsilon$ converges to v_0 as $\epsilon \to 0$ weakly in $D_0[a,\Omega]$, then

$$\liminf_{\epsilon \to 0} \int_{\Omega_\epsilon} \alpha(v_\epsilon, v_\epsilon)(dx) \ge a_0(v_0, v_0) \; .$$

In the following we assume that a is such that $\alpha(u,u)(dx) = \alpha(u,u)(x)m(dx)$ where $\alpha(u,u)(.) \in L^1(\Omega, m)$ for all $u \in D_0[a,\Omega]$.

As in [18] we can prove also that a_0 is strongly local and we denote by $\alpha_0(u,u)(dx)$ its energy density. As a consequence of the boundness and coercivity of a_0 with respect to a and we obtain that

$$\lambda\alpha(u,u)(dx) \le \alpha_0(u,u)(dx) \le \Lambda\alpha(u,u)(dx)$$

[18], then $\alpha_0(u,u)(dx) = \alpha_0(u,u)(x)m(dx)$ where $\alpha_0(u,u)(.) \in L^1(\Omega,m)$ for all $u \in D_0[a,\Omega]$. Moreover, since a is of a Riemannian form, also a_0 is a Riemannian form with the same domain as a with respect to a distance which is of the form $\Lambda^{-1}d$ where d is the distance associated with a. We have so the following result:

Theorem 2.4. *Let u_ϵ be the solution of the problems (2.1_ϵ). There exists a subsequence of ϵ (that does not depend on f), denoted again by ϵ such that (for every f) $P^\epsilon u_\epsilon$ strongly converges in $L^2(\Omega,m)$ to u_0; where u_0 is the solution of the problem*

$$(2.1_0) \qquad \int_\Omega \alpha_0(u,v)(x)\,dx = \langle f, v\rangle_{(D_0[a,\Omega])', D_0[a,\Omega]}$$

$$u_0 \in D_0[a,\Omega] \quad , \quad \forall\, v \in D_0[a,\Omega]$$

where $a_0(u,v) = \int_\Omega \alpha_0(u,v)m(dx)$ is a Riemannian type Dirichlet form and

$$\lambda\alpha(u,u)(x) \le \alpha_0(u,u)(x) \le \Lambda\alpha(u,u)(x)$$

a.e. in Ω.

We observe that the convergence result in Theorem 2.4 holds again if in the problems (2.1_ϵ) we replace f by a sequence f_ϵ converging to f in $(D_0[a,\Omega])'$; then we obtain:

Theorem 2.5. *Let w_ϵ be solutions of the problems*

$$(2.1'_\varepsilon) \qquad \int_{\Omega_\epsilon} \alpha(w_\epsilon, v)\; m(dx) = \int_{\Omega_\epsilon} \frac{f}{\theta}\, v\; m(dx)$$

$$u_\epsilon \in V_\epsilon(\Omega) \qquad \forall\, v \in V_\epsilon(\Omega)$$

where $f \in L^2(\Omega,m)$. Denote by u_ϵ the sequence of solutions of (2.1_ϵ). We have that $\lim_{\epsilon\to 0} \|u_\epsilon - w_\epsilon\|_\epsilon = 0$. Moreover there exists a subsequence of ϵ (that does not *depend on f), denoted again by ϵ, such that (for every f) $P^\epsilon w_\epsilon$ converges strongly in $L^2(\Omega, m)$ and weakly in $D_0[a,\Omega]$ to u_0; where u_0 is the solution of the problem*

$$\int_\Omega \alpha_0(u_0, v)\, m(dx) = \int_\Omega fv\, m(dx) \tag{2.1$_0$}$$

$$u_0 \in D_0[a,\Omega] \qquad \forall\, v \in D_0[a,\Omega]$$

where α_0 is as in Theorem 2.4.

An easy consequence of Theorem 2.5 is the convergence of the global energy of the solution w_ϵ of (2.1$'_\varepsilon$) (with respect to α) in Ω_ϵ to the global energy of the solution u_0 of (2.1$_0$) (with respect to α_0) in Ω i.e.

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \alpha(w_\epsilon, w_\epsilon)\, m(dx) = \int_\Omega \alpha_0(u_0, u_0)\, m(dx)\ . \tag{2.3}$$

We also prove a local result of convergence of energies.

We first observe that denoting by $w_{\epsilon,\lambda}$ the solution of the problem

$$\int_{\Omega_\epsilon} \alpha(w_{\epsilon,\lambda}, v)\, m(dx) + \lambda \int_{\Omega_\epsilon} \frac{w_{\epsilon,\lambda}}{\theta} v\, m(dx) = \int_{\Omega_\epsilon} \frac{f}{\theta} v\, m(dx) \tag{2.1$'_{\epsilon,\lambda}$}$$

$$w_{\epsilon,\lambda} \in V_\epsilon(\Omega) \qquad \forall\, v \in V_\epsilon(\Omega)$$

(where $\lambda > 0$). Assume that $P^\epsilon w_{\epsilon,\lambda}$ strongly converges in $L^2(\Omega, m)$ to $u_{0,\lambda}$; then $u_{0,\lambda}$ is the solution of the problem

$$\int_\Omega \alpha_0(u_{0,\lambda}, v)\, m(dx) + \lambda \int_{\Omega_\epsilon} u_{0,\lambda} v\, m(dx) = \int_\Omega fv\, m(dx) \tag{2.1$'_{0,\lambda}$}$$

$$u_{0,\lambda} \in D_0[a,\Omega] \qquad \forall\, v \in D_0[a,\Omega]\ .$$

We also observe that if f is bounded all the $w_{\epsilon,\lambda}$ are uniformly (with respect to ϵ) bounded on Ω_ϵ and that (2.3) hold again for $w_{\epsilon,\lambda}$ and $u_{0,\lambda}$.

Let now f be bounded and $\phi \geq 0$ be in $D_0[a,\Omega]$ such that $A_0\phi \in L^2(\Omega)$ where A_0 is the operator corresponding to the form A_0. Let ϕ_ϵ be the solution of the problem (2.1$'_\varepsilon$) with $f = \theta^{-1} A_0 f$ (we recall that $\theta \geq \sigma > 0$); we have that ϕ_ϵ converges to ϕ in $L^2(\Omega)$ and $\int_{\Omega_\epsilon} \alpha(\phi_\epsilon, \phi_\epsilon)\, m(dx) \leq C$; moreover

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \alpha(\phi_\epsilon^-, \phi_\epsilon^-)\, m(dx) = 0\ .$$

Using $w_{\epsilon,\lambda}\phi_\epsilon$ in (2.1$'_{\epsilon,\lambda}$) we obtain that

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \alpha(w_{\epsilon,\lambda}, w_{\epsilon,\lambda})\phi_\epsilon\, m(dx) \geq \int_\Omega \alpha_0(u_{0,\lambda}, u_{0,\lambda})\phi\, m(dx)\ . \tag{2.4$_\lambda$}$$

Using now as test function $w_{\epsilon,\lambda}\phi$ we can prove that

$$\lim_{\epsilon\to 0} \theta_\epsilon \alpha(w_{\epsilon,\lambda}, w_{\epsilon,\lambda}) = \chi_\lambda \in L^1(\Omega) \tag{2.5$_\lambda$}$$

weakly* in the space of measures. From (2.5$_\lambda$) and (2.4$_\lambda$) we obtain that

$$\chi \geq \alpha_0(u_{0,\lambda}, u_{0,\lambda})$$

so the global convergence of energies give us the equality

$$\chi = \alpha_0(u_{0,\lambda}, u_{0,\lambda})\ .$$

Then

$$\lim_{\epsilon\to 0} \theta_\epsilon \alpha(w_{\epsilon,\lambda}, w_{\epsilon,\lambda}) = \alpha_0(u_{0,\lambda}, u_{0,\lambda}) \tag{2.6$_\lambda$}$$

weakly* in the space of measures. The result can be generalized to the case $\lambda > 0$, $f \in L^2(\Omega)$ by approximation on f, so it is easily to deduce that the result holds also for $\lambda = 0$. We have so proved:

Theorem 2.6. *Let w_ϵ, u_ϵ, u_0 be as in Theorem 2.5, then*

$$\lim_{\epsilon\to 0} \theta_\epsilon \alpha(w_\epsilon, w_\epsilon) = \alpha_0(u_0, u_0) \tag{2.6}$$

$$\lim_{\epsilon\to 0} \theta_\epsilon \alpha(u_\epsilon, u_\epsilon) = \alpha_0(u_0, u_0) \tag{2.7}$$

weakly* in the space of measures.

As a consequence of Theorems 2.5, 2.6 by an approximation on f we obtain

Theorem 2.7. *Let $f \in (D_0[a,\Omega])'$ and u_ϵ, u_0 be as in Theorem 2.4, then*

$$\lim_{\epsilon\to 0} \theta_\epsilon \alpha(u_\epsilon, u_\epsilon) = \alpha_0(u_0, u_0) \tag{2.7}$$

weakly* in the space of measures.

3. Some examples.

We now give two examples to which the above results may be applied. To the best of our knowledge, these examples have not been treated elsewhere. Hereafter, each hole T_ε is contained in a ball of radius $\varepsilon(1-\delta)$, and calling B_ε the concentric ball of radius ε, the measure of $B_\varepsilon - T_\varepsilon$ is equivalent to the measure of B_ε; from these last two facts, it follows that for weak* $L^\infty(\Omega)$-limit θ of the sequence of characteristic functions θ_ε, there exists a positive constant σ such that $\theta > \sigma$ a.e..

3.1. **Elliptic problems with a weight.** Our first example is an elliptic problem with a weight. Let $w \in L^1_{loc}(\mathbb{R}^N)$ be a weight in the Muckenhoupt's class A_2, *i.e.*, $w \geq 0$ and

$$\sup_B \left(\frac{1}{|B|} \int_B w\, dx)(\frac{1}{|B|} \int_B w^{-1}\, dx \right) \leq C^w_{A_2} ,$$

where B is an arbitrary ball in $\mathbb{R}^N$. Take $Y = (0,1)^N$. Let T_1 and T_2 be two compact sets in Y such that $Y - T_i$, $i = 1, 2$, are open weak John connected sets. For any weight q in A_2, we have

$$\int_{Y-T_i} |u - \bar{u}_i|^2 q\, dx \leq C \int_{Y-T_i} |\nabla u|^2 q\, dx$$

where $\bar{u}_i$ is the average of $u \in H^1(Y - T_i, q)$ on $Y - T_i$, and C only depends on $C^q_{A_2}$, (see theorem 9.7 in [15]).
Call $Y_\varepsilon = \varepsilon Y$, $T_{i,\varepsilon} = \varepsilon T_i$, $Y_{m,\varepsilon} = m\varepsilon + Y_\varepsilon$, $T_{m,i,\varepsilon} = m\varepsilon + T_{i,\varepsilon}$, where m is an N-vector with integer components. By the previous observations we obtain that

$$\int_{Y_{m,\varepsilon}-T_{m,i,\varepsilon}} |u - \bar{u}_{i,\varepsilon}|^2 w\, dx \leq C\varepsilon^2 \int_{Y_{m,\varepsilon}-T_{m,i,\varepsilon}} |\nabla u|^2 w\, dx ,$$

and the assumptions of the previous sections on the holes hold. Let $a_{i,j}(x)$, $i,j = 1, ..., N$, be measurable functions such that $a_{i,j}(x) = a_{j,i}(x)$ a.e. and there exist λ and Λ, $0 < \lambda < \Lambda$, such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^N a_{i,j}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad , \quad \forall \xi \in \mathbb{R}^N .$$

We consider the Riemannian Dirichlet form on $L^2(\mathbb{R}^N, w)$

$$a(u,v) = \int \sum_{i,j=1}^{N} a_{i,j}(x) D_{x_i}u \; D_{x_j}v \; w \; dx \; ,$$

with domain $H^1(\mathbb{R}^N, w)$; here $m(dx) = w \; dx$ and

$$\alpha(u,v)(x) = \sum_{i,j=1}^{N} a_{i,j}(x) D_{x_i}u \; D_{x_j}v \; .$$

Let $\Omega \subset \mathbb{R}^N$ be a relatively compact open set. Let $Y_{m^*,\varepsilon}$ be the collection of the $Y_{m,\varepsilon}$ such that the closure of $(1+\delta)Y_{m,\varepsilon}$, $\delta > 0$ is contained in Ω. Let $T_{m^*,i,\varepsilon}$ be the corresponding subcollection of the $T_{m,i,\varepsilon}$. Let $f_\varepsilon(m)$ be a function defined on $\mathbb{N}^N$ with value in $\{1,2\}$ and call Ω_ε the set $\Omega - \bigcup_{m^*} T_{m^*,f_\varepsilon(m^*),\varepsilon}$. Let us consider the problems: find $u_\varepsilon \in H^1_{0,\partial\Omega}(\Omega_\varepsilon, w)$, such that

$$\int_{\Omega_\varepsilon} \sum_{i,j=1}^{N} a_{i,j}(x) D_{x_i}u_\varepsilon D_{x_j}v \; w \; dx = \int_{\Omega_\varepsilon} fv \; w \; dx \quad , \quad \forall \, v \in H^1_{0,\partial\Omega}(\Omega_\varepsilon, w) \; ,$$

where $f \in L^2(\Omega, w)$ and $H^1_{0,\partial\Omega}(\Omega_\varepsilon, w)$ is the closure in $H^1(\Omega_\varepsilon, w)$ of the space of functions in $H^1(\Omega_\varepsilon, w)$ with support having a positive distance from $\partial\Omega$.
From the results contained in the previous sections and the methods of representation of strongly local Dirichlet forms on open subsets of $\mathbb{R}^N$, [13], we obtain that there exists a subsequence ε_k independent of f such that $P^{\varepsilon_k}u_{\varepsilon_k}$ converges to $u_0 \in H^1(\Omega, w)$ weakly in $H^1(\Omega, w)$ and strongly in $L^2(\Omega, w)$ (the extension operator P^ε is the one defined in Section 2, choosing $\|x\| = \sup_{i=1,..,N} |x_i|$ for the norm in $\mathbb{R}^N$). Moreover there exist some functions $a^0_{i,j}$ (such that $a^0_{i,j} = a^0_{j,i}$), and two structural constants λ_0 and Λ_0 depending on λ, Λ, $0 < \lambda_0 < \Lambda_0$, such that $\lambda_0|\xi|^2 \leq \sum_{i,j=1}^{N} a^0_{i,j}(x)\xi_i\xi_j \leq \Lambda_0|\xi|^2$, $\forall \xi \in \mathbb{R}^N$, and $u_0 \in H^1_0(\Omega, w)$ is the solution of the problem

$$\int_{\Omega} \sum_{i,j=1}^{N} a^0_{i,j}(x) D_{x_i}u_0 \; D_{x_j}v \; w \; dx = \int_{\Omega_\varepsilon} \theta fv \; w \; dx \quad , \quad \forall \, v \in H^1_0(\Omega, w) \; ,$$

where θ is the weak*-$L^\infty(\Omega)$ limit of θ_ε. Moreover, $\theta_{\Omega_\varepsilon} \sum_{i,j=1}^{N} a_{i,j}(x) D_{x_i}u_{\varepsilon_k} D_{x_j}u_{\varepsilon_k} w$ converges to $\sum_{i,j=1}^{N} a^0_{i,j}(x) D_{x_i}u_0 D_{x_j}v \; w$ in the sense of measures.
We stress the fact that the problem is not periodic for two reasons:

1. the distribution of the holes is not periodic,
2. the measure $w \; dx$ is not invariant by translations.

Moreover the holes' boundary does not satisfy the assumptions of the usual extension theorem (in the interior of the holes).

3.2. **Subelliptic problems.** We now give a second example concerning Dirichlet forms defined by subelliptic operators. Let X_i, i=1,...,m, be vectors fields on $\mathbb{R}^N$, $N \geq 2$, with C^∞ coefficients and which satisfy an Hörmander condition. We call $a_{i,j}(x)$, $i,j = 1,...,m$, measurable functions such that $a_{i,j}(x) = a_{j,i}(x)$ a.e. and for two constants λ and Λ, $0 < \lambda < \Lambda$,

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{m} a_{i,j}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad , \quad \forall \, \xi \in \mathbb{R}^m \; .$$

We consider the Riemannian Dirichlet form on $L^2(\mathbb{R}^N)$

$$a(u,v) = \int \sum_{i,j=1}^{m} a_{i,j}(x) X_i u \; X_j v \; dx$$

with domain $H^1(\mathbb{R}^N; X)$; here $m(dx) = dx$ and

$$\alpha(u,v)(x) = \sum_{i,j=1}^{m} a_{i,j}(x) D_{x_i} u D_{x_j} v \; .$$

We associate a Riemannian distance $d(x,y)$ to the Dirichlet form, see for instance [16], and we denote by $B(x,r) = B_d(x,r)$ the ball of center x and radius r for the distance d. We observe that a Poincaré inequality holds on balls with center in an arbitrary compact set K and with radius $r \le r_0(K)$, for a suitable $r_0(K) > 0$, [16]. Let Ω be a relatively compact open set in $\mathbb{R}^N$ with $\Omega \subset K$ and let $B_{j,\varepsilon}$ be as in Section 2. We choose $B_{j,\varepsilon(1-\delta)} - T_{j,\varepsilon}$ as the set of points in $B_{j,\varepsilon(1-\delta)}$ with distance to the boundary of $B_{j,\varepsilon(1-\delta)}$ less than $\delta\varepsilon/8$, $0 < \delta < 1/2$. We observe that $B_{j,\varepsilon(1+\delta)} - T_{j,\varepsilon}$ is a weak John domain, see [15]; moreover the Lebesgue measure on $(\mathbb{R}^N, d)$ is doubling on $B_{j,\varepsilon} - T_{j,\varepsilon}$ and

$$m(B(x,r)) \ge C \left(\frac{r}{\varepsilon}\right)^s m(B_{j,\varepsilon(1+\delta)} - T_{j,\varepsilon})$$

for a suitable $s \ge 2$ where $x \in B_{j,\varepsilon(1+\delta)} - T_{j,\varepsilon}$ and $r \le \varepsilon$. Then taking into account Theorem 9.7 in [15] and Remark 2.1, we see that the Poincaré inequality (P_1) holds. Let Ω_ε be defined as in Section 2. Consider the problems: find $u_\varepsilon \in H^1_{0,\partial\Omega}(\Omega_\varepsilon; X)$ such that

$$\int_{\Omega_\varepsilon} \sum_{i,j=1}^{m} a_{i,j}(x) X_i u_\varepsilon \; X_j v \; dx = \int_{\Omega_\varepsilon} f v \; dx \quad , \quad \forall v \in H^1_{0,\partial\Omega}(\Omega_\varepsilon; X) \; ,$$

for $f \in L^2(\Omega)$, where $H^1_{0,\partial\Omega}(\Omega_\varepsilon, X)$ is the closure of the functions in $H^1(\Omega_\varepsilon, X)$ with support having a positive distance from $\partial\Omega$.

From the results in the previous sections and the methods of representation of strongly local Dirichlet forms on open subsets of $\mathbb{R}^N$, see [13], we obtain that there exists a subsequence ε_k independent of f such that $P^{\varepsilon_k} u_{\varepsilon_k}$ converges to $u_0 \in H^1(\Omega, X)$ weakly in $H^1(\Omega, X)$ and strongly in $L^2(\Omega, X)$ (the extension operator P^ε is the one defined in Section 2 with respect to the distance d).

Moreover there exist some functions $a^0_{i,j}$ (such that $a^0_{i,j} = a^0_{j,i}$), and two structural constants λ_0 and Λ_0 depending on λ, Λ, $0 < \lambda_0 < \Lambda_0$, such that $\lambda_0 |\xi|^2 \le \sum_{i,j=1}^{N} a^0_{i,j}(x) \xi_i \xi_j \le \Lambda_0 |\xi|^2$, $\forall \xi \in \mathbb{R}^N$, and $u_0 \in H^1_0(\Omega, w)$ is the solution of the problem

$$\int_{\Omega} \sum_{i,j=1}^{m} a^0_{i,j}(x) X_i u_0 \; X_j v \; dx = \int_{\Omega_\varepsilon} \theta f v \; dx \quad , \quad \forall\, v \in H^1_0(\Omega, X) \; ,$$

where θ is the weak* - $L^\infty(\Omega)$ limit of θ_ε. Moreover, $\theta_{\Omega_\varepsilon} \sum_{i,j=1}^{m} a_{i,j}(x) X_i u_{\varepsilon_k} \; X_j u_{\varepsilon_k}$ converges to $\sum_{i,j=1}^{m} a^0_{i,j}(x) X_i u_0 \; X_j v$ in the sense of measures.

We stress that the notion of periodicity does not make sense here, since there is no group structure on $\mathbb{R}^N$ connected with the distance d. Here again, the holes' boundaries do not satisfy the assumptions of the usual extension theorem (in the interior of the holes).

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