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Schauder estimates for parabolic and elliptic nondivergence operators of Hörmander type

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Abstract[1]. Let $X_1, X_2, \ldots, X_q$ be a system of real smooth vector fields satisfying Hörmander's rank condition in a bounded domain Ω of $\mathbb{R}^n$. Let $A = \{a_{ij}(t,x)\}_{i,j=1}^q$ be a symmetric, uniformly positive definite matrix of real functions defined in a domain $U \subset \mathbb{R} \times \Omega$. For operators of kind

$$H = \partial_t - \sum_{i,j=1}^q a_{ij}(t,x) X_i X_j - \sum_{i=1}^q b_i(t,x) X_i - c(t,x)$$

we prove local a-priori estimates of Schauder-type, in the natural (parabolic) $C^{k,\alpha}(U)$ spaces defined by the vector fields X_i and the distance induced by them. Namely, for $a_{ij}, b_i, c \in C^{k,\alpha}(U)$ and $U' \Subset U$, we prove

$$\|u\|_{C^{k+2,\alpha}(U')} \leq c \left\{ \|Hu\|_{C^{k,\alpha}(U)} + \|u\|_{L^\infty(U)} \right\} .$$

1. Introduction

This talk describes a joint work with Luca Brandolini (Università di Bergamo), contained in [5].

Let Ω be a bounded domain of $\mathbb{R}^n$, and let $X_1, X_2, \ldots, X_q$ be a system of smooth real vector fields satisfying Hörmander's rank condition in Ω. In this setting, "sum of squares" operators

$$\sum_{i=1}^q X_i^2$$

or their "parabolic" analog

$$\partial_t - \sum_{i=1}^q X_i^2 \tag{1.1}$$

have been widely studied since Hörmander's famous paper [14]: these operators are hypoelliptic, and share with elliptic and parabolic operators several deep analogies.

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In recent years, nondivergence operators modeled on the above classes, namely

$$L=\sum_{i,j=1}^{q}a_{ij}(x)X_iX_j \tag{1.2}$$

or

$$H=\partial_t-\sum_{i,j=1}^{q}a_{ij}(t,x)X_iX_j \tag{1.3}$$

have also been studied, assuming that $A=\{a_{ij}\}_{i,j=1}^{q}$ is a symmetric, uniformly positive definite matrix of real functions defined in Ω (in case (1.2)) or in a bounded domain $U\subset\mathbb{R}\times\Omega$ (in case (1.3)), and $\lambda>0$ is a constant such that:

$$\lambda^{-1}|\xi|^2\le\sum_{i,j=1}^{q}a_{ij}\xi_i\xi_j\le\lambda|\xi|^2 \text{ for every } \xi\in\mathbb{R}^q\ ,$$

uniformly in Ω or U. These classes of operators naturally arise in some problems related to geometry in several complex variables (see [17] and references therein) as well as in some models of human vision (see [11]); moreover, these operators realize a framework where a suitable theory of nonlinear equations modeled on Hörmander's vector fields can be settled.

A system of Hörmander vector fields can be thought as the natural substitute of the "Cartesian" derivatives ∂_{x_i}, in the study of degenerate equations like (1.2) or (1.3). Moreover, it induces a "Carnot–Carathéodory distance", which is (locally) doubling with respect to the Lebesgue measure. These facts allow to define several function spaces shaped on the vector fields, such as Hölder spaces, Sobolev spaces, BMO, VMO *etc.* It is then natural to use these spaces to express the required regularity of the coefficients a_{ij}. Clearly, as soon as the coefficients a_{ij} are not C^∞, the corresponding operator (1.2) or (1.3) is no longer hypoelliptic, and no result can be drawn on it from the classical theory of Hörmander's sums of squares. Nevertheless, many classical results about elliptic and parabolic operators, which do not require, in principle, high regularity of the coefficients, when properly reformulated in the language of vector fields, look like desirable properties of these operators, and reasonable -although nontrivial- conjectures. Two typical instances of this situation are (local) L^p estimates and C^α estimates on the "second order" derivatives X_iX_ju. In [2], [3] we have proved L^p estimates of this kind for operators of type (1.2) or some more general classes, assuming the coefficients a_{ij} in the space VMO, extending the classical results of Rothschild-Stein [19] for Hörmander' sum of squares. In the paper [5], which we are discussing here, we prove local C^α estimates of Schauder type for nonvariational parabolic operators of Hörmander's type. Our main result is the following:

Theorem 1. *Let Ω be a bounded domain of $\mathbb{R}^n$, and let $X_1,X_2,\dots,X_q$ be a system of smooth real vector fields defined in a neighborhood Ω_o of Ω and satisfying Hörmander's rank condition in Ω_o. Let U be a bounded domain of $\mathbb{R}^{n+1}$, $U\subset\mathbb{R}\times\Omega$; let $A=\{a_{ij}(t,x)\}_{i,j=1}^{q}$ be a symmetric, uniformly positive definite matrix of real functions defined in U, and $\lambda>0$ a constant such that*

$$\lambda^{-1}|\xi|^2\le\sum_{i,j=1}^{q}a_{ij}(t,x)\xi_i\xi_j\le\lambda|\xi|^2 \textit{ for every } \xi\in\mathbb{R}^q,(t,x)\in U\ .$$

Let

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x)X_iX_j - \sum_{i=1}^{q} b_i(t,x)X_i - c(t,x) \tag{1.4}$$

with $a_{ij}, b_i, c \in C^{k,\alpha}(U)$ *for some integer* $k \geq 0$ *and some* $\alpha \in (0,1)$*. Then, for every domain* $U' \Subset U$ *there exists a constant* $c > 0$ *depending on* $U, U', \{X_i\}, \alpha, k, \lambda$ *and the* $C^{k,\alpha}$ *norms of the coefficients such that for every* $u \in C^{k+2,\alpha}_{loc}(U)$ *with* $Hu \in C^{k,\alpha}(U)$ *one has*

$$\|u\|_{C^{k+2,\alpha}(U')} \leq c \left\{ \|Hu\|_{C^{k,\alpha}(U)} + \|u\|_{L^\infty(U)} \right\} .$$

Analogous Schauder estimates for stationary operators (1.2) obviously follow from the above theorem, as a particular case.

Let us briefly compare our result with the existing literature. Xu in [21] states local estimates of Schauder type for operators of type (1.2), under an additional assumption on the structure of the Lie algebra generated by the X_i's. Capogna-Han in [10] prove "pointwise Schauder estimates" (in the spirit of Caffarelli's work [8] on fully nonlinear equations) for equations of type (1.2) in Carnot groups. Montanari in [16] proves local Schauder estimates for a particular class of operators of type (1.3), namely tangential operators on CR manifolds, where the vector fields are allowed to be nonsmooth ($C^{1,\alpha}$).

The main feature of the present paper, besides the "evolutionary" case it covers, is that our theory applies to *any* system of Hörmander vector fields.

The general strategy we use is similar to that we have followed in [3], [4]. A basic role is played by C^α continuity of singular and fractional integrals on spaces of homogeneous type (in the sense of Coifman-Weiss [12]), that we prove, coupled with the machinery introduced by Rothschild-Stein [19], that we have adapted to nondivergence operators. In the final part of this talk, we will also discuss the problem of deducing from our main theorem a regularization result, as well as an application of this.

2. Singular integrals on spaces of homogeneous type and continuity on Hölder spaces

As already recalled, a first ingredient of the proof of Theorem 1 consists in some abstract results about singular and fractional integrals on spaces of homogeneous type, which can also be of independent interest. Here we just briefly recall some basic definitions and the statements we prove.

Let (X, d, dx) be a space of homogeneous type in the sense of [12], that is X is a set, d is a quasidistance on X, and μ is a Borel measure satisfying the doubling condition with respect to the d-balls:

$$\mu(B_{2r}(x)) \leq c_\mu \cdot \mu(B_r(x)) \ \ \forall\, x \in X, r > 0 .$$

To simplify notation, the measure $d\mu(x)$ will be denoted simply by dx, and $\mu(A)$ will be written $|A|$. We will also set

$$B(x;y) = B_{d(x,y)}(x) .$$

Hölder spaces can be defined in a natural way, setting, for any $\alpha > 0$, $u : X \to \mathbb{R}$,

$$|u|_{C^\alpha(X)} = \sup \left\{ \frac{|u(x) - u(y)|}{d(x,y)^\alpha} : x, y \in X, x \neq y \right\}$$

$$\|u\|_{C^\alpha(X)} = |u|_{C^\alpha(X)} + \|u\|_{L^\infty(X)}$$

$$C^\alpha(X) = \left\{ u : X \to \mathbb{R} : \|u\|_{C^\alpha(X)} < \infty \right\} .$$

Also, we denote by $C_0^\alpha(X)$ the subspace of boundedly supported $C^\alpha(X)$ functions.

Theorem 2 (C^α continuity of singular integral operators). *Let (X, d, dx) be a bounded space of homogeneous type, and let $k(x, y)$ be a standard kernel, that is a measurable function $k : X \times X \to \mathbb{R}$ such that*

$$|k(x,y)| \le \frac{c}{|B(x;y)|} \quad \forall\, x, y \in X;$$

$$|k(x,y) - k(x_0,y)| \le \frac{c}{|B(x_0;y)|} \left(\frac{d(x_0,x)}{d(x_0,y)} \right)^\beta \tag{2.1}$$

$\forall\, x_0, x, y \in X$, with $d(x_0, y) \ge M d(x_0, x)$, $M > 1, c, \beta > 0$. Let

$$K_\varepsilon f(x) = \int_{d'(x,y)>\varepsilon} k(x,y) f(y)\, dy$$

where d' is any quasidistance on X, equivalent to d, and fixed once and for all. Assume that $\forall\, f \in C^\alpha(X)$ and $x \in X$ the following limit exists:

$$Kf(x) = P.V. \int_X k(x,y) f(y)\, dy = \lim_{\varepsilon \to 0} K_\varepsilon f(x) .$$

Also, assume that:

$$\left| \int_{d'(x,y)>r} k(x,y)\, dy \right| \le c_K$$

$\forall\, r > 0$ (with c_K independent of r) and

$$\lim_{\varepsilon \to 0} \left| \int_{d'(x,y)>\varepsilon} k(x,y)\, dy - \int_{d'(x_0,y)>\varepsilon} k(x_0,y)\, dy \right| \le c_K d(x,x_0)^\gamma \tag{2.2}$$

for some $\gamma \in (0, 1]$. Then the operator K is continuous on $C^\alpha(X)$; more precisely:

$$|Kf|_{C^\alpha(X)} \le c_K \|f\|_{C^\alpha(X)} \text{ for every } \alpha \le \gamma, \alpha < \beta$$

where γ is the number in (2.2) and β is the number in (2.1). Moreover,

$$\|Kf\|_\infty \le c_{K,R,\alpha} \|f\|_\alpha$$

where $R = diam X$.

Theorem 3 (C^α continuity of fractional integral operators). *Let (X, d, dx) be a bounded space of homogeneous type, and assume that X does not contain atoms (that is, points of positive measure). Let $k_\delta(x, y)$ be a "fractional integral kernel", that is:*

$$0 \le k_\delta(x,y) \le c \frac{d(x,y)^\delta}{|B(x;y)|}$$

$\forall\, x, y \in X$, some $c, \delta > 0$;

$$|k_\delta(x,y) - k_\delta(x_0,y)| \le c \frac{d(x_0,y)^\delta}{|B(x_0;y)|} \left(\frac{d(x_0,x)}{d(x_0,y)} \right)^\beta$$

$\forall\, x_0, x, y \in X$, with $d(x_0, y) \geq Md(x_0, x)$, some $M > 1, c, \beta > 0$ ("mean value inequality"). Then the operator

$$I_\delta f(x) = \int_X k_\delta(x, y) f(y)\, dy$$

is continuous on $C^\alpha(X)$, $\forall\, \alpha < \min(\beta, \delta)$.

3. Parabolic Carnot-Carathéodory distance and Hölder spaces

Let Ω be a bounded domain of $\mathbb{R}^n$, and let $X_1, X_2, \ldots, X_q$ be a system of smooth real vector fields defined in a neighborhood Ω_o of Ω and satisfying Hörmander's condition of step s in Ω_o. For $x, y \in \Omega_0$, let $d(x, y)$ be the Carnot-Carathéodory distance induced by the system $\{X_i\}$ (see for instance [18]); it is well known that d is actually a distance and the Lebesgue measure is locally doubling w.r.t. d:

$$|B_{2r}(x)| \leq c\, |B_r(x)| \quad \forall\, x \in \Omega,\ r \leq r_0$$

$$c_1\, |x - y| \leq d(x, y) \leq c_2\, |x - y|^{1/s} \quad \forall\, x,\, y \in \Omega\ , \tag{3.1}$$

for some positive constants c, r_0, c_1, c_2 depending on Ω. (Here s is the step appearing in Hörmander's condition).

Let us now consider the parabolic Carnot-Carathéodory distance d_P corresponding to d, namely

$$d_P\left((t, x), (s, y)\right) = \sqrt{d(x, y)^2 + |t - s|}\ ,$$

defined in the cylinder $\mathbb{R} \times \Omega$. It can be proved that, for any d_P-ball $B_R(t_0, x_0)$,

$$(B_R(t_0, x_0), d_P, dtdx)$$

is a space of homogeneous type.

We can define parabolic Hölder spaces adapted to this context. For any bounded domain $U \subset \mathbb{R} \times \Omega \subset \mathbb{R}^{n+1}$ and any $\alpha > 0$, let:

$$|u|_{C^\alpha(U)} = \sup\left\{ \frac{|u(t, x) - u(s, y)|}{d_P\left((t, x), (s, y)\right)^\alpha} : (t, x), (s, y) \in U, (t, x) \neq (s, y) \right\}$$

$$\|u\|_{C^\alpha(U)} = |u|_{C^\alpha(U)} + \|u\|_{L^\infty(U)}$$

$$C^\alpha(U) = \left\{ u : U \to \mathbb{R} : \|u\|_{C^\alpha(U)} < \infty \right\}\ .$$

Note that, by (3.1), a function $u \in C^\alpha(U)$ is also continuous on U in Euclidean sense. For any positive integer k, let

$$C^{k,\alpha}(U) = \left\{ u : U \to \mathbb{R} : \|u\|_{C^{k,\alpha}(U)} < \infty \right\}$$

with

$$\|u\|_{C^{k,\alpha}(U)} = \sum_{|I| + 2h \leq k} \left\| \partial_t^h X^I u \right\|_{C^\alpha(U)}$$

where, for any multiindex $I = (i_1, i_2, \cdots, i_s)$, with $1 \leq i_j \leq q$, we say that $|I| = s$ and

$$X^I u = X_{i_1} X_{i_2} \cdots X_{i_s} u\ .$$

4. Local Schauder estimates for test functions with small support

We now show how the proof of Theorem 1 is organized. We will focus our attention on our main result in its *basic case* (no lower order terms, no higher order derivatives):

Theorem 4 (Main result in the basic case). *Under the above assumptions on X_1, $X_2, \dots, X_q$ and the matrix $\{a_{ij}(t,x)\}_{i,j=1}^{q}$ let*

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j \ .$$

Then, for every domain $U' \Subset U$ there exists $c > 0$ depending on $U, U', \{X_i\}, \alpha, \lambda$ and $\|a_{ij}\|_{C^\alpha}$ such that for every $u \in C^{2,\alpha}_{loc}(U)$ with $Hu \in C^\alpha(U)$ one has

$$\|u\|_{C^{2,\alpha}(U')} \le c \left\{ \|Hu\|_{C^\alpha(U)} + \|u\|_{L^\infty(U)} \right\} \ .$$

The general case then follows from this result by a tedious repetition of the same general ideas.

Here we will suppose the reader to be familiar with some concepts and results contained in the fundamental papers by Folland [13] and Rothschild-Stein [19].

First of all, by Rothschild-Stein "lifting Theorem", we lift the vector fields $X_i(x)$, defined in $\mathbb{R}^n$, to new vector fields $\widetilde{X}_i(\xi)$ defined on $\mathbb{R}^N$, with $\xi = (x,h), h \in \mathbb{R}^{N-n}$. We also set $\widetilde{a}_{ij}(t,\xi) = \widetilde{a}_{ij}(t,x,h) = a_{ij}(t,x)$, $\widetilde{\Omega} = \Omega \times I$, where I is a neighborhood of the origin in $\mathbb{R}^{N-n}$, $\widetilde{U} = U \times I$ and

$$\widetilde{H} = \partial_t - \sum_{i,j=1}^{q} \widetilde{a}_{ij}(t,\xi) \widetilde{X}_i \widetilde{X}_j \ .$$

The proof of Theorem 4 then proceeds in three steps, the first being the following:

Theorem 5 (Estimates in the lifted space for functions with small support). *There exist r, $c > 0$ such that $\forall\, u \in C_0^{2,\alpha}\left(\widetilde{B}_r(t_0,\xi_0)\right)$,*

$$\|u\|_{C^{2,\alpha}\left(\widetilde{B}_r\right)} \le c \left\{ \left\|\widetilde{H}u\right\|_{C^\alpha\left(\widetilde{B}_r\right)} + \|u\|_{L^\infty\left(\widetilde{B}_r\right)} \right\}$$

where c, r depend on $\{X_i\}, \alpha, \lambda$ and $\|a_{ij}\|_{C^\alpha(U)}$.

Here is a sketch of the proof.

1. We start with the lifted operator:

$$\widetilde{H} = \partial_t - \sum_{i,j=1}^{q} \widetilde{a}_{ij}(t,\xi) \widetilde{X}_i \widetilde{X}_j \ .$$

2. We now freeze the coefficients $\widetilde{a}_{ij}$ (but not the vector fields $\widetilde{X}_i$!) at some point $(t_0,\xi_0) \in \widetilde{U}$, and consider the *frozen lifted operator*:

$$\widetilde{H}_0 = \partial_t - \sum_{i,j=1}^{q} \widetilde{a}_{ij}(t_0,\xi_0) \widetilde{X}_i \widetilde{X}_j \ .$$

3. To study $\widetilde{H}_0$, we will consider its *approximating operator*, defined on $\mathbb{G}' = \mathbb{R} \times \mathbb{G}$:

$$\mathcal{H}_0 = \partial_t - \sum_{i,j=1}^{q} \widetilde{a}_{ij}(t_0, \xi_0) Y_i Y_j$$

where the Y_i are left invariant, 1-homogeneous vector fields on the homogeneous group $\mathbb{G}$ which appears in Rothschild-Stein "lifting and approximation" construction. Since $\{\widetilde{a}_{ij}(t_0, \xi_0)\}$ is a constant matrix with positive eigenvalues, the operator $\sum_{i,j=1}^{q} \widetilde{a}_{ij}(t_0, \xi_0) Y_i Y_j$ can be rewritten as a "sum of squares" operator, so $\mathcal{H}_0$ is hypoelliptic by Hörmander's theorem. Moreover, $\mathcal{H}_0$ is left invariant and homogenous of degree 2 in $\mathbb{G}' = \mathbb{R} \times \mathbb{G}$, hence by Folland's results it has a fundamental solution, denoted by

$$h(t_0, \xi_0, t, u)$$

or briefly

$$h(t, u)$$

which is homogeneous of degree $-Q$, being Q the homogeneous dimension of $\mathbb{G}$. Also, $h(t, u)$ is nonnegative and vanishes for $t < 0$.

4. Next, we use this fundamental solution $h(t, u)$ to build a parametrix for the lifted operator $\widetilde{H}$. The basic computation, which explains how the "lifting and approximation theorem" is exploited, is the following:

$$\widetilde{X}_i \left[h(t, \Theta(\xi, \cdot)) \right] (\eta) = (Y_i h)(t, \Theta(\xi, \eta)) + \left(R_i^{\xi} h \right) ((t, \Theta(\xi, \eta)))$$

where the function $\left(R_i^{\xi} h \right)(t, u)$ can be written as the sum of homogeneous functions of degrees $\geq -Q$, plus a smooth function. The map $\Theta(\xi, \cdot)$ is a diffeomorphism from a neighborhood of ξ onto a neighborhood of the origin in $\mathbb{G}$.

Iterating, we find that

$$\widetilde{H}_0 \left[h(\cdot, \Theta(\xi, \cdot)) \right] (t, \eta) = \delta_{(0,\xi)}(t, \eta) + \text{remainder}$$

where the remainder is a kernel with a locally integrable singularity.

This is the basic idea to prove a suitable representation formula for any test function $f(t, \xi)$ in terms of $\widetilde{H}_0 f$, and then a representation formula for $\widetilde{X}_i \widetilde{X}_j f$ in terms of $\widetilde{H} f$ and f.

To implement this idea rigorously, some labour is required. Skipping many important details (which have been already worked out in [19] and [3]), let us jump to the conclusion:

Theorem 6. *The following representation formula for second derivatives of any test function f, in terms of $\widetilde{H} f$, holds:*

$$\widetilde{X}_r \widetilde{X}_s (af)(t, \xi) = T \widetilde{H} f(t, \xi) + \tag{4.1}$$

$$+ T \sum_{i,j=1}^{q} \left[\widetilde{a}_{ij}(t_0, \xi_0) - \widetilde{a}_{ij}(t, \xi) \right] \widetilde{X}_i \widetilde{X}_j f(t, \xi) +$$

$$+ \sum_{i,j=1}^{q} \widetilde{a}_{ij}(t_0, \xi_0) \left\{ \sum_{k=1}^{q} T_{ij}^{k} \widetilde{X}_k f(t, \xi) + T_{ij} f(t, \xi) \right\}.$$

where T, T_{ij}, T_{ij}^{k} are frozen singular integrals, a concept we now define.

Definition 1. If $h(s,u)$ is, as above, the fundamental solution of $\mathcal{H}_0$, homogeneous of degree $-Q$, we say that $k(t,\xi,\eta)$ is a *frozen singular kernel* if for every positive integer m there exists a positive integer H_m such that

$$\begin{aligned} k(t,\xi,\eta) &= \sum_{i=1}^{H_m} a_i(\xi) b_i(\eta)(D_i h)(t,\Theta(\eta,\xi)) + \\ &+ a_0(\xi) b_0(\eta)(D_0 h)(t,\Theta(\eta,\xi)) \end{aligned}$$

where:

a_i, b_i ($i=0,1,\dots H_m$) are test functions,

D_i are differential operators such that: for $i=1,\dots,H_m$, D_i is homogeneous of degree ≤ 2 (so that $D_i h$ is a homogeneous function of degree $\geq -Q-2$), and D_0 is a differential operator such that $D_0 h$ has m derivatives with respect to the vector fields Y_i ($i=1,\dots,q$).

We say that T is a *frozen singular integral* if $k(t,\xi,\eta)$ is a frozen singular kernel and

$$\begin{aligned} Tf(t,\xi) &= \lim_{\varepsilon\to 0} \int_{\widetilde{d}'_P((t,\xi),(s,\eta))>\varepsilon} k(t-s,\xi,\eta)\, f(s,\eta)\, ds\, d\eta + \\ &+ \alpha(t_0,\xi_0)\beta(t,\xi) f(t,\xi)\ , \end{aligned}$$

where α is bounded and β is smooth.

Note that the singular integral kernel is shaped on the fundamental solution of the frozen homogeneous operator $\mathcal{H}_0$ (hence the name of *frozen* singular integral); nevertheless, the representation formula (4.1) holds for the nonhomogeneous, unfrozen operator $\widetilde{H}$.

Roughly speaking, what we have called "frozen singular integral" is the sum of many *singular and fractional* integral operators, plus an integral operator with smooth kernel. Applying our abstract results in spaces of homogeneous type we are able to prove the following:

Theorem 7 (C^α continuity of frozen singular integrals)**.** *If T is a frozen singular integral and $\widetilde{B}_r$ a $\widetilde{d}_P$-ball in $\mathbb{R}^{N+1}$, then T is continuous on $C^\alpha\left(\widetilde{B}_r\right)$:*

$$\|Tf\|_{C^\alpha\left(\widetilde{B}_r\right)} \leq c\,\|f\|_{C^\alpha\left(\widetilde{B}_r\right)}\ .$$

Now, taking C^α norms of both sides of (4.1) and applying the above continuity theorem, plus standard properties of Hölder norms, we first get

$$\begin{aligned} &\left\|\widetilde{X}_k \widetilde{X}_h f\right\|_{C^\alpha\left(\widetilde{B}_r\right)} \leq \\ &\leq c\left\{\left\|\widetilde{H} f\right\|_{C^\alpha\left(\widetilde{B}_r\right)} + \sum_{i,j=1}^{q}\left\|\left[\widetilde{a}_{ij}(t_0,\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\widetilde{X}_i\widetilde{X}_j f\right\|_{C^\alpha\left(\widetilde{B}_r\right)} + \right. \\ &\left. + \sum_{l=1}^{q}\left\|\widetilde{X}_l f\right\|_{C^\alpha\left(\widetilde{B}_r\right)} + \|f\|_{C^\alpha\left(\widetilde{B}_r\right)}\right\} \end{aligned}$$

and then, for r small enough (classical "Korn's trick" which is used in Schauder theory):

$$\left\|\widetilde{X}_k\widetilde{X}_h f\right\|_{C^\alpha(\widetilde{B}_r)} \le c\left\{\left\|\widetilde{H}f\right\|_{C^\alpha(\widetilde{B}_r)} + \sum_{l=1}^{q}\left\|\widetilde{X}_l f\right\|_{C^\alpha(\widetilde{B}_r)} + \|f\|_{C^\alpha(\widetilde{B}_r)}\right\}$$

and finally, with some more work,

$$\|f\|_{C^{2,\alpha}(\widetilde{B}_r)} \le c\left\{\left\|\widetilde{H}f\right\|_{C^\alpha(\widetilde{B}_r)} + \|f\|_{L^\infty(\widetilde{B}_r)}\right\}$$

which is Step 1 of the proof of our basic result.

5. Interpolation inequalities for Hölder norms and local Schauder estimates in the lifted variables

The second step in the proof of Theorem 4 amounts to prove the following C^α-estimates for $\widetilde{H}$ on a ball, for functions not necessarily vanishing at the boundary:

Theorem 8. *There exist positive constants r, c, β such that $\forall\, u \in C^{2,\alpha}\left(\widetilde{B}_r(t_0,\xi_0)\right)$, $0 < t < s < r$,*

$$\|u\|_{C^{2,\alpha}(\widetilde{B}_t)} \le \frac{c}{(s-t)^\beta}\left\{\left\|\widetilde{H}u\right\|_{C^\alpha(\widetilde{B}_s)} + \|u\|_{L^\infty(\widetilde{B}_s)}\right\}$$

where c, r depend on $\{X_i\}, \alpha, \lambda$ and $\|a_{ij}\|_{C^\alpha(U)}$, β depends on $\{X_i\}, \alpha$.

This follows from Step 1 (Theorem 5) by standard properties of cutoff functions and suitable interpolation inequalities for Hölder norms. The construction of well-shaped cutoff functions is fairly standard, also in this context, and we will not give further details. The second tool is much more delicate, due to the lack of dilations in this general context:

Theorem 9 (Interpolation inequality)**.** *There exist positive constants c, R, γ depending on α, $\{X_i\}$ such that $\forall\, f \in C^{2,\alpha}\left(\widetilde{B}_R\right)$, $0 < \rho < R, 0 < \delta < 1/3$,*

$$\|Df\|_{C^\alpha(\widetilde{B}_\rho)} \le \delta\left[\left\|D^2 f\right\|_{C^\alpha(\widetilde{B}_R)} + \|\partial_t f\|_{C^\alpha(\widetilde{B}_R)}\right] + \frac{c}{\delta^\gamma (R-\rho)^{2\gamma}}\ \|f\|_{L^\infty(\widetilde{B}_R)}\ .$$

The proof of this Theorem exploits:

- representation formulas (for first order derivatives $\widetilde{X}_i u$) already used in Step 1;
- our results about fractional integrals in spaces of homogeneous type (Theorems 2 and 3);
- our continuity result for frozen operators of type 0 (Theorem 7);
- cutoff functions.

When one applies this interpolation inequality to the proof of "Step 2", it is crucial the fact that the "large constant" $c(\delta)$ which multiplies $\|f\|_{L^\infty}$ be controlled by *some negative power of δ*. This fact cannot be derived by easy homogeneity arguments, but must be gained by a careful quantitative control of the bounds on the integral kernels involved in our representation formulas.

6. Hölder spaces and lifting

We have now to show how we can transfer to the original space $\mathbb{R}^{n+1}$ the Schauder estimates we have proved in the lifted space $\mathbb{R}^{N+1}$. In other words, starting from Theorem 8, we have to prove the analogue:

Theorem 10. *There exist positive constants r, c, β such that $\forall\, u \in C^{2,\alpha}(B_r(t_0, x_0))$, $0 < t < s < r$,*

$$\|u\|_{C^{2,\alpha}(B_t)} \leq \frac{c}{(s-t)^{\beta}} \left\{ \|Hu\|_{C^{\alpha}(B_s)} + \|u\|_{L^{\infty}(B_s)} \right\}$$

where c, r depend on $\{X_i\}, \alpha, \lambda$ and $\|a_{ij}\|_{C^{\alpha}(U)}$, β depends on $\{X_i\}, \alpha$.

This fact involves the delicate relation between the CC-distance $d(x, y)$ induced by a system $X_1, X_2, \cdots, X_q$ of Hörmander's vector fields in $\mathbb{R}^n$, and the CC-distance $\widetilde{d}(\xi, \eta)$ induced by the lifted vector fields $\widetilde{X}_1, \widetilde{X}_2, \cdots, \widetilde{X}_q$ in $\mathbb{R}^N$.

Let $d_P, \widetilde{d}_P$ be the corresponding parabolic distances; denote by $C^{\alpha}_X(U), C^{\alpha}_{\widetilde{X}}\left(\widetilde{U}\right)$ the Hölder spaces induced by d_P and $\widetilde{d}_P$, respectively. We are interested in the following question.

$$\forall\, f : U \to \mathbb{R}\,, \quad \text{set} \quad \widetilde{f} : \widetilde{U} \to \mathbb{R} \quad \text{with} \quad \widetilde{f}(t, x, h) = f(t, x)\,.$$

Then, can we say that

$$f \in C^{\alpha}_X(U) \Longleftrightarrow \widetilde{f} \in C^{\alpha}_{\widetilde{X}}\left(\widetilde{U}\right)\ ?$$

It is well-known that $\widetilde{d}\,((x, h), (y, k)) \geq d(x, y)$; this obviously implies

$$\left|\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}(\widetilde{U})} \leq |f|_{C^{\alpha}(U)}\ .$$

However, the reverse inequality

$$|f|_{C^{\alpha}_X(U)} \leq c \left|\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}(\widetilde{U})}$$

is not trivial. We are actually able to prove it, making use of an integral formulation of Hölder continuity, analogous to the classical integral characterization of C^{α}, and a relation between the volume of d-balls and $\widetilde{d}$-balls. Let

$$M_{\alpha, B_R(t_0,x_0)}(f) = \sup_{(t,x)\in B_R, r>0} \inf_{c\in\mathbb{R}} \frac{1}{r^{\alpha}|B_r(t,x)|} \int_{B_r(t,x)\cap B_R(t_0,x_0)} |f(s,y) - c|\, ds\, dy\,.$$

If $f \in C^{\alpha}_X(B_R(t_0, x_0))$, then $M_{\alpha, B_R(t_0,x_0)}(f) \leq c\,|f|_{C^{\alpha}_X(B_R(t_0,x_0))}$. But the converse is also true:

Lemma 1. *If $f \in L^1_{loc}(B_R(t_0, x_0))$ is a function such that $M_{\alpha,B_R(t_0,x_0)}(f) < \infty$, then there exists a function f^*, a.e. equal to f, such that $f^* \in C^{\alpha}_X(B_R(t_0, x_0))$ and*

$$|f^*|_{C^{\alpha}_X(B_R)} \leq c M_{\alpha,B_R(t_0,x_0)}(f)$$

for some c independent of f.

The original proof of this result in the Euclidean case is due to Campanato [9], and substantially works also in this context. The second ingredient we use is essentially due to Sanchez-Calle [20] (the parabolic version we have written is a straightforward adaptation):

Lemma 2. *Given a point* $(t, x, h) \in \mathbb{R}^{N+1}$,

$$\left|\widetilde{B}_r(t,x,h)\right| \simeq |B_r(t,x)| \cdot \left|\left\{h' \in \mathbb{R}^{N-n} : (\tau, z, h') \in \widetilde{B}_r(t,x,h)\right\}\right|$$

provided $(\tau, z) \in B_{\delta r}(t,x)$ *for some fixed* $\delta < 1$. *The equivalence holds with respect to* $r > 0$, *and the symbol* $|\cdot|$ *denotes the volume of a set in the suitable dimension.*

The above two ingredients enable us to prove the following:

Proposition 1. *If* $f, \widetilde{f}$ *are as above, then*

$$\left|\widetilde{f}\right|_{C^\alpha_{\widetilde{X}}(\widetilde{B}_R)} \leq |f|_{C^\alpha_X(B_R)} \leq c\left|\widetilde{f}\right|_{C^\alpha_{\widetilde{X}}(\widetilde{B}_R)}.$$

Moreover,

$$\left|\widetilde{X}_{i_1}\widetilde{X}_{i_2}\cdots\widetilde{X}_{i_k}\widetilde{f}\right|_{C^\alpha_{\widetilde{X}}(\widetilde{B}_R)} \leq |X_{i_1}X_{i_2}\cdots X_{i_k}f|_{C^\alpha_X(B_R)} \leq c\left|\widetilde{X}_{i_1}\widetilde{X}_{i_2}\cdots\widetilde{X}_{i_k}\widetilde{f}\right|_{C^\alpha_{\widetilde{X}}(\widetilde{B}_R)}$$

for $i_j = 1, 2, \cdots, q$.

Combining "Step 2" with the last Proposition, we immediately get "Step 3":

Proof of Theorem 10.

$$\begin{aligned}\|u\|_{C^{2,\alpha}_X(B_t)} &\leq c\,\|\widetilde{u}\|_{C^{2,\alpha}_{\widetilde{X}}(\widetilde{B}_t)} \leq \frac{c}{(s-t)^\beta}\left\{\left\|\widetilde{H}\widetilde{u}\right\|_{C^\alpha_{\widetilde{X}}(\widetilde{B}_s)} + \|\widetilde{u}\|_{L^\infty(\widetilde{B}_s)}\right\} \leq \\ &\leq \frac{c}{(s-t)^\beta}\left\{\|Hu\|_{C^\alpha_X(B_s)} + \|u\|_{L^\infty(B_s)}\right\}.\end{aligned}$$

□

Finally, by a covering argument, Theorem 4 follows.

7. Regularization of solutions

Once we have proved Theorem 1, a more subtle question poses, namely the possibility of using the above a-priori estimates to show that, whenever a function $u \in C^{2,\alpha}_{loc}(U)$ solves $Hu = f$ in U with $C^{k,\alpha}(U)$ coefficients and data, then actually $u \in C^{k+2,\alpha}_{loc}(U)$. This result is contained in the following:

Theorem 11. *Under the assumptions of Theorem 1, for every* $\alpha \in (0,1)$, *if* $u \in C^{2,\alpha}_{loc}(U)$ *and* $Hu \in C^{k,\alpha}(U)$ *for some even integer* k, *then* $u \in C^{2+k,\alpha}_{loc}(U)$. *Moreover, for every domain* $U' \Subset U$ *there exists a constant* $c > 0$ *depending on* U, U', $\{X_i\}$, α, k, λ *and* $\|a_{ij}\|_{C^{k,\alpha}(U)}$, $\|b_i\|_{C^{k,\alpha}(U)}$, $\|c\|_{C^{k,\alpha}(U)}$ *such that*

$$\|u\|_{C^{2+k,\alpha}(U')} \leq c\left\{\|Hu\|_{C^{k,\alpha}(U)} + \|u\|_{L^\infty(U)}\right\}. \tag{7.1}$$

Note that we already know that (7.1) holds, assuming $Hu \in C^{k,\alpha}(U)$ *and* $u \in C^{2+k,\alpha}_{loc}(U)$: we would like to replace the last assumption with $u \in C^{2,\alpha}_{loc}(U)$. The scheme of the proof is the following:

1. regularize coefficients and data: $a^\varepsilon_{ij}, f^\varepsilon$;
2. solve the Dirichlet problem with boundary datum u, for the equation with C^∞ coefficients and data; by hypoellipticity, this solution u^ε is C^∞, so it satisfies our a-priori estimates in $C^{k+2,\alpha}$;

3. apply some compactness argument to find a sequence u^{ε_n} converging to some $v \in C^{k+2,\alpha}_{loc}$ (provided the constants in the a-priori estimates and the $C^{k,\alpha}$-norm of $a^{\varepsilon}_{ij}, f^{\varepsilon}$ are bounded uniformly in ε);

4. apply some maximum principle and barrier argument to conclude that actually $v = u$, so that $u \in C^{k+2,\alpha}_{loc}$.

A solvability result in the smooth case (point 2) as well as the suitable barrier argument (point 4) are classical results by Bony [1].

The point is to provide a suitable mollification technique, allowing to control $C^{k,\alpha}$-norms. In our general context (without dilations and translations) there is not an obvious way to build good mollifiers. We have done this exploiting the heat kernel for the model operator:

$$\mathbf{H} = \partial_t - L = \partial_t - \sum_{i=1}^{q} X_i^2 .$$

By known results of Kusuoka-Stroock (see [15] §4), there exists a fundamental solution $h(t,x,y)$ of $\mathbf{H}$, satisfying suitable Gaussian estimates. We use this "Gaussian kernel" to build a family of mollifiers adapdet to the vector fields X_i.

Theorem 12 (mollifiers)**.** *Let $\eta \in C_0^\infty(\mathbb{R})$ be a positive test function with $\int \eta(t)dt = 1$ and let*

$$\phi_\varepsilon(t,x,y) = \varepsilon^{-1} h(\varepsilon,x,y)\eta\left(\frac{t}{\varepsilon}\right) .$$

$\forall f \in C^\alpha\left(\mathbb{R}^{n+1}\right)$, $\varepsilon \in (0,1)$, *set*

$$f_\varepsilon(t,x) = \int_{\mathbb{R}^{n+1}} \phi_\varepsilon(t-s,x,y) f(s,y)\, ds\, dy .$$

Then, there exists a constant c depending on α, $\{X_i\}$, such that

$$\|f_\varepsilon\|_{C^\alpha} \le c \|f\|_{C^\alpha} .$$

Moreover,

$$\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_{L^\infty(\mathbb{R}^{n+1})} = 0 .$$

To prove the above theorem, the idea is to apply again our abstract result on C^α-continuity of singular integrals on spaces of homogeneous type, showing that the kernels ϕ_ε satisfy the axioms of our theory with constants uniformly bounded with respect to ε. This is possible exploiting the properties of the heat kernel h proved in [15]. We are also able to prove an analogous control on the $C^{k,\alpha}$-norm of the mollified function, but unfortunately *only for even k*:

Proposition 2. $\forall \alpha \in (0,1)$, *k even integer, U, U′ bounded open sets, with $U' \Subset U$, there exists a constant c such that* $\forall f \in C^{k,\alpha}(U)$, $\varepsilon \in (0,1)$,

$$\|f_\varepsilon\|_{C^{k,\alpha}(U')} \le c \|f\|_{C^{k,\alpha}(U)} .$$

With these tools, our regularization result can be proved following the scheme we have recalled. The limitation "k even" in the control of $C^{k,\alpha}$-norms of mollified functions is the reason of the analogous limitation in our regularization result.

We end by pointing out an application of the above results and techniques. In a recent paper in collaboration with Brandolini, Lanconelli, Uguzzoni (see [6] and [7]) we prove that operators (1.4) with C^α coefficients, possess a fundamental solution $h(t,x;\tau,y)$, which satisfies sharp Gaussian estimates. By construction, this fundamental solution has only a weak regularity (roughly speaking, $\partial_t h$ and $X_i X_j h$ just *exist*, without good continuity properties); however, applying our mollification technique, with a proof very similar to that of our regularization theorem, in the above paper we prove that actually

$$h(\cdot;\tau,y) \in C^{2,\alpha}_{loc}(\mathbb{R}^{n+1} \setminus \{(\tau,y)\})$$

with norm depending only on the vector fields, the C^α norms of the coefficients, and the ellipticity constant λ.

For the complete results proved about h, the reader is referred to [6].

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