Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 6(2007), pp. 83–91.

p-Lagrangians on fractals

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Abstract[1]. In this paper, we consider measure-valued p-Lagrangians and the associated nonlinear energy forms. This Lagrangian formalism is very useful for the study of the dynamics of intrinsically irregular structures as the most common fractals.

1. Introduction

In the classical case, a nonlinear energy form on an open subset Ω of $\mathbb{R}^N$ has this type of expression

$$E^p(u,u) = \int_\Omega |\nabla u|^p \, d\mu \; ,$$

where μ is the N-dimensional Lebesgue measure and $u \in C^1(X)$.

Recently it has been developed an unifying framework in order to extend this notion to the case of *non Euclidean structures*. With the term *non Euclidean structures* we want to describe degenerate or irregular structures. The degeneracy can be mainly of two types: a first degeneracy is due to the presence of weights or subelliptic metrics, the second degeneracy is due the presence of non differentiabile structures such as the case of fractal sets. In both these cases, these two seemingly distant worlds will be treated by suitable metric tools. The main idea is to generalized the notion of Dirichlet forms which arise in the quadratic case to nonlinear energy forms which will be expressed in terms of measure-valued Lagrangians, by stating the gradient-like behaviour in a suitable form. More precisely, the nonlinear energy form E^p on a structure X is obtained by integrating on X a local energy measure $\mathcal{L}^{(p)}(u,u)$, the so called p-Lagrangian, that is,

$$E^p(u,u) = \int_X d\mathcal{L}^{(p)}(u,u) \; .$$

In presence of weights or subelliptic metrics, the p-Lagrangians are absolutely continuous with respect to the corresponding volume measures like the classical case. We point out that in our approach the p-Lagrangians may also be singular in order to treat the case of fractal sets where the p-Lagrangians are tipically singular with respect to the volume measure.

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Keywords. Nonlinear energy forms, Lagrangians, fractals, Harnack inequality.
AMS Subject Classification. 35J60, 28A80, 35B45, 35B65.

The layout of the paper is as follows. In Section 2 we recall the definitions of homogeneous p-Lagrangians. In Section 3 we present some concrete examples of fractals where the p-Lagrangians have been constructed. In the last Section we investigate the validity of Harnack type inequalities for p-Laplacians associated to p-Lagrangians on fractals. In particolar, we show that Harnack type inequalities for p-Laplacians associated to p-Lagrangians hold also in the case of singularity.

2. Homogeneous p-Lagrangians

A homogeneous p-Lagrangian can be considered as a nonlinear extension of a local Dirichlet form. The notion has been introduced in the paper of Malý and Mosco [11] and developed by Biroli and Vernole in [1] and [2].

Let X be a locally compact Hausdorff topological space and μ a nonnegative bounded Radon measure on X with $supp\mu = X$. Let $\tilde{\mathcal{L}}^{(p)}$ be a Radon measure valued nonnegative map defined on a dense subalgebra $\mathcal{C}^{(p)}$ of the space $C_b(X)$ of bounded continuous functions on X. We make the following assumptions on $\tilde{\mathcal{L}}^{(p)}$, $(p > 1)$:

L1) $\tilde{\mathcal{L}}^{(p)}$ is positive semidefinite and convex in the space $\mathcal{M}$ of Radon measure.

L2) $\tilde{\mathcal{L}}^{(p)}$ is homogeneous of degree p.

L3) $\tilde{\mathcal{L}}^{(p)}$ is such that

$$||u|| = (\int_X |u|^p \, d\mu + \int_X d\tilde{\mathcal{L}}^{(p)}(u))^{1/p} \tag{2.1}$$

is a norm in $\mathcal{C}^{(p)}$.

L4) $\tilde{\mathcal{L}}^{(p)}$ is strongly local: if $u - v =$ constant on $supp\varphi$, then

$$\int_X \varphi(x) \, d\tilde{\mathcal{L}}^{(p)}(u) = \int_X \varphi(x) \, d\tilde{\mathcal{L}}^{(p)}(v)$$

for any $\varphi \in C(X)$, $u, v \in \mathcal{C}^{(p)}$.

L5) for every u, $v \in \mathcal{C}^{(p)}$ there exists in the weakly* topology of $\mathcal{M}$ the following limit:

$$\lim_{t\to 0} \frac{\tilde{\mathcal{L}}^{(p)}(u+tv) - \tilde{\mathcal{L}}^{(p)}(u)}{t} = \langle \partial\tilde{\mathcal{L}}^{(p)}(u), v\rangle \, .$$

We define $\mathcal{L}^{(p)} : \mathcal{C}^{(p)} \times \mathcal{C}^{(p)} \to \mathcal{M}$ as

$$\mathcal{L}^{(p)}(u, v) = \langle \partial\tilde{\mathcal{L}}^{(p)}(u), v\rangle \, . \tag{2.2}$$

L6) The chain rules hold: if $u, v \in \mathcal{C}^{(p)}$ and $g \in C^1(\mathbb{R})$, with g' bounded on $\mathbb{R}$, then

$$g(u) \ : \ x \to g(u(x))$$

belongs to $\mathcal{C}^{(p)}$ and

$$\mathcal{L}^{(p)}(g(u), v) = |g'(u)|^{p-2} g'(u)\mathcal{L}^{(p)}(u, v) \, ,$$
$$\mathcal{L}^{(p)}(v, g(u)) = g'(u)\mathcal{L}^{(p)}(v, u) \, .$$

Definition 2.1. In the assumptions L1,..,L6, the measure $\mathcal{L}^{(p)}(u, v)$ in (2.2) will be called *homogeneous p-Lagrangian.*

We denote by $\mathcal{D}^{(p)}$ the (abstract) completion of $\mathcal{C}^{(p)}$ with respect to the norm (2.1). Moreover, we assume that $\tilde{\mathcal{L}}^{(p)}$ is closable in $L^p(X, \mu)$, that is, $\mathcal{D}^{(p)}$ is injected in $L^p(X, \mu)$. We extend the homogeneous p-Lagrangian to $\mathcal{D}^{(p)}$ and we still denote it by $\mathcal{L}^{(p)}(u, v)$.

3. Fractals

In this Section we present some concrete examples of fractals where p-Lagrangians have been constructed. We begin by recalling the definition of self-similar fractals.

In the D-dimensional Euclidean space $\mathbb{R}^D$, $D \geq 1$, we consider the Euclidean distance $d_e(x,y) \equiv |x-y|$; let $B_e(x,r) := \{y \in \mathbb{R}^D : |x-y| < r\}$, $x \in \mathbb{R}^D$, $r > 0$, be the Euclidean balls. We suppose that $\Psi = \{\psi_1, \dots, \psi_N\}$ is a given set of contractive similitudes $\psi_i : \mathbb{R}^D \to \mathbb{R}^D$, with contraction factors $\alpha_i^{-1} < 1$, that is,

$$|\psi_i(x) - \psi_i(y)| = \alpha_i^{-1}|x-y|$$

for every $x, y \in \mathbb{R}^D$, $i = 1, \dots, N$. In [9], it is proved that there exists a unique closed bounded set K - the so-called *self-similar fractal* - which is invariant under $\Psi = \{\psi_1, \dots, \psi_N\}$, that is,

$$K = \bigcup_{i=1}^{N} \psi_i(K) \ . \tag{3.1}$$

The real number d_f, uniquely determined by the relation

$$\sum_{i=1}^{N} \alpha_i^{-d_f} = 1 \ ,$$

is the *similarity dimension* of K. Moreover, there exists a unique Borel regular measure μ in $\mathbb{R}^D$, supported on K and unit total mass, which is *invariant* with respect to the given $\Psi = \{\psi_1, \dots, \psi_N\}$, that is, μ satisfies

$$\mu = \sum_{i=1}^{N} \alpha_i^{-d_f} \psi_{i\#}\mu \tag{3.2}$$

where $\psi_{i\#}\mu(\cdot) := \mu(\psi_i^{-1}(\cdot))$, $i = 1, \dots, N$ with $\mathrm{supp}\psi_{i\#}\mu = \psi_i(\mathrm{supp}\mu)$ (see [9]). More specific metric informationon K and μ are available when the family $\Psi = \{\psi_1, \dots, \psi_N\}$ satisfies the following *open set condition*: there exists a bounded open set $U \subset \mathbb{R}^D$, such that

$$\bigcup_{i=1}^{N} \psi_i(U) \subset U \quad , \quad \text{with} \quad \psi_i(U) \cap \psi_j(U) = \emptyset \ \text{ if } i \neq j \ . \tag{3.3}$$

In fact, under this assumption, the invariant measure μ coincides with the restriction to K of the d_f-dimensional Hausdorff measure of $\mathbb{R}^D$, $H^{d_f}\lfloor K$, normalized:

$$\mu = (H^{d_f}(K))^{-1} H^{d_f}\lfloor K$$

and d_f is also called the *fractal dimension* of K.

We will use the notations $\psi_{i_1 \dots i_n} := \psi_{i_1} \circ \psi_{i_2} \circ \dots \circ \psi_{i_n}$, $A_{i_1 \dots i_n} := \psi_{i_1 \dots i_n}(A)$ for arbitrary n-tuples of indices $i_1, \dots, i_n \in \{1, \dots, N\}$ and arbitrary $A \subset K$.

We define the boundary Γ of K as

$$\Gamma = \bigcup_{i \neq j} \psi_i^{-1}(K_i \cap K_j) \ .$$

In the following, we shall assume that Γ is a finite set,

$$\#\Gamma < \infty \tag{3.4}$$

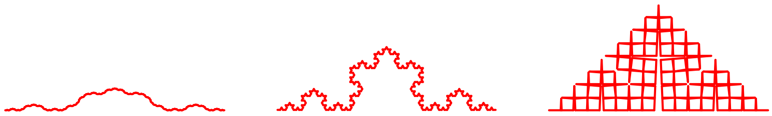

FIGURE 1. The curves $K^{(l)}$ for $l = 3.8$, $l = 3$, and $l = 2.2$ respectively.

and, for every $n \geq 1$ and every $i_1, \dots, i_n \neq j_1, \dots, j_n$,

$$K_{i_1 \dots i_n} \cap K_{j_1 \dots j_n} = \Gamma_{i_1 \dots i_n} \cap \Gamma_{j_1 \dots j_n} . \tag{3.5}$$

The first example of p-Lagrangians on self-similar fractal has been constructed for *Koch curve type fractals*. More precisely, the construction of the nonlinear energies has been given in [3] (as the limit of a sequence of energy forms defined on graph approximations to the fractal, similar to the construction of Dirichlet forms on p.c.f. self-similar sets, developed by J. Kigami [10]) and the construction of Lagrangians has been given in [5].

We recall that the Koch curve type fractals $K^{(l)}$ is the invariant set uniquely determined by the family $\Psi^{(l)} = \{\psi_1^{(l)}, \dots, \psi_4^{(l)}\}$ of contractive similitudes $\psi_i^{(l)} : \mathbb{C} \to \mathbb{C}$, $i = 1, \dots, 4$, with contraction factor l^{-1}, $2 < l \leq 4$,

$$\psi_1^{(l)}(z) = \frac{z}{l} \quad , \quad \psi_2^{(l)}(z) = \frac{z}{l} e^{i\theta(l)} + \frac{1}{l} , \tag{3.6}$$

$$\psi_3^{(l)}(z) = \frac{z}{l} e^{-i\theta(l)} + \frac{1}{2} + i\sqrt{\frac{1}{l} - \frac{1}{4}} \quad , \quad \psi_4^{(l)}(z) = \frac{z-1}{l} + 1 , \tag{3.7}$$

where

$$\theta(l) = \arcsin\left(\frac{\sqrt{l(4-l)}}{2} \right) . \tag{3.8}$$

We remark that for $l = 3$ we obtain the usual Koch curve.

The existence of nonlinear energy forms on a class of fractals called *weakly completely symmetric fractals* has been proved in [8] (see [17] also) and the construction of Lagrangians has been given in [6]. This class - contained in the class of post critically finite fractals (for definition, see [10]) - does not includes all nested fractals (for example, the snowflake or the pentagasket) and includes fractals which are not nested fractals. The Sierpinski gasket and many variants of it are the most significant examples of fractals belonging to this class. We recall that the Sierpinski gasket is the invariant set uniquely determined by the family of contractive homotheties $\{\psi_1, \psi_2, \psi_3\}$ with contraction factor $1/2$ and fixed points $\{q_1, q_2, q_3\}$ (vertices of an equilateral triangle T) respectively, that is,

$$\psi_1(z) = \frac{1}{2}(z - q_1) + q_1 \ , \ \psi_2(z) = \frac{1}{2}(z - q_2) + q_2 \ , \ \psi_3(z) = \frac{1}{2}(z - q_3) + q_3$$

We remark that the proof of the existence of the nonlinear energy forms on this class of fractals is not constructive and it does not give us the exact representation

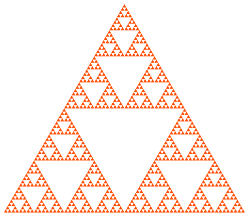

FIGURE 2. The Sierpinski gasket

of these forms. However, it is possible to have some estimates which are enough for obtaining Harnack type inequalities (see the following section).

Recently, nonlinear energy forms have been constructed on *products of Koch type fractals* in [7]. A first example is the so-called Koch-roof, that is, the cartesian product of the usual Koch curve $K^{(3)}$ in $\mathbb{R}^2$ and the unit interval $I = [0, 1]$ in $\mathbb{R}$.

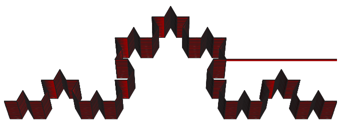

FIGURE 3. The Koch-roof.

Generalizing the previous example, nonlinear energy forms and related p-Lagrangians have been constructed on more general fractal structures by founding on the products of two or more Koch type fractals.

We remark that the construction of nonlinear energy forms on other types of fractals is still an open problem: hence, the following theory applies to all the available examples here.

4. Harnack type inequalities

In this Section we investigate the validity of Harnack type inequalities inequalities for p-Laplacians associated to p-Lagrangians on fractals. A fondamental tool in the proofs of Harnack type inequalities is the construction of suitable cut-off functions with bounded gradient, which, in turn, relies on the absolute continuity of the Lagrangian with respect to the underlying volume measure. It is worthy to point out that the p-Lagrangians are absolute continuous with respect to the underlying volume measure only in the case of Koch curve type fractals and of products of them: therefore, Harnack type inequalities for positive harmonics relative to the nonlinear energy forms hold by using the results of [2] (see [7] also).

In all the other cases, the Lagrangians are tipically singular with respect to the underlying volume measure. We overcome this difficulty by using the approach of "metric fractals" introduced by Mosco in [13]. We now recall the definition.

Definition 4.1. A *metric fractal* is a quadruple $X \equiv (X, d, \mu, \mathcal{L}^{(p)})$ on a connected topological space X, with the following properties MF1,...,MF4:

MF1) d is a quasi-distance on X (that is, d is a function on $X \times X$ with the properties of a usual distance, except for the triangle inequality satisfied in the form $d(x, y) \leq c_T(d(x, z) + d(z, y))$ with a fixed constant $c_T \geq 1$) and $X = (X, d)$ is a complete quasi-metric space;

MF2) μ is a doubling measure on X, with fixed constants $\nu > 0$, $c_G > 0$ and $R_0 \in (0, \infty]$ (that is, it is a positive Borel measure μ supported on X, such that there exist constants $\nu > 0$, $c_G > 0$ and $R_0 \in (0, \infty]$, with

$$0 < \mu(B_R) \leq c_G \left(\frac{R}{r}\right)^{\nu} \mu(B_r) < \infty \tag{4.1}$$

for every $x \in X$, $B_R \equiv B(x, R) := \{y \in X : d(x, y) < R\} \subset\subset X$, $0 < r \leq R < R_0$;

MF3) $\mathcal{L}^{(p)}$ is a measure-valued homogeneous p-Lagrangian according the Definition 2.1;

MF4) d, μ and $\mathcal{L}^{(p)}$ are related by the inequalities

$$\frac{1}{\mu(B_R)} \int_{B_R} |u - u_{B_R}| \, d\mu \leq c_P R \left(\frac{1}{\mu(B_{qR})} \int_{B_{qR}} d\mathcal{L}^{(p)}(u, u) \right)^{1/p} , \tag{4.2}$$

(where, for every ball B, $u_B = \mu(B)^{-1} \int_B u \, d\mu$), and

$$p\text{-cap}(B_R, B_{2qR}) \leq c_C \, \frac{\mu(B_R)}{R^p} , \tag{4.3}$$

for every $0 < R < R_0$ and every $B_R \subset B_{2qR} \subset\subset X$, with constants c_P, c_C, $q \geq 1$ independent of u, x and R.

Above,

$$p\text{-cap}(B_R, B_{2qR}) :=$$

$$:= \inf \left\{ \int_X d\mathcal{L}^{(p)}(\Phi, \Phi) \ : \ \Phi \in \mathcal{C}^{(p)} \ , \ \Phi \geq 1 \ \text{ on } \ B_R \ , \ \operatorname{supp} \Phi \subset B_{2qR} \right\} .$$

The constants c_T, ν, c_G, c_P, c_C and q will be referred to, in the following, as the *structural constants* of X. To sum up a "metric fractal" is a connected topological space X, equipped with a quasimetric d (for which X is complete), a doubling measure μ and a measure-valued p-Lagrangian $\mathcal{L}^{(p)}$; moreover, d, μ, $\mathcal{L}^{(p)}$ are related by scaled Poincaré inequalities and by scaled estimates of the p-capacity on d-balls.

By using Morrey type estimates (obtained in [4]), we have proved the Harnack inequality for non negative local supersolutions of p-Laplacians - associated to p-Lagrangians - on metric fractals whose homogeneous dimension ν (defined in (4.1)) is less than p (see [6]; in the linear case, see [15] and [16]).

We begin by recalling the definition of local supersolution.

Definition 4.2. A *local supersolution* u in X is a function $u \in \mathcal{D}^{(p)}_{loc}$, such that

$$\int_X d\mathcal{L}^{(p)}(u,\phi) \geq 0$$

for every nonnegative $\phi \in \mathcal{C}^{(p)}$ with compact support in X. Here $\mathcal{D}^{(p)}_{loc}$ denotes the space of the functions $u \in L^p(X,\mu)$, such that, for every relatively compact open subset A, there exists a function $\tilde{u} \in \mathcal{D}^{(p)}$ such that $u = \tilde{u}$ on A.

Theorem 4.3. *Let X be a metric fractal, with $\nu < p$. Let $u \in \mathcal{D}^{(p)}_{loc}$ be a non-negative local supersolution in X. Then, there exists a constant c, depending only on the structural constants of X, such that*

$$\sup_{B_R} u \leq c \inf_{B_R} u$$

for every ball $B_{R/\sigma} \subset B_{2qR/\sigma} \subset\subset X$, $0 < 4c_T(1+c_T^2)R < R_0$.

Self-similar fractals with self-similar p-Lagrangians provide interesting examples of metric fractals - possibly with singular Lagrangian (see [13] and [14], for various “classical” and “semi-classical” examples of metric fractals).

More precisely, let be K a self-similar set (that is, the invariant set of a given family $\Psi = \{\psi_1, \ldots, \psi_N\}$ satisfying (3.1), (3.3), (3.4) and (3.5) with the invariant measure μ (3.2)) with a self-similar p-Lagrangian with domain $\mathcal{D}^{(p)}$ in $L^p(K,\mu)$, that is, for every $u \in \mathcal{D}^{(p)}$ and for every $\varphi \in C(K)$, we have

$$\int_K \varphi \, d\mathcal{L}^{(p)}(u,u) = \sum_{i=1}^N \rho_i^{(p)} \int_K \varphi \circ \psi_i d\mathcal{L}^{(p)}(u \circ \psi_i, u \circ \psi_i) \tag{4.4}$$

with the real constants $\rho_i^{(p)} > 0$, $i = 1, \ldots, N$, satisfying $\rho_i^{(p)} = \mu(K_i)^\tau$, $i = 1, \ldots, N$, for some real constant $\tau < 1$, independent of $i = 1, \ldots, N$.

Given a variational fractal $K \equiv (K, \mu, \mathcal{L}^{(p)})$, we consider quasi-distances d on K with Euclidean scaling

$$d(x,y) = |x-y|^\delta \ , \ \ x, y \in K \tag{4.5}$$

indexed by a real parameter $\delta > 0$.

Following [12], we choose d by requiring d^p to obey on K the same scaling as $\mathcal{L}^{(p)}$ itself: more precisely, in [5], it has been proved that there exists one and only one constant $\delta > 0$, such that, the following relation holds

$$d^p(x,y) = \sum_{i=1}^N \rho_i^{(p)} d^p(\psi_i(x), \psi_i(y)) \ , \tag{4.6}$$

for every $x, y \in K$ and such δ is uniquely determined by the identity

$$\sum_{i=1}^N \rho_i^{(p)} \alpha_i^{-p\delta} = 1 \tag{4.7}$$

and is given by

$$\delta = \frac{d_f(1-\tau)}{p} \ . \tag{4.8}$$

We will call d the *intrinsic metric*.

In [6], for a self-similar fractal with a self-similar Lagrangian endowed with its intrinsic metric d, we have proved that if a global Poincaré inequality and a global estimate of capacity hold, then also the scaled Poincaré inequalities and the capacity inequalities on the homogeneous (intrinsic) balls in MF4 hold. More precisely, we have proved the following Theorem in [6].

Theorem 4.4. *Let* $K \equiv (K, \mu, \mathcal{L}^{(p)})$ *be a self-similar fractal with a self-similar Lagrangian endowed with its intrinsic metric* d*. Moreover, we suppose that there exists two constants* c_Π *and* c_Γ *such that*

$$\int_K |u - u(z)|^p \, d\mu \le c_\Pi \int_{K-\Gamma} d\mathcal{L}^{(p)}(u, u) \tag{4.9}$$

for every $u \in \mathcal{D}^{(p)}$ *and every* $z \in \Gamma$ *and*

$$p\text{-cap}(\Gamma_1, \Gamma_2) \le c_\Gamma \tag{4.10}$$

for all $\Gamma_1 \neq \emptyset$ *and* $\Gamma_2 \neq \emptyset$ *such that* $\Gamma_1 \cup \Gamma_2 = \Gamma$*. Then,* $(K, d, \mu, \mathcal{L}^{(p)})$ *is a metric fractal according to the Definition 4.1.*

By using the previous Theorems, we obtain that Harnack type inequalities hold on the class of weakly completely symmetric fractals (where self-similar p-Lagrangians are tipically singular with respect to the volume measure). In fact, by Theorem 4.4, the weakly completely symmetric fractals are metric fractals since global inequalities (4.9) and (4.10) hold (see [6]). Moreover, their homogeneous dimension ν is less than p : indeed, in Theorem 5.9 in [8], it is proved that $r_p < 1$ and so the renormalization factors $\rho_i^{(p)} = r_p^{-1} > 1$ and $\tau < 0$ in (4.4), therefore,

$$\nu = \frac{d_f}{\delta} = \frac{p}{1-\tau} < p \, ,$$

where, we have taken into account the expression (4.8) of δ. In conclusion, the weakly completely symmetric fractals are metric fractals with homogeneous dimension $\nu < p$: by Theorem 4.3, Harnack type inequalities hold.

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