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Uniform estimates of fundamental solution and regularity of vanishing viscosity solutions of mean curvature equations in H^n

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Abstract[1]. In this paper we are concerned with a family L_ε of elliptic operators in non divergence form, which approximate an hypoelliptic one. It is well known that L_ε admits a fundamental solution Γ_ε. Here we establish some a priori estimates uniform in ε of it, using a modification of the lifting technique of Rothschild and Stein. These estimates can be used in particular while studying regularity of viscosity solutions of mean curvature equation in the Heisenberg group.

1. Introduction

Let $X_1, \cdots, X_m$ be smooth real vector fields on an open set $\Omega \subset \mathbb{R}^N$ satisfying the Hörmander condition for the hypoellipticity ([38]), and linearly independent at every point. Operators represented as a sum of squares of these types of vector fields have been intensively studied after the classical works of Folland [28], Roschild and Stein [52], Nagel Stein Wainger [49]. After that also divergence and non divergence form operators have been diffusively studied: Harnack's inequality, regularity results for solutions, and estimates of the fundamental solutions have been established. We refer the reader to [9], [10], [11], [19], [43], [44], [45], for some results concerning divergence form operators and [7], [8], [61],[12] for divergence form operators. In particular estimates of the fundamental solution have been proved in [4], [5], [6]. We select an uniformly elliptic matrix $(a_{ij})_{ij=1\cdots m}$ with ellipticity coefficients $\lambda\ \Lambda > 0$, such that

$$\lambda|\xi|^2 \leq a_{ij} \leq \Lambda|\xi|^2 \ ,$$

then an uniformly subelliptic operator L is the operator formally represented as:

$$L = \sum_{ij=1}^{m} a_{ij} X_i X_j \ . \tag{1.1}$$

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It can be studied in terms of its Riemannian approximation. Indeed for every fixed point x_0 we can select, from the family of the commutators of the given vectors, $N-m$ vector fields numbered $X_{m+1},\cdots,X_N$ such that

$$X_1,\cdots,\ X_m,\ X_{m+1},\cdots,\ X_N \tag{1.2}$$

span the tangent space at x for every x in a neighborhood of x_0. We will call Riemannian approximation of the basis $X_1,\cdots,X_m$ the basis

$$X_1^\varepsilon = X_1,\cdots,\ X_m^\varepsilon = X_m,\ X_{m+1}^\varepsilon = \varepsilon X_{m+1},\cdots,\ X_N^\varepsilon = \varepsilon X_N\ , \tag{1.3}$$

since, when ε goes to 0 we recover the previous basis. Now we complete the matrix $(a_{ij})_{ij=1\cdots m}$ to a new matrix $(a_{ij})_{ij=1\cdots N}$, bounded by the same ellipticity coefficients as the previous one. Then the operator

$$L_\varepsilon = \sum_{ij=1}^{N} a_{ij} X_i^\varepsilon X_j^\varepsilon \tag{1.4}$$

is called elliptic regularization of the operator in (1.1). It is an uniformly elliptic operator in Ω, which formally converges to L, as ε goes to 0. We note that the natural distance associated to the elliptic operator L_ε is not the standard Euclidean one, but for every ε it is the control distance d_ε associated to the vector fields $X_1,\cdots,X_m,\ \varepsilon X_{m+1},\cdots,\varepsilon X_N$. Here $B_\varepsilon(x,r)$ will denote the sphere of the metric d_ε.

With these notations it is well known that the fundamental solution of the operator L_ε can be locally estimated in terms of d_ε as

$$|\Gamma_\varepsilon(x,y)| \le C_\varepsilon\ \frac{d_\varepsilon^2(x,y)}{|B_\varepsilon(x,d_\varepsilon(x,y))|}\ ,$$

for every (x,y) in a neighborhood of 0, and for a suitable constant C_ε (see [52], [49]). It is also well known that the control distance d_ε converges as ε goes to 0 to the control distance associated to the vector fields $X_1,\cdots,X_m$ (see [52], [49]) However the dependence of C_ε on the variable $\varepsilon>0$ is completely unknown, at our knowledge.

Hence we will afford this problem in this paper (and in grater detail in [17]) and we will prove that

Theorem 1.1. *For every compact set $K\subset\Omega$ and $p\in\mathbb{N}$ there exist positive constants C, C_p independent of ε such that*

$$|X_{i_1}^\varepsilon\cdots X_{i_p}^\varepsilon\Gamma_\varepsilon(x,y)| \le C_p\ \frac{d_\varepsilon^{2-p}(x,y)}{|B_\varepsilon(x,d_\varepsilon(x,y))|}\quad,\quad i_1,\cdots,i_p\in\{1,\cdots,N\}\ , \tag{1.5}$$

for every $x,y\in K$ with $x\neq y$. If $p=0$ we mean that no derivative are applied on Γ_ε.

1.1. **Applications to the mean curvature equation in the Heisenberg group.** Since the constants in Theorem 1.1 are independent of ε, then this result can play a crucial role in studying the properties of linear and non linear Hörmander type operators. In particular this approximation can be used to study interior regularity of viscosity solutions of nonlinear problems, when the vector fields $(X_i)_{i=1,\cdots,m}$ depend on the solution: $X_i=X_i(u)$. Indeed there is a large literature regarding the existence of solutions of non linear elliptic problems, but nothing is known for non linear sub-elliptic ones. One way is the finding solutions for the elliptic approximation, and then passing to the limit as ε goes to 0. This technique has been used for example in [20], [21], for different problems, arising in Complex Geometry or mathematical finance. In [15], [16], it was introduced a very general method for studying the high regularity of solutions of this type of nonlinear equation. The non linear operator is approximated with a linear one, uniform estimates are provided for the fundamental solution of the linear equation, and finally a parametrix method provides the regularity of the solution of the non linear equation. In particular uniform estimates of linear equations play a crucial role for studying these type of non linear problems.

Also the equation which describes the mean curvature of a graph in the Heisemberg group is of this type. We recall that H^n is a Lie group whose underlying manifold has dimension N. Hence, if we denote $(x_0, x_1, \cdots, x_{2n})$ the coordinates of the space, the horizontal basis for left invariant vector fields can be represented as

$$X_i = \partial_{x_i} \quad , \quad X_{i+1} = \partial_{x_{i+1}} + x_i \partial_{x_{2n}} \qquad \text{for } 0 \le i \le 2n-2 \text{ even} . \tag{1.6}$$

The notion of intrinsic regular surface has been studied in [30], [18]. It is the graph of a function $u : \mathbb{R}^{2n} \to \mathbb{R}$, and can represented as

$$\{(x_0, x_1, \cdots, x_{2n}) \ : \ x_0 = u(x_1, \cdots, x_{2n})\} .$$

Moreover the function u is regular with respect to the projection on its domain of the vector fields in (1.6) (see [18] and [1]). Since X_0 has null projection of the domain of u, the regularity of this function will be described in terms of the vector fields:

$$X_1 = \partial_{x_1} + u(x)\partial_{x_{2n}} \quad , \quad X_i = \partial_{x_i} , \tag{1.7}$$

$$X_{i+1} = \partial_{x_{i+1}} + x_i \partial_{x_{2n}} \qquad \text{for } 2 \le i \le 2n-2 \text{ even} .$$

In particular X_1 is a non linear vector field, since it depends on u. However, if the dimension n is strictly bigger than 1, we can compute $[X_2, X_3] = \partial_{2n}$. So that, independently of the regularity of u, then the vector fields $X_1, \cdots, X_{2n-1}$ span the whole space at every point, so that they satisfy a non regular Hörmander type condition. In particular if we choose $m = 2n-1$ and $N = 2n$, and we are in the preceding setting. We will call intrinsic gradient of u the vector

$$\nabla u = (X_1 u \cdots X_m u)$$

The notion of mean curvature has been recently established as the first variation of the area formula by [25], [3], [51], [40], [47], [57] Properties of regular minimal surfaces have been investigated in [34], [50], [23], [22], [33], [26], [2] and [48]). However the problem of regularity is still open. In case of intrinsic graphs in H^n, with $n > 1$ its expression reduces to the following one:

$$L_u = \sum_{i,j=1}^{m} a_{ij} X_i X_j u = 0 \quad , \quad \text{and} \qquad a_{ij} = \delta_{ij} - \frac{X_i X_j}{\sqrt{1 + |\nabla u|^2}} \ . \tag{1.8}$$

The approximating Riemannian problems express the mean curvature in a Riemannian metrics, which approximates the subriemannian Heisenberg one (see [] for the relation between Riemannian and subriemannian curvature). It is defined completing the family X_i of vector fields with $X_N^\varepsilon = \varepsilon\partial_{2n}$. For every ε the new basis will be dependent of on ε, and on the solution so that it is expressed as

$$X_i^\varepsilon = X_i \quad \text{for } i \le m \qquad X_N^\varepsilon = \varepsilon\partial_{2n} \;, \tag{1.9}$$

and the associated gradient will be

$$\nabla_\varepsilon = (X_1^\varepsilon, \cdots, X_N^\varepsilon)\;.$$

Consequently the mean curvature equation has the same expression as (1.8), but depends on ε, and all the N vector fields:

$$L_\varepsilon = \sum_{i,j=1}^N a_{ij}^\varepsilon X_i^\varepsilon X_j^\varepsilon \quad , \quad \text{and} \quad a_{ij}^\varepsilon = \delta_{ij} - \frac{X_i^\varepsilon u\, X_j^\varepsilon u}{\sqrt{1+|\nabla_\varepsilon u|^2}} \;. \tag{1.10}$$

Calling C_E^1 the standard Euclidean C^1 norm, we will say that an Euclidean Lipschitz continuous function u is a vanishing viscosity solution of (1.8) in an open set Ω, if there exists a sequence u_ε of solutions of (1.10) in Ω such that for every compact set $K \subset \Omega$

- $||u_\varepsilon||_{C_E^1(K)} \le C$
- $u_\varepsilon \to u$ as $\varepsilon \to 0$ punctually a.e. in Ω.

The existence of viscosity solutions of the main curvature flow has been proved by [13]. Its stationary solutions are indeed Lipschitz continuous viscosity solutions in the previous sense.

With this definition we will prove the following (see also [14])

Theorem 1.2. *The Lipschitz continuous viscosity solutions of* (1.8) *are* C^∞.

2. Proof of Theorem 1.1

In this section we sketch the proof of uniform estimates for the fundamental solution of L_ε in (1.4). By simplicity we provide here the proof under the additional assumption that the vector fields $X_1, \cdots, X_m$ are defined on R^N, generate a free and nilpotent Lie algebra and that for every $i = 1, \cdots, m$ each X_i agrees at origin with ∂_{x_i} (for the complete proof we refer to [18]). In this case the collection

$$X_{m+1}, \cdots,\; X_N \tag{2.1}$$

of the commutators complete the collection $X_1, \cdots, X_m$ to a basis of R^N. We will call s the step of the Lie algebra.

2.1. **The lifting procedure.** In order to prove Theorem 1.1 we first note that the notion of degree of each vector field is not well defined for the approximating operator 1.4. Indeed at every point the vector fields $(X_i)_{i=m+1,\cdots,N}$, are the complete list of the commutators, so that it would be natural to associate to them a degree grater that 1, but multiplying them by ϵ, we get the vector fields defining L_ε, which could be considered of degree 1. In order to overcome this difficulty we will introduce a new lifting method, since the classical lifting procedure of Rothshild and Stein [52] can not be directly applied here. In our procedure the vector fields $(\varepsilon X_i)_{i=m+1,\cdots,N}$ will be lifted to new vector fields linearly independent of the commutators. In order to do so, we define $N-m$ new vector fields free and nilpotent of step s, in term of completely new variables. These vector fields will be denoted

$$\widetilde{X}_{m+1}, \cdots,\; \widetilde{X}_N \qquad \text{in } \mathbb{R}^{\widetilde{N}} \;, \tag{2.2}$$

and the dimension of the new lifted space will be denoted by $\widetilde{N}$. In this way the new vector fields have the same step of the staring ones and

$$X_1, \cdots, X_m, \cdots, \widetilde{X}_{m+1} + \varepsilon X_{m+1}, \cdots, \widetilde{X}_N + \varepsilon X_N$$

are linearly independent from their commutators. We call this list of vectors $(\widetilde{X}_i^\varepsilon)_{i=1,\cdots,N}$ and define a lifted regularized operator as

$$\widetilde{L}_\varepsilon = \sum_{i=1}^{N} a_{ij} \widetilde{X}_i^\varepsilon \widetilde{X}_j^\varepsilon \qquad \text{in } \mathbb{R}^{\widetilde{N}} .$$

2.2. **Uniform estimates of the lifted operator.** We will need a third operator $\widetilde{L}$, with the same structure of $\widetilde{L}_\varepsilon$, but independent of ε. We simply eliminate the dependence on ε in the vector defining $\widetilde{L}_\varepsilon$ and choose the new family as

$$\widetilde{X}_1 = X_1, \cdots, \widetilde{X}_m = X_m, \cdots, \widetilde{X}_{m+1}, \cdots, \widetilde{X}_N .$$

Then we define

$$\widetilde{L} = \sum_{i=1}^{N} a_{ij} \widetilde{X}_i \widetilde{X}_j \qquad \text{in } \mathbb{R}^{\widetilde{N}} .$$

Since the operator $\widetilde{L}$ is independent of ε, its fundamental solution $\widetilde{\Gamma}$ is obviously estimated independently of ε, in terms of the control distance $\widetilde{d}$ associated to the vector fields $\widetilde{X}_i$. Precisely the following proposition holds (see [6]):

Proposition 2.1. *For every compact set $K \subset \Omega$ and $p \in \mathbb{N}$ there exists a positive constant C_p such that*

$$|\widetilde{X}_{i_1} \cdots \widetilde{X}_{i_p} \widetilde{\Gamma}(\widetilde{x}, \widetilde{y})| \le C_p \, \widetilde{d}(\widetilde{x}, \widetilde{y})^{2-\widetilde{Q}-p} \quad , \quad \textit{with } i_1, \cdots, i_p \in \{1, \cdots, N\} \tag{2.3}$$

for every $\widetilde{x}, \widetilde{y} \in K$ with $\widetilde{x} \neq \widetilde{y}$. Here $\widetilde{Q}$ denotes the local homogeneous dimension of the space, with respect to the vectors $(\widetilde{X}_i)_{i=1,\cdots,N}$.

We will see that the estimates of the fundamental solution of $\widetilde{L}$ provide us with estimates of the fundamental solution of $\widetilde{L}_\varepsilon$ independent of ε. Indeed it is possible to define a Lie algebra isomorphism ψ_ε between the vector fields defining $\widetilde{L}_\varepsilon$ and $\widetilde{L}$:

$$\psi_\varepsilon(\widetilde{X}_i) = \widetilde{X}_i^\varepsilon \quad , \quad i = 1, \cdots, N .$$

The function ψ_ε induces, via the exponential mapping a change of variables on $\mathbb{R}^{\widetilde{N}}$

$$\Phi_\varepsilon : \mathbb{R}^{\widetilde{N}} \to \mathbb{R}^{\widetilde{N}} \quad , \quad \Phi_\varepsilon = Exp \circ \psi_\varepsilon \circ \Theta_0 .$$

Then if $\widetilde{d}$ is the distance associated of $\widetilde{L}$ and $\widetilde{d}_\varepsilon$ the distance associated to $\widetilde{L}_\varepsilon$ then the follow relation holds:

$$\widetilde{d}_\varepsilon(\widetilde{x}, \widetilde{y}) = \widetilde{d}(\Phi_\varepsilon(\widetilde{x}), \Phi_\varepsilon(\widetilde{y})) .$$

Analogously the fundamental solution $\widetilde{\Gamma}_\varepsilon$ of the operator $\widetilde{L}_\varepsilon$ can be represented in terms of the fundamental solution $\widetilde{\Gamma}$ of the operator $\widetilde{L}$

$$\widetilde{\Gamma}_\varepsilon(\widetilde{x}, \widetilde{y}) = \widetilde{\Gamma}(\Phi_\varepsilon(\widetilde{x}), \Phi_\varepsilon(\widetilde{y})) .$$

Besides

$$\widetilde{X}_i^\varepsilon \widetilde{\Gamma}_\varepsilon(\widetilde{x}, \widetilde{y}) = (\widetilde{X}_i \widetilde{\Gamma})(\Phi_\varepsilon(\widetilde{x}), \Phi_\varepsilon(\widetilde{y})) .$$

Hence $\widetilde{\Gamma}_\varepsilon$ satisfies the estimates (1.5) with the same constants as $\widetilde{L}$. Precisely

Proposition 2.2. *For every compact set $K \subset \Omega$ and $p \in \mathbb{N}$ there exists a positive constant C_p such that for every $i_1 \cdots, i_p \in \{1, \cdots, N\}$, one has*

$$|\widetilde{X}^{\varepsilon}_{i_1} \cdots \widetilde{X}^{\varepsilon}_{i_p} \widetilde{\Gamma}_{\varepsilon}(\widetilde{x}, \widetilde{y})| \leq C_p\, \widetilde{d}(\Phi_{\varepsilon}(\widetilde{x}), \Phi_{\varepsilon}(\widetilde{x}))^{2-Q-p} = C_p(\widetilde{d}_{\varepsilon}(\widetilde{x}, \widetilde{y})^{2-Q-p} \tag{2.4}$$

for every $\widetilde{x}, \widetilde{y} \in K$ with $\widetilde{x} \neq \widetilde{y}$.

In order to simplify notations we will give the following definition

Definition 2.3. We will say that k is a regular kernel of type λ with constants C_p with respect to the vectors $X_1, \cdots, X_\nu$, and the distance d in an open set W and we will denote $k \in F_\lambda(X, d, W)$ if for every $x, y \in W$ with $x \neq y$

$$|X_{i_1} \cdots X_{i_p} k(x, y)| \leq C_p \, \frac{d^{\lambda - p}(x, y)}{|B(x, d(x, y))|} \quad , \quad i_1, \cdots, i_p \in \{1, \cdots, \nu\} \, . \tag{2.5}$$

2.3. **Projection and uniform estimates of the operator L_ε.** with a generalization of the projection-result of Nagel Stein Wainger [49] we will deduce the uniform estimates of the fundamental solution of L_ε from the uniform estimates of $\widetilde{L}_\varepsilon$.

The generic point of the lifted space $\mathbb{R}^{\widetilde{N}}$ will be $\widetilde{x} = (x, \hat{x})$, where $x \in \mathbb{R}^N$ denotes the initial variables, $\hat{x} \in R^{\widetilde{N}-N}$ the variables added in the lifting procedure.

A parametrix of the fundamental solution of L_ε will be defined as the restriction on $\mathbb{R}^N$ of the fundamental solution $\widetilde{\Gamma}_\varepsilon$. Following [52], we define restriction operator R an operator which maps kernels in $\widetilde{\Omega}$ to kernels on Ω as follows:

$$Rf(y) = \int f(y, \hat{y}) a(\hat{y}) \, d\hat{y} \tag{2.6}$$

where a is any function of class C^∞ with compact support and integral equal to 1.

With this definition there is a natural relation between kernels of type $F_\lambda(\widetilde{X}_\varepsilon, \widetilde{d}_\varepsilon, U)$, in $R^{\widetilde{N}}$ and their restriction on $\mathbb{R}^N$, which belongs to $F_\lambda(X_\varepsilon, d_\varepsilon, W)$. Indeed

Proposition 2.4. *If $\widetilde{k}(\widetilde{x}, \widetilde{y})$ is a kernel of class $F_\lambda(\widetilde{X}_\varepsilon, \widetilde{d}_\varepsilon, U)$, then $k(x, y) = R\widetilde{k}(\widetilde{x}, \widetilde{y})$ is a kernel of class $F_\lambda(X_\varepsilon, d_\varepsilon, W)$, which satisfies inequalities* (2.5) *with the same constants as k.*

2.4. **Idea of the proof of Theorem 1.1.** The idea of the proof of this theorem is based on the parametrix method. It is contained in [17], and it is similar to Theorem 2 in [53]. Indeed by Propositions 2.2 and 2.4, the kernel

$$K_{\varepsilon,0}(x, y) = R(\widetilde{\Gamma}_\varepsilon(\widetilde{x}, \widetilde{y}))$$

satisfies (2.5) with constants independent of ε. Then it is possible to modify this kernel with the parametrix method, and to built, for every p a new kernel $K_{\varepsilon,p} \in F_p(X_\varepsilon, d_\varepsilon, W)$ satisfying (2.5), with constants independent of ε such that

$$L^y_\varepsilon(K_{\varepsilon,p}(x, y)) = a(x)\delta_y(x) + H_{\varepsilon,p}(x, y) \tag{2.7}$$

with δ_y the Dirac distribution at y, and $H_{\varepsilon,p} \in F_p(X_\varepsilon, d_\varepsilon, W)$. For p arbitrarily large, the function $H_{\varepsilon,p}$ is a kernel arbitrary regular. Hence $aK_{\varepsilon,p}(x, y))$ differs from the fundamental solution by a regular kernel. So that it provides the required estimate of the fundamental solution.

3. Regularity of viscosity solutions of mean curvature equation

We prove in this section that a viscosity solution u of the mean curvature equation in H^n, with $n > 1$ are smooth, using a Caccioppoli type technique and the previously established estimates of the fundamental solution.

By definition the function u is approximated on the set Ω by a family of smooth function u_ε, solution of (1.10). Then the non linear vector fields $(X_i^\varepsilon)_{i=1,\cdots,N}$, defined as in (1.9) in terms of u_ε are well defined and smooth. Recall that the associated gradient is ∇_ε. The vector of all intrinsic second order derivatives $X_i^\varepsilon X_j^\varepsilon$ will be denoted ∇_ε^2, and the vector of all the intrinsic derivatives of order j will be ∇_ε^j. Will also denote $\nabla = (X_1, \cdots, X_m)$, the part of the gradient dependent only on the first m vectors, (which are independent of ε), and ∇^j the vector of all the associated derivatives of order j. Finally ∇_E^j will be the vector of all Euclidean derivatives of order j.

Proposition 3.1 (**Caccioppoli type inequality).** *Let us assume that u_ε is a C^∞ function, classical solution of $L_\varepsilon u_\varepsilon = 0$. Let us also assume that there exists a compact set $K_0 \subset \Omega$ and a constant C_0 such that*

$$||u||_{C^1_E(K_0)} \leq C_0 \ .$$

Then there exists a constant C such that for every $\varphi \in C_0^\infty$ ones has

$$\int \left(|\partial_{2n} \nabla_\varepsilon u_\varepsilon|^2 + |\nabla_\varepsilon^2 u_\varepsilon|^2\right) \varphi^2 \leq$$

$$\leq C \int \left(|\partial_{2n} u_\varepsilon|^2 + (\nabla_\varepsilon u_\varepsilon)^2\right) \left(\varphi^2 + |\nabla_\varepsilon \varphi|^2 + \varphi |\partial_{2n}\varphi|\right) \, dx \ .$$

From the previous Caccioppoli type estimate for the gradient, and an intrinsic Sobolev inequality [46], it is possible to deduce the Hölder continuity of the gradient. Indeed, the vector fields X_i^ε being smooth, for every ε is defined a control distance d_ε, and intrinsic Hölder continuity classes $C_\varepsilon^{1,\alpha}$.

Proposition 3.2. *Let us assume that u_ε is a C^∞ function, classical solution of $L_\varepsilon u_\varepsilon = 0$. Let us also assume that there exists a compact set K and a constant C_0 such that*

$$||u_\varepsilon||_{C^1_E(K)} \leq C_0 \ ,$$

then for every compact set $K_0 \subset\subset K$ there exists a constant C and a constant $\alpha < 1$ such that

$$\|u_\varepsilon\|_{C_\varepsilon^{1,\alpha}(K)} + \|\nabla_\varepsilon^2 u_\varepsilon\|_{L^2(K)} \leq C \ . \tag{3.1}$$

Finally we use a parametrix method, and prove that the solution is C^∞:

Theorem 3.3. *Let us assume that u_ε is a classical solution of $L_\varepsilon u_\varepsilon = 0$. Let us also assume that condition* (3.1) *is satisfied on a compact set $K \subset \Omega$. Then, for any compact set $K_0 \subset\subset K$, for every $j \in N$ and $\alpha < 1$, there exists a constant $C > 0$ depending only on C_0, K, j, α such that*

$$||u_\varepsilon||_{C_{\bar{u}}^{j,\alpha}} \leq C \ .$$

Here we give a short sketch of the proof of the last theorem, which is based on the estimates proved in the first section. We use a technique similar to the one in [21].

For a fixed solution u_ε, the operator L_ε in (1.8) is a linear operator, and we could directly apply to it the estimates of the previous section. However, the vector fields X_1^ε explicitly depends on u_ε, we have estimates uniform in ε only on its $C_\varepsilon^{1,\alpha}-$norm. In order to obtain estimates uniform in ε also for all derivatives, we approximate X_1^ε with a new vector, only dependent on the first derivatives of u_ε. Precisely the intrinsic first order Taylor polynomial of u_ε with starting point $\bar{x}$ is

$$P^1_{\bar{x}} u_\varepsilon(x) = u(\bar{x}) + \sum_{i=1}^N X_i^\varepsilon u_\varepsilon(\bar{x})(x-\bar{x})_i$$

Correspondingly a first order approximation of X_1 is

$$X_{1\bar{x}}^\varepsilon = \partial_{x_1} + P^1_{\bar{x}} u_\varepsilon \partial_{x_{2n}} \ .$$

All the other vector, which have polynomial coefficients, remain unchanged

$$X_{i\bar{x}}^\varepsilon = X_i^\varepsilon \quad , \quad \text{for } 2 \le i \le N \ .$$

This choice allows to define frozen operator

$$L_{\varepsilon,\bar{x}} = \sum_{ij=1}^N a_{ij}^\varepsilon(\bar{x}) X_{i,\bar{x}}^\varepsilon X_{j,\bar{x}}^\varepsilon \ ,$$

which satisfies all the assumptions of Theorem 1.1.

3.1. **Regularity of the solution.** The operator $L_{\varepsilon,\bar{x}}$ is an uniformly subelliptic operator in divergence form, with C^∞ coefficients, and depending on ε. The existence of a fundamental solution and its estimates for uniformly subelliptic operator (independent of ε) has been proved in the previous section. by [5].We can represent a solution of equation (1.10) in terms of its fundamental solution

Proposition 3.4. *Let us assume that u_ε is a fixed solution of $L_\varepsilon u_\varepsilon = 0$ in Ω. Then for any $\varphi \in C_0^\infty(\Omega)$ the function $u_\varepsilon\varphi$ can be represented as*

$$(3.2) \qquad u_\varepsilon\varphi(x) = \int_\Omega \Gamma^\varepsilon_{\bar{x}}(x,y) N_1(x,y)\, dy + \int_\Omega \Gamma^\varepsilon_{\bar{x}}(x,y) L_\varepsilon u_\varepsilon(y)\, \varphi(y)\, dy +$$

$$+ \sum_{ij=1}^{2n} \int_\Omega \Gamma^\varepsilon_{\bar{x}}(x,y) \left(a_{ij}^\varepsilon(\bar{x}) - a_{ij}^\varepsilon(y)\right) X_i^\varepsilon X_j^\varepsilon u(y)\varphi(y)\, dy +$$

$$+ \sum_{i=1}^{2n} a_{i2n-1}^\varepsilon(\bar{x}) \int_\Omega \Gamma^\varepsilon_{\bar{x}}(x,y) \, (X_1^\varepsilon u_\varepsilon(y) - X_1^\varepsilon u_\varepsilon(\bar{x})) X_n^\varepsilon X_i^\varepsilon u_\varepsilon(y) -$$

$$-(X_n^\varepsilon u_\varepsilon(y) - X_n^\varepsilon u_\varepsilon(\bar{x})) X_1^\varepsilon X_i^\varepsilon u_\varepsilon(y))\, \varphi(y)\, dy +$$

$$+ \sum_{i=1}^{2n} a_{i\,2n-1}^\varepsilon(\bar{x}) \int_\Omega X_{i,\bar{x}}^\varepsilon \Gamma^\varepsilon_{\bar{x}}(x,y)(u_\varepsilon(y) - P^1_{\bar{x}} u_\varepsilon(y)) \partial_{2n} u(y)\varphi(y)\, dy +$$

$$+ \sum_{i=1}^{2n} a_{i\,2n-1}^\varepsilon(\bar{x}) \int_\Omega \big(X_{1,\bar{x}}^\varepsilon \Gamma^\varepsilon_{\bar{x}}(x,y)(u_\varepsilon(y) - P^1_{\bar{x}} u_\varepsilon(y)) X_n^\varepsilon X_i^\varepsilon u_\varepsilon(y) -$$

$$- X_{n,\bar{x}}^\varepsilon \Gamma^\varepsilon_{\bar{x}}(x,y)(u_\varepsilon(y) - P^1_{\bar{x}} u_\varepsilon(y)) X_1^\varepsilon X_i^\varepsilon u_\varepsilon(y)\big)\, \varphi(y)\, dy$$

where δ is the Kroeneker function and $N_1(\zeta, z_0)$ is a suitable kernel, depending on $\nabla_\varepsilon u_\varepsilon$ and on the derivatives of φ, of order at most 2.

Idea of the proof of Theorem 3.3. We can represent u_ε as in Proposition 3.4, and by the assumption (3.2) and the estimates of the fundamental solution $\Gamma^\varepsilon_{\bar{x}}$, we deduce that, for every $p \in \mathbb{N}$, for every K_1 such that $K \subset\subset K_1 \subset\subset K_0$ there exists a constant C_1 independent of ε such that

$$\|\nabla^2_\varepsilon u_\varepsilon\|_{L^p(K_1)} < C_1 \ .$$

This, estimates, together with the Sobolev embedding theorem proved in [46], implies that for every $\alpha < 1$, and for every K_2 such that $K \subset\subset K_2 \subset\subset K_1$ there exists a constant C_2 independent of ε such that

$$\|\nabla_\varepsilon u_\varepsilon\|_{C^{1,\alpha}(K_2)} < C_2 \ .$$

Applying again the representation formula in Proposition 3.4 we can now deduce that for very K_3 such that $K \subset\subset K_3 \subset\subset K_2$ there exists a constant C_3 independent of ε such that

$$\|\nabla^2_\varepsilon u_\varepsilon\|_{C^{2,\alpha}(K_3)} < C_2 \ .$$

In this way we get the thesis for $j = 2$. Since u_ε is smooth, it is possible to prove that its gradient is solution of the same differential equation, with a suitable second member. Hence the preceding argument can be iterated and proves the thesis.

Idea of the proof of Theorem 1.2. If u is a vanishing viscosity solution, there exists a family u_ε of smooth solutions of $L_\varepsilon u_\varepsilon = 0$ and for every compact set K_0 there exists a constant C_0 such that

$$\|u_\varepsilon\|_{C^1_E(K_0)} \le C_0 \ .$$

Then, by Propositions 3.1 and 3.2 and by Theorem 3.3 for every compact $K \subset\subset K_0$, for every j and for every α there exists a constant $C > 0$ independent of ε such that

$$\|u_\varepsilon\|_{C^{j,\alpha}(K)} < C \ .$$

Now we note that $X_1, \cdots, X_m$ satisfy the Hörmander condition for hypo-ellipticity with step 2. Hence, for every j the Euclidean derivatives of order j, ∇^j_E can be represented as a linear combination of intrinsic derivatives of order at most $2j$, and with coefficients independent of ε. Hence there exists a constant C_E independent of ε such that

$$\|u_\varepsilon\|_{C^j_E(K)} < C_E \sum_{i\le 2j} \|\nabla^i u_\varepsilon\| < C_E \sum_{i\le 2j} \|\nabla^i_\varepsilon u_\varepsilon\| \le C C_E$$

(recall that ∇ is expressed in terms of the first m derivatives, while ∇_ε in terms of all the derivatives from 1 to N). Letting ε going to 0 in this last inequality, we get the thesis.

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