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## Isoperimetric inequalities in the Heisenberg group and in the plane

Luca CAPOGNA

**Abstract**<sup>1</sup>. We formulate the isoperimetric problem for the class of  $C^2$  smooth cylindrically symmetric surfaces in the Heisenberg group in terms of Legendrian foliations. The known results for the sub-Riemannian isoperimetric problem yield a new isoperimetric inequality in the plane: For any strictly convex,  $C^2$  loop  $\gamma \in \mathbb{R}^2$ , bounding a planar region  $\omega$ , we have

$$I(\omega)^{3/4} \leq \frac{\sqrt{\pi}}{3^{1/4}(8\pi)^{3/4}} L_3,$$

where  $I(\omega) = \int_{\omega} |z|^2 dz$  is the moment of inertia and  $L_3$  is the length of the curve  $\gamma^3$ . Moreover if equality is achieved then  $\gamma$  is a circle.

### 1. INTRODUCTION

Any absolutely continuous planar curve  $\gamma = (\gamma_1, \gamma_2)$  can be lifted to a family of Legendrian curves  $(\gamma_1, \gamma_2, \gamma_3)$  in the Heisenberg group  $\mathbb{H}^1$  by setting  $(d/ds)\gamma_3 = 2(\gamma, i\gamma')$ . On the other hand, any Legendrian curve yields a unique planar curve by projection in the first two variables. If  $M \subset \mathbb{H}^1$  is a cylindrically symmetric smooth surface then it is foliated by rotations  $(e^{i\theta}\gamma, \gamma_3)$ , with  $\theta \in (0, 2\pi]$  of a Legendrian lift of a Jordan curve  $\gamma$ . In [8], Cheng, Hwang, Malchiodi and Yang have observed that the curvature of  $\gamma$  agrees with the horizontal mean curvature  $\mathcal{H}_0$  of the surface  $M$ . This basic link between the planar geometry of  $\gamma$  and the horizontal (sub-Riemannian) geometry of  $M$  allows to rephrase differential geometry questions for cylindrically symmetric surfaces in  $\mathbb{H}^1$  in terms of plane geometry.

In this note we provide two simple, related examples of this approach and study the *isoperimetric problem* and the *volume constrained horizontal mean curvature flow* in  $\mathbb{H}^1$  in terms of corresponding questions for their Legendrian foliations. The first problem was introduced by Pierre Pansu in [19] and [20], where he formulated a conjecture on the shape of the isoperimetric profile of  $\mathbb{H}^1$ . This conjecture is still open, although considerable progress has been achieved in recent years. The horizontal mean curvature flow arises as the  $L^2$  gradient flow of the sub-Riemannian perimeter and was first introduced in the literature by Citti and Sarti in [9], where

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<sup>1</sup>Author's address: L. Capogna, University of Arkansas, Department of Mathematics, Fayetteville, AR 72701, USA; e-mail: lcapogna@uark.edu .

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it appears as a model of amodal completion in the geometry of the first layer  $V^1$  of the visual cortex.

We find that the sharp isoperimetric inequality in  $\mathbb{H}^1$  recently established by Danielli, Garofalo, Nhieu [11] yields a new sharp isoperimetric type inequality for  $C^2$  convex loops in the plane. We also show that the corresponding gradient flow displays some monotonicity properties. Although these results are very simple and hardly surprising we feel that they represent an instance of the correlation between horizontal geometry of  $\mathbb{H}^1$  and planar geometry we alluded to earlier, and may yield some insight into the study of Pansu conjecture and the horizontal mean curvature flow.

For a more detailed discussion of the topics treated in this note we refer the reader to the results and the literature quoted in [9], [3], [4], and [6].

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## 2. SURFACES IN $\mathbb{H}^1$

Here we briefly recall the definition of the Lie groups  $\mathbb{H}^1$  and their sub-Riemannian structure. We refer the reader to [26] and [6] for a more detailed description.

**Algebraic structure** The underlying manifold of  $\mathbb{H}^1$  is  $\mathbb{C} \times \mathbb{R}$  and the main feature of its Lie algebra  $\mathfrak{h}$  is the stratification  $\mathfrak{h} = V^1 \oplus V^2$  where  $V^2$  is one dimensional and  $[V^1, V^1] = V^2$ ,  $[V^1, V^2] = 0$ . We identify  $V^1$  with  $\mathbb{C}$  and use *exponential coordinates*  $x = (x_1, x_2, x_3) = (z, x_3)$ , with  $z = x_1 + \mathbf{i}x_2 \in \mathbb{C}$ , and  $x_3 \in \mathbb{R}$ . The Campbell-Baker-Hausdorff formula yields the group law  $(z, x_3)(z', x'_3) = (z + z', x_3 + x'_3 - 2\text{Im}(zz'))$ . The group identity  $(0, 0)$  is denoted by  $0$ . An homogeneous structure is given by the non-isotropic dilations

$$\delta_s(x) = (sx_1, sx_2, s^2x_3).$$

The Haar measure in  $\mathbb{H}^1$  is the Lebesgue measure in  $\mathbb{R}^3$  and we denote by  $\text{Vol}(\Omega)$  the measure of any Borel set  $\Omega \subset \mathbb{H}^1$ . If  $x_1 + \mathbf{i}x_2, y_1 + \mathbf{i}y_2 \in \mathbb{C}$  we set  $(x_1 + \mathbf{i}x_2, y_1 + \mathbf{i}y_2) = x_1y_1 + x_2y_2$ . If  $\gamma = (\gamma_1, \gamma_2)$  is a planar curve then we let  $\gamma^2 = \gamma\gamma$  be the curve obtained through complex multiplication by setting  $\gamma = \gamma_1 + \mathbf{i}\gamma_2$ .

**Horizontal sub-bundle** Set  $X_1 = \partial_{x_3} - 2x_2\partial_{x_1} - 2x_1\partial_{x_2}$ , and  $X_2 = \partial_{x_2} + 2x_1\partial_{x_3}$ , to be a left-invariant basis for the non-integrable *horizontal* fibration  $H\mathbb{H}^1$  obtained by left-translating the first layer  $V^1$  of the Lie algebra stratification, i.e. its fibers are  $H(x) = xV^1$ . Note that  $H\mathbb{H}^1$  is a contact distribution, in fact

$$H(x) = \text{Ker}[dx_3 - 2(x_1dx_2 - x_2dx_1)].$$

Denote by  $\pi_H : H\mathbb{H}^1 \rightarrow \mathbb{R}^2$  the projection

$$\pi_H(x, v) := \pi_H(x, v_1X_1 + v_2X_2 + v_3X_3) = v_1X_1 + v_2X_2.$$

**Sub-Riemannian structure.** The sub-Riemannian structure of  $\mathbb{H}^1$  is given through a choice of left-invariant, positive definite quadratic form  $g_0(\cdot, \cdot)$  defined on

the horizontal bundle  $H\mathbb{H}^1$ , which allows to define the length of horizontal vectors  $v \in H(x)$ ,  $|v|_0 = \sqrt{\langle v, v \rangle_0} = \sqrt{g_0(x)(v, v)}$ . Without loss of generality we choose a sub-Riemannian metric  $g_0$  so that  $X_i$ 's form an orthonormal set. The general case can be recovered through a change of variables. The *Carnot-Carathéodory* (CC) metric on  $\mathbb{H}^1$  is defined as the shortest time it takes to go from two points, traveling at unit speed along horizontal paths: for  $x, y \in \mathbb{H}^1$  and  $\delta > 0$  let  $C(\delta) = \{\gamma : [0, \delta] \rightarrow \mathbb{H}^1 : \gamma(0) = x, \gamma(\delta) = y \text{ and } \gamma' = a_1 X_1 + a_2 X_2, \text{ with } |\gamma'|_0^2 = \sum_{i=1}^2 |a_i|^2 \leq 1\}$ . Then define

$$(2.1) \quad d(x, y) = \inf\{\delta : C(\delta) \neq \emptyset\}.$$

Next, we extend  $g_0$  to a left-invariant Riemannian metric  $g_1$  defined so that the two layers  $V^2$  and  $V^1$  of the Lie algebra  $\mathfrak{h}$  are orthogonal.

The *horizontal Levi Civita connection* is defined in the following way: Let  $\nabla^1$  be the Levi Civita connection of the metric  $g_1$  and for all horizontal sections  $V, W \in \Gamma(H\mathbb{H}^1)$  set  $\nabla_W^0 V = \pi_H(\nabla_W^1 V)$ . It is immediate to see (see [6]) that this definition is independent of the choice of the Riemannian extension  $g_1$  of the sub-Riemannian metric  $g_0$  as long as the horizontal layer is orthogonal to the rest of the stratification.

**Horizontal derivatives.** For any sufficiently smooth function  $\phi$  defined in an open set of  $\mathbb{H}^1$  we denote by  $\nabla_0 \phi = (X_1 \phi, X_2 \phi)$  its (*left-invariant*) *horizontal gradient*. The *horizontal Hessian* is given by the  $2 \times 2$  matrix  $(D_0^2 \phi)_{ij} = X_i X_j \phi$ . We denote by  $(D_0^2 \phi)^*$  its symmetrized form. The *sub-Laplacian* is a second-order, divergence form degenerate elliptic operator  $\mathcal{L}\phi = X_1^2 \phi + X_2^2 \phi$ . The *infinite sub-Laplacian* is  $\mathcal{L}_\infty \phi = \sum_{i,j=1}^2 X_i X_j \phi X_i \phi X_j \phi$ .

**Submanifolds** Let  $M \subset \mathbb{H}^1$  be a  $C^2$  surface. Set  $HTM = H\mathbb{H}^1 \cap TM$  to be the horizontal tangential bundle. The dimension of each fiber  $HT_x M$  is 1 in every point but for the *characteristic set*

$$\Sigma(M) = \{x \in M \mid H(x) = T_x M\}.$$

A theorem of Derridj [12] (see also Balogh [2]) states that for each  $0 < \epsilon \leq 1$ , the  $g_\epsilon$  surface measure ( $d\sigma_\epsilon$ ) of  $\Sigma(M)$  is zero. Choose a unit vector field  $e_1$  in  $HT(M \setminus \Sigma(M))$ . Let  $\vec{n}_1$  denote the  $g_1$  normal. The horizontal second fundamental form of  $M$  at  $x \in M \setminus \Sigma(M)$  is the scalar

$$(2.2) \quad h^0 = II^0(e_1, e_1) = \langle \nabla_{e_1}^0 e_1, \nu_h \rangle_1 = -\langle \nabla_{e_1}^0 \nu_h, e_1 \rangle_0,$$

where  $\nu_h = \pi_H(\vec{n}_1)/|\pi_H(\vec{n}_1)|_0$  is the *horizontal normal* to  $M$ , which is defined only at non-characteristic points (see [6] and [10] for more details). We define the *horizontal mean curvature* of  $M$  outside  $\Sigma(M)$  to be  $\mathcal{H}_0 = h_{11}^0$ . If  $M$  is the level set of a  $C^2$  function,  $M = \{x \in \mathbb{H}^1 \mid u(x) = 0\}$  then it is shown in [10], [6] and [7] that for any  $x \in M \setminus \Sigma(M)$ ,  $\nu_h = \sum_{i=1}^2 (\nu_h)_i X_i$ , with  $(\nu_h)_i = X_i u / |\nabla_0 u|$ ,

$$(2.3) \quad \mathcal{H}_0 = \operatorname{div}_0 \nu_h = \sum_{i=1}^2 X_i (\nu_h)_i = |\nabla_0 u|^{-1} \left( \mathcal{L}u - \frac{\mathcal{L}_\infty u}{|\nabla_0 u|^2} \right).$$

If  $u(z, x_3) = x_3 - f(|z|)$  then

$$(2.4) \quad \mathcal{H}_0 = \frac{4r^2 f'' + f'^3 / r}{(f'^2 + 4r^2)^{3/2}},$$

where  $r = |z|$ . If  $f'(0) = 0$ , then the point  $(0, f(0))$  is characteristic.

**Perimeter and horizontal first variations** The sub-Riemannian perimeter of the hypersurface  $M \subset \mathbb{H}^n$  is given by

$$P_0(M) = \int_M d\mu = \int_M |\pi_H(\vec{n}_1)|_0 d\sigma_1 ,$$

where  $d\sigma_1$  denotes the surface measure induced by  $g_1$  and  $d\mu = |\pi_H(\vec{n}_1)|_0 d\sigma_1$ , is the sub-Riemannian perimeter measure. If  $M$  is a bounded level set of a  $C^2$  function  $u : \mathbb{H}^n \rightarrow \mathbb{R}$  then

$$P_0(M) = \int_M \frac{|\nabla_0 u|}{|\nabla_1 u|} d\sigma_1 .$$

A number of formulations for the first variation formula has been proved in the recent literature (see [22], [10] and [6] for a more detailed list of references): *Consider a one-parameter family of diffeomorphism  $F_t(x) = F(x, t) : \mathbb{H}^n \times (-\epsilon, \epsilon) \rightarrow \mathbb{H}^n$ , with  $F(x, 0) = x$  and  $(d/dt)F(x, 0) = Z(x) \in T_x \mathbb{H}^n$ . For all  $f \in C^1(M)$  such that  $\operatorname{div}_1(f\nu_h)\langle Z, \vec{n}_1 \rangle_1 \in L^1_{loc}(M, d\sigma_1)$  then*

$$(2.5) \quad \frac{d}{dt} \int_{M_t} f |\pi_H(\vec{n}_1)|_0 d\sigma_1 = \int_M \operatorname{div}(f\nu_h)\langle Z, \vec{n}_1 \rangle_1 d\sigma_1 .$$

*In particular, if  $Z$  is horizontal,  $f = 1$  and  $\mathcal{H}_0 \in L^1_{loc}(M, |\pi_H(\vec{n}_1)|_0 d\sigma_1)$  then the right-hand side of (2.5) is  $\int_M \mathcal{H}_0 \langle Z, \nu_h \rangle_0 d\mu$ .*

**Legendrian foliation** To better understand the geometry of surfaces in  $\mathbb{H}^1$  we recall a notion introduced in [8]: For all  $x \in M \setminus \Sigma(M)$  the space  $HT_x M$  is one dimensional, we call the flow lines of  $HTM$  the *Legendrian foliation* of  $M \setminus \Sigma(M)$ . If  $M$  is a level set of a defining function  $u$ , then  $HT_x M$  is the span of  $\nu_h^\perp = (-X_2 u X_2 + X_1 u X_1) / |\nabla_0 u|$ . In fact, this vector field is clearly horizontal and since  $\langle \nu_h^\perp, \vec{n}_1 \rangle_1 = \langle \nu_h^\perp, \nu_h \rangle_0 = 0$  it is tangent to  $M$ . The crucial property of the Legendrian foliation is that for each of its leaves  $c = (\gamma_1, \gamma_2, \gamma_3)$  the curvature of the planar curve  $\gamma = (\gamma_1, \gamma_2)$  is proportional to  $\mathcal{H}_0(c)$ : In fact, we have  $(d/ds)(\gamma, \gamma_3) = \nu_h^\perp$ . and since, when  $n = 1$ , the second fundamental form is a scalar, then letting  $s$  denote arc length one has

$$(2.6) \quad \begin{aligned} II^0 &= h_{1,1}^0 = - \left\langle \frac{d}{ds} \nu_h(c(s))|_{s=0}, c'(s) \right\rangle_1 = \\ &= - \left\langle \frac{d}{ds} \nu_h(c(s))|_{s=0}, \nu_h^\perp \right\rangle_1 = - \left\langle \frac{d}{ds} \mathbf{i} \gamma', \gamma' \right\rangle = k . \end{aligned}$$

Here we have denoted by  $\langle \cdot, \cdot \rangle$  the Euclidean product in  $\mathbb{R}^2$ , by  $\mathbf{i} \gamma'$  the normal to  $\gamma$  and by  $k$  the Euclidean curvature of the planar curve  $\gamma$  which is the projection of a leaf  $c$  in the Legendrian foliation. The observation in (2.6) has been made independently by a number of researchers, see [8], [9], [1], and [14]. For future reference we list two elementary properties of the leaves of the Legendrian foliation.

**Lemma 2.1.** *Let  $M$  be a  $C^2$ -surface  $\mathbb{H}^1$ . If  $(\gamma, \gamma_3)$  is a curve in the Legendrian foliation of  $M \setminus \Sigma$ , parametrized by arc-length  $ds$ , then*

$$(2.7) \quad (\gamma_3)_s := \partial_s(\gamma_3) = -2 \langle \gamma, \mathbf{i} \gamma_s \rangle = -2 \frac{\langle z, \nabla_z u \rangle}{|\nabla_0 u|} |_{z=\gamma_z(s)} ,$$

and

$$(2.8) \quad \gamma_s(s) := \partial_s \gamma(s) = \frac{1}{|\nabla_0 u|} (-X_2 u + \mathbf{i} X_1 u) = \frac{1}{|\nabla_0 u|} (\mathbf{i} \nabla_z u - 2 \partial_{x_3} u z) |_{z=\gamma(s)} ,$$

where  $\nabla_z u = (\partial_{x_1} u, \partial_{x_2} u)$ .

3. THE ISOPERIMETRIC PROBLEM IN  $\mathbb{H}^1$  AND PANSU CONJECTURE

In [19] Pansu proved an isoperimetric inequality for bounded,  $C^1$  opens sets  $\Omega \subset \mathbb{H}^1$ :

$$(3.1) \quad (\text{Vol}(\Omega))^{3/4} \leq \left(\frac{3}{2\pi}\right)^{1/4} P_{\mathbb{H}^1}(\Omega) .$$

The constant on the right-hand side does not represent an optimal choice, in fact one can define the isoperimetric constant of the Heisenberg group as the best constant  $C_{best}(\mathbb{H}^1)$  for which the isoperimetric inequality (3.1) holds, i.e.

$$(3.2) \quad C_{best}(\mathbb{H}^1) = \sup_{\Omega} \frac{\min\{\text{Vol}(\Omega)^{3/4}, \text{Vol}(\mathbb{H}^1 \setminus \Omega)^{3/4}\}}{P_{\mathbb{H}^1}(\Omega)} ,$$

where the supremum is taken on all finite perimeter subsets of the Heisenberg group.

The isoperimetric profile(s) of the Heisenberg group are defined as

**Definition 3.1.** An *isoperimetric profile* for  $\mathbb{H}^1$  consists of a family of bounded finite perimeter sets  $\Omega_{best} = \Omega_{best}(V)$ ,  $V > 0$ , with  $\text{Vol}(\Omega_{best}(V)) = V$  and

$$\text{Vol}(\Omega_{best})^{3/4} = C_{best}(\mathbb{H}^1) P_{\mathbb{H}^1}(\Omega_{best}) .$$

In [20], Pansu conjectured that any set in the isoperimetric profile of  $\mathbb{H}^1$  is, up to translation and dilation, a so-called bubble set  $\mathcal{B}(o, R)$ . These are sets obtained by rotating around the  $x_3$ -axis a CC geodesic joining two points at height  $\pm\pi R^2/2$ . More precisely, these cylindrically symmetric surfaces have profile curve

$$x_3 = f_R(r) = \pm \frac{1}{4} (r\sqrt{R^2 - r^2} + R^2 \arccos r/R) .$$

Following [6] we observe that

$$(3.3) \quad |\mathcal{B}(o, R)| = 4\pi \int_0^R r f(r) dr = \frac{3}{16} \pi^2 R^4$$

and

$$P_{\mathbb{H}^1}^1(\mathcal{B}(o, R)) = 2 \int_{B(o, R)} |\nabla_0 u| = 4\pi \int_0^R r \sqrt{f'(r)^2 + r^2/4} dr = \frac{1}{2} \pi^2 R^3 .$$

**Pansu's Conjecture** [20]:

$$(3.4) \quad C_{best}(\mathbb{H}^1) = \frac{\text{Vol}(\mathcal{B}(o, R))^{3/4}}{P_{\mathbb{H}^1}^1(\mathcal{B}(o, R))} = \frac{3^{3/4}}{4\sqrt{\pi}}$$

for any  $R$ , and equality is obtained if and only if  $\Omega$  is a bubble set.

Although Pansu's conjecture is still unsolved, several partial results and special cases have been established over the years (see [6] for a detailed list of references). Here we recall a special case of a result due to Danielli, Garofalo and Nhieu [11] on the characterization of isoperimetric profiles in a class of domains having suitable symmetry and regularity properties. Consider the half-spaces  $\mathbb{H}_+^1 = \{(z, x_3) \in \mathbb{H}^1 : x_3 > 0\}$  and  $\mathbb{H}_-^1 = \{(z, x_3) \in \mathbb{H}^1 : x_3 < 0\}$ , set  $B_R = B((0, 0), R)$  for the ball of radius  $R$  in  $\mathbb{R}^2$  and consider the class

$$(3.5) \quad \mathcal{E} = \{E \subset \mathbb{H}^1 : E \text{ satisfies (i) and (ii)}\} ,$$

where

$$(i) \quad |E \cap \mathbb{H}_+^1| = |E \cap \mathbb{H}_-^1| , \text{ and}$$

(ii) there exist  $R > 0$ , and radial functions  $f, g : \overline{B_R} \rightarrow [0, \infty)$ , with  $f, g \in C^1(B_R) \cap C(\overline{B_R})$ ,  $f = g = 0$  on  $\partial B_R$ , and such that

$$\partial E \cap \mathbb{H}_+^1 = \{(z, x_3) \in \mathbb{H}_+^1 : |z| < R, x_3 = f(z)\}$$

and

$$\partial E \cap \mathbb{H}_-^1 = \{(z, x_3) \in \mathbb{H}_-^1 : |z| < R, x_3 = -g(z)\}.$$

**Theorem 3.2** (Danielli-Garofalo-Nhieu). *Let  $V > 0$ , and define  $R > 0$  so that  $V = |\mathcal{B}(o, R)|$  (see (3.3)). Then the variational problem*

$$\min_{E \in \mathcal{E}: \text{Vol}(E)=V} P_{\mathbb{H}^1}^1(E)$$

*has a unique solution in  $\mathcal{E}$  given by the bubble set  $\mathcal{B}(o, R)$ .*

**Corollary 3.3.** *Denote by  $\tilde{\mathcal{E}}$  the class of sets of the form  $y\delta_\lambda(E)$  for some  $E \in \mathcal{E}$ ,  $\lambda > 0$  and  $y \in \mathbb{H}^1$ . Then*

$$(3.6) \quad (\text{Vol}(E))^{3/4} \leq C_{\text{best}}(\mathbb{H}^1) P_{\mathbb{H}^1}^1(E)$$

*for all  $E \in \tilde{\mathcal{E}}$ , where  $C_{\text{best}}(\mathbb{H}^1) = 3^{3/4}/(4\sqrt{\pi})$ , with equality if and only if  $E = y\mathcal{B}(o, R)$  for some  $R > 0$  and  $y \in \mathbb{H}^1$ .*

In [17], Leonardi and Masnou show, among other things, that such  $u_o$  is a critical point (but not the unique minimizer) of the horizontal perimeter, when the class of competitors is restricted to  $C^2$  domains with defining function  $x_3 = \pm f(|z|)$ . The same result has been also noted in [23]. We also recall related results by Ritoré and Rosales [22], and by Monti and Rickly [18].

#### 4. THE ISOPERIMETRIC INEQUALITY AND LEGENDRIAN FOLIATIONS

Let  $M_0$  be the  $C^1$ -smooth union of the graphs of two radial functions  $f$ , and  $g$  as in the previous section, defined over the unit disk in the  $z$ -plane. Necessarily  $f$  and  $g$  are strictly decreasing,  $\lim_{r \rightarrow R} f'(r) = -\infty$ ,  $\lim_{r \rightarrow R} g'(r) = -\infty$ ,  $g'(0) = 0$  and  $f'(0) = 0$ . The only two characteristic points in  $M_0$  are the two poles  $(0, 0, f(0))$ , and  $(0, 0, -g(0))$ . By cylindrical symmetry, the  $(x_1, x_2)$  projections of any curve in the Legendrian foliation are obtained by rotating a single loop  $\gamma$  (containing the origin) around the  $x_3$ -axis.

**Remark 4.1.** If  $(\gamma, \gamma_3)$  is a curve in the Legendrian foliation of  $M_0$ , then by (2.7), we have that  $(d/ds)\gamma_3(s) > 0$  for all  $s$  such that  $\gamma(s)$  is not characteristic. Hence, the curve  $(\gamma, \gamma_3)$  will spiral upwards from the “south pole”  $(0, 0, -g(0))$  to the “north pole”  $(0, 0, f(0))$ .

If we set  $R_\theta\gamma = e^{i\theta}\gamma$ , and  $(\gamma, \gamma_3)$  is a curve in the Legendrian foliation of  $M$ , then  $(R_\theta\gamma, \gamma_3)$  is a leaf in the Legendrian foliation for all  $\theta \in \mathbb{R}$ .

**Lemma 4.2.** *Let  $M_0$  be as above and denote by  $\Omega_0$  the region in  $\mathbb{H}^1$  bounded by  $M_0$ . If  $(\gamma, \gamma_3)$  is any curve in the Legendrian foliation of  $M_0$  and  $\omega \subset \mathbb{R}^2$  is the planar region bounded by the loop  $\gamma$ , then we have*

$$\begin{aligned} \text{Vol}(\Omega_0) &= 8\pi \int_\omega |z|^2 dz = 8\pi I, \\ P_{\mathbb{H}^1}(M_0) &= 4\pi \int_\gamma |\gamma|^2 ds = \frac{4}{3} \pi L_3, \end{aligned}$$

$$\int_{M_0^+} \mathcal{H}_0 d\mu = 4\pi \int_0^L \mathcal{H}_0 |\gamma|^2 ds ,$$

where  $ds$  is arc-length,  $I$  denotes the moment of inertia of  $\omega$  and  $L_3$  the length of the curve  $\gamma^3 = (\gamma_1 + \mathbf{i}\gamma_2)^3$ .

*Proof.* Let  $\gamma : [0, 1] \rightarrow \mathbb{C}$  be the projection of one curve in the Legendrian foliation of  $M_0$ . This curve will be formed of two pieces  $\gamma_+$ , and  $\gamma_-$ , corresponding to  $M_0^+$  and  $M_0^-$ , the upper and lower half of  $M_0$ . Denote by  $L = L(\gamma)$  the length of  $\gamma$  and by  $ds$  its arc-length. As we have seen earlier,  $\gamma$  is a convex loop containing the origin. From the cylindrical symmetry assumption we can parametrize the upper hemisphere  $M_0^+$  as

$$(4.1) \quad \Xi : (s, \theta) \rightarrow (R_\theta \gamma_+(s), f(|\gamma_+(s)|)) \quad , \quad \text{with } (s, \theta) \in [0, L] \times [0, 2\pi] ,$$

where  $R_\theta$  denotes the rotation of angle  $\theta$  in  $\mathbb{C}$ . Since  $s \rightarrow \Xi(s, \theta)$  is a Legendrian lift of  $R_\theta \gamma_+$ , lying on  $M_0^+$ , the following identity hold as a consequence of (2.7), and (2.8),

$$(4.2) \quad \sqrt{|f'|^2 + 4|\gamma_+|^2} \langle \gamma_+, \gamma'_+ \rangle = -2|\gamma_+|^2 ,$$

which follows from

$$(4.3) \quad -f' \frac{\langle \gamma_+, \gamma'_+ \rangle}{|\gamma_+|} = 2\langle \gamma'_+, \mathbf{i}\gamma_+ \rangle \quad \text{and} \quad \gamma'_+ = \frac{\mathbf{i}z f'/r - 2z}{\sqrt{|f'|^2 + 4|z|^2}} \Big|_{z=\gamma_+}$$

A simple computation yields  $|\Xi_s \times \Xi_\theta| = \langle \gamma_+, \gamma'_+ \rangle \sqrt{|f'|^2 + 1}$ .

$$(4.4) \quad P_{\mathbb{H}^1}(M_0^+) = \int_{M_0^+} d\mu = \int_0^{2\pi} \int_0^L \frac{|\nabla_0(x_3 - f(|z|))|}{|\nabla(x_3 - f(|z|))|_{z=\gamma_+}} |\Xi_s \times \Xi_\theta| ds d\theta = \\ = 4\pi \int_0^L |\gamma_+|^2 ds ,$$

and

$$\int_{M_0^+} \mathcal{H}_0 d\mu = \int_0^{2\pi} \int_0^L \mathcal{H}_0(\gamma_+) \frac{|\nabla_0(x_3 - f(|z|))|}{|\nabla(x_3 - f(|z|))|_{z=\gamma_+}} |\Xi_s \times \Xi_\theta| ds d\theta = \\ = 4\pi \int_0^L \mathcal{H}_0 |\gamma_+|^2 ds .$$

Next, we compute the volume of the upper hemisphere  $\Omega^+$ , in terms of  $\gamma_+$ : Using the fact that

$$\text{Vol}(\Omega^+) = \pi \int_{x_3-\min}^{x_3-\max} r^2 dx_3 = \pi \int_{x_3-\min}^{x_3-\max} |\gamma_+|^2 dx_3 ,$$

we use the identity  $x_3 = f(|\gamma_+(s)|)$ , and change variables to  $ds$ , obtaining  $dx_3/ds = f' \langle \gamma_+, \gamma''_+ \rangle / |\gamma_+|$ , and

$$\text{Vol}(\Omega) = \pi \int_0^{L(\gamma_+)} |\gamma_+|^2 f' \frac{\langle \gamma_+, \gamma''_+ \rangle}{|\gamma_+|} ds .$$

Since  $f' \langle \gamma_+, \gamma''_+ \rangle / |\gamma_+| = 2\langle \gamma_+, \mathbf{i}\gamma''_+ \rangle$ ,

$$\text{Vol}(\Omega) = \pi \int_0^{L(\gamma_+)} 2|\gamma_+|^2 \langle \gamma_+, \mathbf{i}\gamma''_+ \rangle ds = 8\pi \int_\omega |z|^2 dz .$$

Repeating the same argument for  $\Omega^-$  and  $\gamma_-$  we prove that  $\text{Vol}(\Omega) = 8\pi I$ , where  $I$  is the moment of inertia of  $\omega$ .

□

**Remark 4.3.** Observe that for any curve  $\gamma$  bounding a planar region  $\omega$ , if we set  $\text{Area}(\gamma) = \text{Area}(\omega)$  then

$$\text{Area}(\gamma) = \int_{\omega} dx_1 \wedge dx_2 = \frac{1}{2} \int_{\gamma} (x_1 dx_2 - x_2 dx_1) = -\frac{1}{2} \int_{\gamma} \langle \gamma, \mathbf{i} \gamma' \rangle ds .$$

In particular, denoting by  $ds$  the arc-length of  $\gamma$  and by  $dx$  a parametrization of  $\gamma$  in the unit interval one has

$$(4.5) \quad \begin{aligned} \text{Area}(\gamma^2) &= -\frac{1}{2} \int_0^1 \left\langle \gamma^2, \frac{\mathbf{i}(\gamma^2)'}{|\gamma^2|} \right\rangle 2|\gamma||\gamma'| dx = \\ &= -\int_0^1 \langle \gamma^2, \mathbf{i} \gamma \gamma' \rangle dx = -\int_{\gamma} |\gamma|^2 \langle \gamma, \mathbf{i} \gamma' \rangle ds = 2I . \end{aligned}$$

In the last equality we have used Stokes' theorem and the fact that

$$d([x_1^2 + x_2^2]x_1 dx_2 - [x_1^2 + x_2^2]x_2 dx_1) = 2(x_1^2 + x_2^2)dx_1 \wedge dx_2 .$$

In view of the previous Lemma and Corollary 3.3 we have

**Proposition 4.4.** *In the class of cylindrically symmetric  $C^1$  domains described above, for any leaf  $(\gamma, \gamma_3)$  of the Legendrian foliation one has the inequality*

$$(4.6) \quad I(\omega)^{3/4} \leq \frac{\sqrt{\pi}}{3^{1/4} (8\pi)^{3/4}} L_3 .$$

Moreover, if equality is achieved then  $\gamma$  is a circle.

Viceversa, if  $\gamma$  is a Jordan curve through the origin,  $C^2$  outside of  $(0, 0)$ , then any Legendrian lift  $(\gamma, \gamma_3)$  will be a  $C^1$  horizontal curve joining two points on the  $x_3$ -axis. Rotating by  $2\pi$  this curve around the  $x_3$ -axis may or may not yield a  $C^1$  cylindrically symmetric surface which is the union of two graphs over a disk. In case it does then (4.6) yields a isoperimetric type relation between the length of  $\gamma^3$  and the moment of inertia of  $\gamma$ . The following is a particular instance where this construction can be applied

**Theorem 4.5.** *The inequality (4.6) holds for all Jordan curves  $\gamma : [0, L_1] \rightarrow \mathbb{C}$ , strictly convex and  $C^2$ . If equality is achieved then  $\gamma$  is a circle.*

*Proof.* Let  $P$  and  $Q$  be two points on the curve such that the diameter of  $\gamma$  is  $|P - Q|$ . Translate  $\gamma$  so that the origin  $P$  is mapped to the origin. Since  $\gamma$  is strictly convex then  $|\gamma(s)|^2$  is strictly increasing from  $s = 0$  up to a parameter  $L_2 \in (0, L_1)$  with  $\gamma(L_2) = P$ , and strictly decreasing in  $(L_2, L_1)$ .

Define the *normal angle* function  $\theta(s)$  so that  $\mathbf{i} \gamma'(s) = e^{i\theta(s)}$ . Since  $\gamma$  is strictly convex then the support function with center at the origin satisfies  $h(\theta) = \langle \gamma(\theta), e^{i\theta} \rangle > 0$  and consequently

$$\frac{d}{d\theta} \gamma_3(\theta) = h(\theta) > 0 .$$

We split  $\gamma$  in two portions  $\gamma_-$  (defined in  $(0, L_1)$ ) and  $\gamma_+$  (defined in  $(L_1, L_2)$ ), both containing the origin and such that  $|\gamma|^2$  is increasing on  $\gamma_-$  and decreasing on  $\gamma_+$ . We define the function  $f : [0, |\gamma(L_2)|] \rightarrow \mathbb{R}$  whose graph is the upper hemisphere as  $f(r) = \gamma_3(s)$  where  $s \in (L_1, L_2)$  is the unique value such that

$|\gamma(s)| = r$ ; the function  $g : [0, |\gamma(L_2)|] \rightarrow \mathbb{R}$  defining the lower hemisphere is constructed analogously. It is now easy to verify through (4.3) that the surface  $M$  composed of the two graphs of  $f$  and  $-g$  is  $C^1$  regular and thus we can apply Proposition 4.4 and conclude the proof.  $\square$

Theorem 4.5 should be compared with certain sharp isoperimetric inequalities, e.g.

$$4\pi^2 I(\omega) \leq L(\gamma)^3 ,$$

(with equality only if  $\gamma$  is an equilateral triangle) studied by Polyá and Szegő [21], Sachs [24], [25] and Laugesen *et al.* [16].

## 5. A VOLUME PRESERVING, PERIMETER-SHRINKING FLOW

A possible approach to the isoperimetric problem consists in studying a flow which shrinks the perimeter of a set while maintaining its volume constant. In the Euclidean case this kind of flow is obtained as a non-local renormalization of the mean curvature flow. This method yields a partial answer to the Euclidean isoperimetric problem: In dimension higher than two, Huisken [15] and, for the plane, Gage [13] have showed that convex initial data flow into spheres (or circles) through the volume constrained flow. Of course in the Euclidean case there are well known direct methods which provide easier proofs of the convex isoperimetric problem. In the Heisenberg group setting, where no direct methods are known, it seems natural to study the Heisenberg group analogue of the renormalized flow, i.e., we will consider the system

$$(5.1) \quad \left\langle \frac{\partial}{\partial t} x, \vec{n}_1 \right\rangle = \langle -\mathcal{H}_0 \nu_h + c \nu_h, \vec{n}_1 \rangle ,$$

where  $x : M \times (0, T) \rightarrow \mathbb{H}^1$  is a one parameter family of smooth embeddings and  $M_t = x(M, t)$  is a surface with horizontal mean curvature  $\mathcal{H}_0$ , horizontal normal  $\nu_h$  and

$$c = \frac{\int_{M_t} \mathcal{H}_0 d\mu}{\int_{M_t} d\mu} .$$

The equality between the right-hand and left-hand sides in the PDE system (5.1) may be interpreted in the pointwise sense outside of the characteristic set  $\Sigma(M_t)$  but not elsewhere, as  $\mathcal{H}_0$  is not defined along the characteristic locus and indeed it may be even have a distributional mass supported in  $\Sigma(M_t)$ . Two different approaches to the study of (5.1) may be found in the papers [4] and [3]. Here we confine ourselves with studying non-characteristic regions of  $M_t$ . Note that solutions of  $M_t$  satisfy  $(d/dt) P_{\mathbb{H}^1}(M_t) \leq 0$ , while  $(d/dt) \text{Vol}(\Omega_t) = 0$ . This can be easily checked using Hölder inequality, and (2.5), in fact

$$\frac{d}{dt} P_{\mathbb{H}^1}(M_t) = - \int_{M_t} \mathcal{H}_0^2 d\mu + P_{\mathbb{H}^1}^{-1}(M_t) \left( \int_{M_t} \mathcal{H}_0 d\mu \right)^2 \leq 0 ,$$

and  $(d/dt) \text{Vol}(\Omega_t) = \int_{M_t} (\mathcal{H}_0 - \mathcal{H}_0) d\mu = 0$ .

Throughout this paper we will assume the short time existence of smooth solutions to (5.1). The study of short-time existence is contained in the forthcoming paper [5].

**5.1. A special class of initial data.** We consider the simple model case of initial data  $M_0$  in the class  $\mathcal{E}$  defined in (3.5) with the additional symmetry assumption  $g = f$  (the upper and the lower hemispheres are mapped into each other by reflection  $x_3 \rightarrow -x_3$ ). Let  $\gamma : [0, 1] \rightarrow \mathbb{C}$  be the projection of one curve in the Legendrian foliation of  $M_0$ . This curve will be formed of two pieces, symmetric with respect to a half-line passing through the origin, corresponding to the upper and lower half of  $M_0$ .

Let us assume that we have a smooth flow  $\{M_t\}$ , solution of (5.1), with initial data  $M_0$ . In view of the comparison principle proved in [4] and [3], it is not restrictive to assume that the solution  $M_t$  with initial value  $M$  has cylindrical symmetry for all times  $t$  for which the flow exists and is  $C^2$ -smooth, i.e. the flow  $\{M_t\}$  is composed of level sets of functions

$$(5.2) \quad u(x, t) = \begin{cases} x_3 - f(|z|, t) & \text{if } x_3 \geq 0, \\ -x_3 - f(|z|, t) & \text{if } x_3 \leq 0, \end{cases}$$

with  $f$  a  $C^2$  function defined for  $0 \leq t \leq T$  and  $0 \leq r \leq R(t)$ , for some continuous function  $R(t) > 0$ .

**5.2. Induced planar flow.** In the paper [3], Bonk and the author have derived an evolution equation for the leaves of the Legendrian foliation of an evolving flow  $M_t$ . Here we recall the main result: We assume that the legendrian foliation of  $M_t$  evolves in a  $C^2$  smooth flow  $\gamma(\cdot, t)$  induced by (5.1) and we determine the equation of this planar flow.<sup>2</sup>

**Theorem 5.1.** *Let  $\{M_t\} \subset \mathbb{H}^1$  be a family of  $C^2$ -surfaces that has the form (5.2). Denote by  $\gamma(s, t)$  a curve in its legendrian foliation, so that  $M_t$  is obtained by rotating  $(\gamma, \gamma_3)$  around the  $x_3$ -axis. If  $\{M_t\}$  evolves by (5.1), then the family of curves  $\gamma$  evolves according to the law*

$$(5.3) \quad \left\langle \mathbf{i}_{\gamma_s}, \frac{\partial \gamma}{\partial t} \right\rangle = \frac{1}{2}(k - c + \alpha),$$

where  $s$  is arc-length,  $\mathbf{i}_{\gamma_s}$  is the interior normal to  $\gamma$ ,  $k$  denotes the mean curvature of  $\gamma$  and  $\alpha$  is any function such that

$$(5.4) \quad (i) \quad \frac{\partial}{\partial s} \left( \frac{\alpha}{\langle \gamma, \partial_s \gamma \rangle} \right) = \frac{1}{2} \frac{2(k - c)}{|\gamma|^2}$$

$$(ii) \quad \lim_{s \rightarrow 0 \text{ or } s \rightarrow L} [\alpha(s, t) - k(s, t) + c] = 0 \quad \text{for all } t.$$

**Remark 5.2.** The flow (5.3) tracks simultaneously *all* flows  $R_{\theta(t)}\gamma(\cdot, t)$ . This accounts for the lack of uniqueness of  $\alpha$  which is defined only up to the transformation

$$(5.5) \quad \alpha \rightarrow \alpha - \langle \gamma, \partial_s \gamma \rangle \partial_t \theta(t).$$

To prove the latter it suffices to substitute  $\gamma = R_{\theta(t)}\gamma$  in (5.3) and observe that

$$\partial_t R_{\theta(t)}\gamma = \mathbf{i} \partial_t \theta R_{\theta(t)}\gamma + R_{\theta(t)}\partial_t \gamma,$$

<sup>2</sup>To link the leaves of the foliation at different times we may choose the following method: For each  $0 < t < T$  let  $\gamma(\cdot, t)$  be the  $(x_1, x_2)$  projection of the unique leaf in the Legendrian foliation of  $M_t$  that intersects the  $x_1$  axis. However notice that for any choice of  $\theta \in C^2(\mathbb{R})$  the flow  $R_{\theta(t)}\gamma$  is also the projection of a  $C^2$  Legendrian flow.

and  $v \rightarrow R_\theta v$  is unitary (it is a rotation). Clearly, properties (i) and (ii) in (5.4) are not affected by the transformation (5.5).

**5.3. Monotonicity properties.** We continue the study of geometric properties of the Legendrian foliation flows induced by (5.1) with initial data as in (5.2) and prove the following simple monotonicity results.

**Theorem 5.3.** *Let  $t \rightarrow \gamma(\cdot, t)$  be a  $C^2$ -flow induced by (5.1), with initial data as in (5.2).*

- (i) *The length of the complex curve  $w = \gamma^3$  decreases monotonically in time unless  $\gamma$  is a circle.*
- (ii) *The moment of inertia  $I(\omega_t) = \int_{\omega_t} |z|^2 dA$  is constant in time. Here we have let  $\omega_t$  denote the portion of plane contained in the loop  $\gamma$  and  $dA$  denote the area element in  $\mathbb{C}$ .*

**Remark 5.4.** If  $M_0$  is the “bubble” set, then its Legendrian foliation is composed of circles, and hence  $k = k_{\text{const}}$  at the initial time. This implies that  $k - c = 0$  and  $\alpha = 0$ . Consequently, the curves in the Legendrian foliation of the “bubble” sets are stationary solutions of (5.1).

**Remark 5.5.** Although similar to the area preserving curve shrinking flow in the plane (see [13]), the flow in (5.1) does not preserve area and does not decrease length. In fact, if we denote by  $L(t)$  and by  $A(t)$  the length and area defined by  $\gamma$  at time  $t > 0$  we have

$$\frac{d}{dt}L(t) = \int \left\langle k \vec{n}_1, \frac{\partial \gamma}{\partial t} \right\rangle = \frac{1}{2} \left( - \int k^2 + \frac{\int k \int k |\gamma|^2}{\int |\gamma|^2} - \int k \alpha \right),$$

and

$$\frac{d}{dt}A(t) = \int \left\langle \vec{n}_1, \frac{\partial \gamma}{\partial t} \right\rangle = \frac{1}{2} \left( - \int k + \frac{\int k |\gamma|^2}{\int |\gamma|^2} L(t) - \int \alpha \right).$$

Since, in general,  $k$  blows up like  $1/r$  near the origin, both the integrals  $\int k$  and  $\int k^2$  must be interpreted as principal values.

Note that

$$\int \alpha ds = \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{L(t)-\epsilon} \frac{\alpha}{\langle \gamma, \partial_s \gamma \rangle} \left( \frac{|\gamma|^2}{2} \right)_s ds.$$

Integrating by parts we obtain

$$\int \alpha ds = \lim_{\epsilon \rightarrow 0} \left[ \frac{\alpha}{\langle \gamma, \partial_s \gamma \rangle} \frac{|\gamma|^2}{2} \Big|_{\epsilon}^{L(t)-\epsilon} - \int_{\epsilon}^{L(t)-\epsilon} \frac{2(k-c)}{|\gamma|^2} \frac{|\gamma|^2}{2} ds \right].$$

Substituting back into the formula for  $(d/dt) A(t)$  we obtain

$$\frac{d}{dt}A(t) = -\frac{1}{2} \lim_{\epsilon \rightarrow 0} \frac{\alpha}{\langle \gamma, \partial_s \gamma \rangle} \frac{|\gamma|^2}{2} \Big|_{\epsilon}^{L(t)-\epsilon}.$$

Thanks to (2.8) we have that the above limit can be computed as

$$\begin{aligned} (5.6) \quad & \frac{d}{dt}A(t) = \\ & = -\frac{1}{8} \lim_{\epsilon \rightarrow 0} (k-c) [-|\nabla_0 u|(\gamma(L-\epsilon), \gamma_3(L-\epsilon)) - |\nabla_0 u|(\gamma(\epsilon), \gamma_3(\epsilon))] = \\ & = \frac{1}{4} \lim_{r \rightarrow 0} \left( \frac{4r^2 f'' + f'^3/r}{[f'^2 + 4r^2]^{3/2}} - c \right) \sqrt{f'^2 + 4r^2}. \end{aligned}$$

Since  $f'(r, t) = f''(0, t)r + o(r) = r(f''(0, t) + o(1))$ , the latter yields

$$\frac{d}{dt}A(t) = \frac{1}{4} \lim_{r \rightarrow 0} \frac{f''(r, t) + (f''(0, t) + o(1))^3}{(f''(0, t) + o(1))^2 + 4} = \frac{1}{4} f''(0, t).$$

We can obtain a similar (less simple) formula for the variation of the length.

At this point we proceed with the proof of Theorem 5.3.

*Proof.* To verify the first point (i) we observe that the length of  $\gamma^3$  is given by  $3 \int_0^1 |\gamma|^2 |\gamma'| dx$  and the latter is a multiple of the  $X$ -perimeter of  $M_t$  (see (4.4)) which decreases monotonically according to the formula

$$\begin{aligned} \frac{d}{dt}P_{\mathbb{H}^1}(M_t) &= \int_{M_t} \mathcal{H}_0 \langle \nu_h, \frac{\partial}{\partial t} x \rangle d\mu(x) = - \int_{M_t} (\mathcal{H}_0^2 - c\mathcal{H}_0) d\mu = \\ (5.7) \quad &= -4\pi \int_0^L k^2 |\gamma|^2 + 4\pi \frac{(\int_0^L k |\gamma|^2)^2}{\int_0^L |\gamma|^2} \leq 0. \end{aligned}$$

We want to stress that (5.7) can be obtained by looking at the evolution of the curves in the Legendrian foliation only, without any reference to the flow of the surface in  $\mathbb{H}^1$ . In fact, a direct computation yields

$$\begin{aligned} (5.8) \quad &2 \frac{d}{dt} \int_0^L |\gamma|^2 ds = \\ &= 4 \int_0^L \langle \gamma, \frac{\partial}{\partial t} \gamma \rangle ds + 2 \int_0^1 |\gamma|^2 \frac{\langle \partial_x \gamma, \partial_t \gamma \rangle}{|\partial_x \gamma|} dx = \\ &= (\text{integrating by parts in } dx) \\ &= 2 \int_0^L \langle \gamma, \mathbf{i} \partial_s \gamma \rangle (k - c + \alpha) ds - \int_0^L |\gamma|^2 k (k - c + \alpha) ds. \end{aligned}$$

We show that the first term and the  $\alpha$  component of the second term on the right hand side sum to zero. In fact,

$$\begin{aligned} &\int_0^L \alpha (2 \langle \gamma, \mathbf{i} \partial_s \gamma \rangle - k |\gamma|^2) ds = \\ &= \int_0^L \frac{\alpha}{\langle \gamma, \partial_s \gamma \rangle} (2 \langle \gamma, \mathbf{i} \partial_s \gamma \rangle - k |\gamma|^2) \langle \gamma, \partial_s \gamma \rangle ds = \\ &= \int_0^L \frac{\alpha}{\langle \gamma, \partial_s \gamma \rangle} \partial_s (\langle \gamma, \mathbf{i} \partial_s \gamma \rangle |\gamma|^2) ds = \\ &= \lim_{\epsilon \rightarrow 0} \alpha \frac{\langle \gamma, \mathbf{i} \partial_s \gamma \rangle}{\langle \gamma, \partial_s \gamma \rangle} |\gamma|^2 \Big|_{\epsilon}^{L-\epsilon} - \int_0^L \frac{\partial}{\partial s} \left( \frac{\alpha}{\langle \gamma, \partial_s \gamma \rangle} \right) \langle \gamma, \mathbf{i} \partial_s \gamma \rangle |\gamma|^2 ds = \\ &= -2 \int_0^L (k - c) \langle \gamma, \mathbf{i} \partial_s \gamma \rangle ds. \end{aligned}$$

In the latter we have used

$$(5.9) \quad \lim_{s \rightarrow 0 \text{ or } s \rightarrow L} \left( k - \frac{\int_0^L k |\gamma|^2 ds}{\int_0^L |\gamma|^2 ds} + \alpha \right) = 0.$$

and argued as above to show

$$\begin{aligned}
 & \lim_{\epsilon \rightarrow 0} \alpha \frac{\langle \gamma, \mathbf{i} \partial_s \gamma \rangle}{\langle \gamma, \partial_s \gamma \rangle} |\gamma|^2 \Big|_{\epsilon}^{L-\epsilon} \approx \\
 & \approx \lim_{r \rightarrow 0} \left( \frac{4f''r^2 + f'^3/r}{(f'^2 + 4r^2)^{3/2}} - c \right) f'r \approx \\
 (5.10) \quad & \approx \lim_{r \rightarrow 0} \left( \frac{1}{r} \frac{4f'' + (f''(0,t) + o(1))^3}{[(f''(0,t) + o(1))^2 + 4]^{3/2}} - c \right) r^2 (f''(0,t) + o(1)) = 0.
 \end{aligned}$$

To verify (ii) we recall that the moment of inertia  $I$  is proportional to the volume of the surface  $M_t$ , and using the familiar formula  $(d/dt)\text{Vol}(M_t) = \int_{M_t} \langle Z, n \rangle d\sigma$ , for flows along a vector field  $Z$ , we have

$$\frac{d}{dt} \text{Vol}(M_t) = \int_{M_t} \langle \nu_h, \frac{\partial}{\partial t} x \rangle d\mu = \int_{M_t} \left( -\mathcal{H}_0 + \frac{\int_{M_t} \mathcal{H}_0 d\mu}{\int_{M_t} d\mu} \right) = 0.$$

Once again, we can prove conservation of  $I$  by considering only the Legendrian foliation, without involving the flow of  $M_t$ . In fact we have

$$\frac{d}{dt} I = \int_0^L |\gamma|^2 \left\langle \frac{\partial \gamma}{\partial t}, \vec{n}_1 \right\rangle ds = - \int_0^L |\gamma|^2 (k - c + \alpha) ds.$$

To see that the right hand side vanishes we do integration by parts and argue as above, obtaining

$$\int \alpha |\gamma|^2 ds = \lim_{\epsilon \rightarrow 0} \left[ \frac{1}{2} \frac{|\gamma|^4 \alpha}{\langle \gamma, \partial_s \gamma \rangle} \Big|_{\epsilon}^{L(t)-\epsilon} - \frac{1}{2} \int_{\epsilon}^{L(t)-\epsilon} 2(k - c) |\gamma|^2 ds \right],$$

concluding the proof. □

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