Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 6(2007), pp. 119–131.

A generation result for a class of elliptic operators with unbounded coefficients in L^p spaces

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Abstract[1]. We consider strongly elliptic second-order differential operators with possibly unbounded lower order coefficients. We comment and give a sketch of the proof of a generation result of C_0-semigroups on $L^p(\mathbb{R}^N)$, $1 < p < +\infty$, recently obtained by the authors. An explicit characterization of the domain is given.

1. Introduction

Linear elliptic operators with regular and bounded coefficients have nowadays a satisfactory theory including existence, uniqueness and regularity for the solutions to the corresponding equations in several Banach spaces, such as L^p spaces, Hölder spaces and so on. On the other hand, elliptic operators with unbounded coefficients are still object of intensive investigation, as the recent literature shows. In general, there exist different approaches to show that elliptic operators with unbounded coefficients generate C_0-semigroups in L^p. On this subject we mention [2], [3], [8], [10], [12], [14], [16] and the list of references therein. However, only some of them give a precise description of the domain of the generator. In [6], [7] and in [17] only the special case of $p = 2$ is considered.

Here, we consider second order elliptic operators in divergence form

$$Au := \sum_{i,j=1}^{N} D_i(q_{ij} D_j u) - \langle F, Du\rangle - Vu \,, \tag{1.1}$$

where $q_{ij} \in C_b^1(\mathbb{R}^N)$ and $Q(x) = (q_{ij}(x))$ is a $N \times N$ symmetric, real matrix satisfying a uniform ellipticity condition, whereas the drift F and the potential V are possibly unbounded functions.

If Q is a symmetric and positive definite real matrix, $F(x) = Bx$, with B real matrix, and $V = 0$ then A is the so called Ornstein-Uhlenbeck operator. This operator is well-studied in literature, both with analytic and probabilistic tools. It is well known that it is possible to associate a semigroup $(T_t)_{t\geq 0}$ which has the

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Keywords. Elliptic operator, unbounded coefficients, C_0-semigroup, characterization of the domain.

AMS Subject Classification. 35J70, 35K15, 35K65.

following explicit representation formula due to Kolmogorov

$$(T_t(f))(x) = \frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^N} e^{-\langle Q_t^{-1}y,y\rangle/4} f(e^{tB}x - y)\, dy$$

where $Q_t = \int_0^t e^{sB} Q e^{sB^*}\, ds$.

In $L^p(\mathbb{R}^N)$, $1 < p < +\infty$, the domain of A as generator of $(T_t)_{t\geq 0}$ is given by $\{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{\rm loc}(\mathbb{R}^N) : Au \in L^p(\mathbb{R}^N)\}$, see e.g. [11], and it coincides with

$$\{u \in W^{2,p}(\mathbb{R}^N) : \langle F, Du\rangle \in L^p(\mathbb{R}^N)\}\ , \tag{1.2}$$

as proved in [13]. Moreover, if $F \neq 0$ then $(T_t)_{t\geq 0}$ is not analytic (see [15] and [11]).

A generalization of the class of Ornstein-Uhlenbeck operators is given by the class of operators with globally Lipschitz continuous drift and still $V \equiv 0$. In [12] it is proved that under the further assumption

$$\langle F, Dq_{ij}\rangle \in L^\infty(\mathbb{R}^N) \quad , \quad i,j = 1,...,N\ , \tag{1.3}$$

the corresponding operator A, endowed with domain (1.2), generates a C_0-semigroup in $L^p(\mathbb{R}^N)$, $1 < p < +\infty$. Here, the characterization of the domain follows from regularity results for the solution to the non homogenous Cauchy problem associated with A. In [16] the authors are able to prove the generation result in [12] replacing the global Lipschitz continuity of F with a slightly more general condition. It is interesting to notice that, even if the strategy and the techinque are completely different, nevertheless (1.3) is still assumed. Moreover, [16] provides a one dimensional counterexample showing that if F grows as $|x|^{1+\varepsilon}$ then the corresponding operator may not generate a C_0-semigroup in $L^p(\mathbb{R})$.

In [14] a second order operator in the general form (1.1) is considered and the description of the domain of the generator is given in $L^p(\mathbb{R}^N)$ assuming the conditions $|DV| \leq \gamma V^{3/2}$, $|F| \leq \kappa V^{1/2}$ and ${\rm div} F \leq \beta V$. We observe that the first two assumptions are the same as those of Cannarsa and Vespri in [3], whereas the last one replaces an additional bound on the constant κ assumed in [3]. In [14], with a more direct approach, it is proved that A generates an analytic semigroup on $L^p(\mathbb{R}^N)$, $1 < p < +\infty$, with domain $\{u \in W^{2,p}(\mathbb{R}^N) : Vu \in L^p(\mathbb{R}^N)\}$. The cases $p = 1$ and $p = +\infty$ are also studied.

In this note, we mainly refer to our paper [4], in which we prove that if suitable growth conditions on F, V and their first order derivatives are assumed, then (1.1) generates a C_0-semigroup in $L^p(\mathbb{R}^N)$, $1 < p < +\infty$, with domain

$$\mathcal{D}_p := \{u \in W^{2,p}(\mathbb{R}^N) : \langle F, Du\rangle \in L^p(\mathbb{R}^N),\, Vu \in L^p(\mathbb{R}^N)\}\,.$$

As a by-product, we obtain regularity results for the solutions to the elliptic equation related to A.

We state two generation results, Theorem 2.1 for $p = 2$, and Theorem 2.2 for the general case $1 < p < +\infty$, due to the different hypotheses, weaker if $p = 2$. Due to the presence of some parameters in the assumptions, our result can be seen as a "continuous interpolation" between [12] and [14].

In Section 2 we include notations, statements of the main results and comments on the assumptions. In the subsequent sections we give a sketch of the proof of Theorem 2.2, with details of the main steps.

Here, we just remark that the main tool to show that $(A, \mathcal{D}_p)$ generates a C_0-semigroup, is represented by good a priori estimates for the functions in $\mathcal{D}_p$, which

yield the closedness of $(A, \mathcal{D}_p)$. Using basically integrations by parts and other elementary tools, we prove those for Du and Vu.
The variational method fails to estimate the second order derivatives of u when $p \neq 2$. Therefore, we are forced to use a different technique which works under stronger hypotheses. The idea is the following. We introduce a change of variable, depending on V, together with a localization argument in order to obtain a family of new operators $\{A_{x_n}\}_{n\in\mathbb{N}}$ with globally Lipschitz drift coefficients and bounded potentials. To each operator A_{x_n} we apply the right estimate already known from the literature, thus obtaining local estimates in the original setting, with uniform constants. Using a covering argument we get the global estimate we were looking for. Once the estimate on the second order derivatives is obtained, the surjectivity of the operator $\lambda - A$ follows by an approximation procedure. Finally, in the last section we describe some properties of the above semigroups.

2. Notation and statement of the main results

Throughout this paper $C_c^\infty(\mathbb{R}^N)$ is the space of real-valued C^∞-functions on $\mathbb{R}^N$ with compact support and $C_b^1(\mathbb{R}^N)$ is the space of real-valued functions on $\mathbb{R}^N$, which are bounded and continuous together with their first order derivatives. We denote by $\|\cdot\|_\infty$ the sup-norm in $\mathbb{R}^N$ and by $\operatorname{spt}\phi$ the support of a given function ϕ.
For $p \geq 1$ and $k \in \mathbb{N}$, $L^p(\mathbb{R}^N)$ and $W^{k,p}(\mathbb{R}^N)$ are the usual Lebesgue and Sobolev spaces, respectively. The norm of $L^p(\mathbb{R}^N)$ is denoted by $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$ denotes that of $W^{k,p}(\mathbb{R}^N)$. Given a function u on $\mathbb{R}^N$, we denote its gradient and its Hessian matrix by Du and D^2u, respectively. Moreover, we set

$$|Du|^2 = \sum_{i=1}^N (D_i u)^2 \quad , \quad |D^2u|^2 = \sum_{i,j=1}^N (D_{ij}u)^2 \ ,$$

where, clearly, $D_i = D_{x_i}$ and $D_{ij} = D_{x_i x_j}$.
The ball in $\mathbb{R}^N$ centered in x with radius $r > 0$ is indicated by $B(x, r)$. To shorten the notation, if $x = 0$ we will write B_r instead of $B(0, r)$.
If J is a set, $\operatorname{card} J$ is its cardinality.

In the following $Q = Q(x) = (q_{ij}(x))$ is a $N \times N$ symmetric real matrix such that $q_{ij} \in C_b^1(\mathbb{R}^N)$ and

$$\langle Q(x)\xi, \xi\rangle := \sum_{i,j=1}^N q_{ij}(x)\,\xi_i\xi_j \geq \nu|\xi|^2 \quad , \quad \nu > 0 \ , \tag{2.1}$$

for every $x, \xi \in \mathbb{R}^N$. Moreover, we consider $F \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ and $V \in C^1(\mathbb{R}^N)$ and we assume that V is bounded from below. Without loss of generality, we suppose that $V \geq 1$. We deal with the elliptic operator

$$Au := A_0 u - \langle F, Du\rangle - Vu \ , \tag{2.2}$$

where $A_0 u(x) := \sum_{i,j=1}^N D_i(q_{ij}(x) D_j u(x))$.
For $1 < p < +\infty$, we define the space $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$ as

$$\mathcal{D}_p := \{u \in W^{2,p}(\mathbb{R}^N) \,:\, \langle F, Du\rangle \in L^p(\mathbb{R}^N), Vu \in L^p(\mathbb{R}^N)\} \ , \tag{2.3}$$

$$\|u\|_{\mathcal{D}_p} := \|u\|_{2,p} + \|\langle F, Du\rangle\|_p + \|Vu\|_p \ . \tag{2.4}$$

We endow $\mathcal{D}_p$ also with the graph norm of the operator A, namely $\|u\|_A := \|Au\|_p + \|u\|_p$. In the case $p = 2$, besides the previous assumptions on the coefficients, we require that the following growth conditions hold

$$\text{(H1)} \qquad |DV| \leq \alpha V^{3/2} + c_\alpha \ ,$$

$$\text{(H2)} \quad \operatorname{div} F \leq \beta V + c_\beta \quad , \quad \sum_{i,j=1}^{N} D_i F_j(x)\xi_i\xi_j \geq -\tau\, V(x)|\xi|^2 - c_\tau|\xi|^2 \quad , \quad \xi,\ x \in \mathbb{R}^N \ ,$$

$$\text{(H3)} \qquad \langle F, DV\rangle \leq \gamma V^2 + c_\gamma \ ,$$

$$\text{(H4)} \qquad |F(x)| \leq \theta(1+|x|^2)^{1/2} V(x) \quad , \quad x \in \mathbb{R}^N$$

with $\alpha, \beta, \gamma, \tau, \theta > 0$ and $c_\alpha, c_\beta, c_\gamma, c_\tau \geq 0$ satisfying

$$1 - \frac{\beta}{2} - \tau > 0 \quad , \quad \frac{M}{4}\,\alpha^2 + \frac{\beta}{2} + \frac{\gamma}{2} < 1 \ , \tag{2.5}$$

where $M := \sup_{x\in\mathbb{R}^N} \max_{|\xi|=1}\langle Q(x)\xi, \xi\rangle$. We note that the second inequality in (H2) is a dissipativity condition for the function $-F$.

The following generation result holds. Here we do not provide the proof, but we refer the reader to [4, Section 5].

Theorem 2.1 (p=2)**.** *Suppose that* (H1)*,* (H2)*,* (H3)*,* (H4) *and* (2.5) *hold. Then the operator* $(A, \mathcal{D}_2)$ *generates a* C_0*-semigroup on* $L^2(\mathbb{R}^N)$*. If* $c_\beta = 0$*, then the semigroup is contractive.*

In order to obtain an analogous result in the general case $p > 1$, we use a different technique, which works under more restrictive assumptions on the coefficients of A. Precisely, we replace assumptions (H1), (H2) and (H4) with the following ones

$$\text{(H1')} \qquad |DV(x)| \leq \alpha \frac{V^{2-\sigma}(x)}{(1+|x|^2)^{\mu/2}} \quad , \quad x \in \mathbb{R}^N$$

$$\text{(H2')} \qquad |DF| \leq \frac{1}{\sqrt{N}}\,(\beta V + c_\beta) \ ,$$

$$\text{(H4')} \qquad |F(x)| \leq \theta(1+|x|^2)^{\mu/2} V^\sigma(x) \quad , \quad x \in \mathbb{R}^N$$

respectively, where DF denotes the Jacobian matrix of F and $|DF|^2 = \sum_{k,i=1}^N |D_k F_i|^2$, $\alpha, \beta, \theta > 0$, $c_\beta \geq 0$, $1/2 \leq \sigma \leq 1$ and $0 \leq \mu \leq 1$. Moreover, we suppose that for every $x \in \mathbb{R}^N$

$$\text{(H5)} \qquad |\langle F(x), Dq_{ij}(x)\rangle| \leq \kappa\, V(x) + c_\kappa \ ,$$

holds, with constants $\kappa > 0$ and $c_\kappa \geq 0$.
Analogously to the case $p = 2$, also in this case a smallness condition on the coefficients is required. Let

$$\omega := \begin{cases} \dfrac{M}{4}(p-1)\alpha^2 & , \quad \text{if } (\sigma,\mu) = \left(\dfrac{1}{2}\,, 0\right) , \\ 0 & , \quad \text{otherwise} . \end{cases}$$

Then we assume that

$$\omega+\sqrt{2}\,\frac{\beta+\sqrt{N}\alpha\theta}{p}+\alpha\,\theta\,\frac{p-1}{p}<1 \quad , \quad \text{if } 1<p<2\,, \tag{2.6}$$

$$\omega+\sqrt{2}\,\left(\beta+\sqrt{N}\alpha\theta\right)\left(\frac{1}{p}+\frac{1}{\sqrt{N}}\right)<1 \quad , \quad \text{if } p\geq 2\,.$$

The following generation result holds.

Theorem 2.2 (1<p<+∞)**.** *Suppose that* (H1'), (H2'), (H4'), (H5) *and* (2.6) *are satisfied, for some* $1<p<\infty$*. Then the operator* $(A,\mathcal{D}_p)$ *generates a* C_0*-semigroup on* $L^p(\mathbb{R}^N)$*, which turns out to be contractive if* $c_\beta=0$*.*

We point out that in Theorem 2.2 we do not explicitly assume (H3), since (H1') and (H4') imply

$$|\langle F,DV\rangle|\leq\alpha\theta V^2\,. \tag{2.7}$$

Remark 2.3. Hypothesis (H1) is essential to determine the domain. In fact in [14, Example 3.7] the authors present a Schrödinger operator $A=\Delta-V$ on $L^2(\mathbb{R}^3)$ such that (H1) holds with a too large constant α and the domain is not $W^{2,2}(\mathbb{R}^3)\cap D(V)$. Moreover in [14] it is observed that (H1) holds for example for any polynomial whose homogenous part of maximal degree is positive definite. However (H1) fails for the function $1+x^2y^2$.

Remark 2.4. We note that making particular choices of the parameters μ and σ, we may cover cases already known or discuss new ones. For example, if $\mu=0$ and $\sigma=1/2$, then we get exactly the framework of [14]

$$|F|\leq\theta V^{1/2} \quad , \quad |DV|\leq\alpha V^{3/2}$$

and therefore of [3]. If we take V constant, then we reduce to the case where F is globally Lipschitz studied in [12]. Setting $\mu=0$ and $\sigma=1$ we have the case

$$|F|\leq\theta V \quad , \quad |DV|\leq\alpha V\,,$$

which, according to our knowledge, seems to be new. From the second condition above, one deduces that V grows at most exponentially. Observe, however, that the exponent α is small, by (2.6). In any case, we can treat in this way polynomials V as in Remark 2.3.

If we optimize assumption (H4') choosing $\mu=\sigma=1$, analogously to (H4) in the case $p=2$, then (H1') becomes $|DV(x)|\leq\alpha\,\left(V(x)/(1+|x|^2)^{1/2}\right)$, which is much more restrictive than (H1). This shows that the cases $p=2$ and $p\neq 2$ are quite different. Such a difference is also confirmed by the fact that when $p=2$ we do not require any condition on $\langle Dq_{ij},F\rangle$.

The assumptions for $p\neq 2$ are determined by our approach to estimate the second order derivatives of a test function u in terms of u and Au (see Section 5).

3. Operators with globally Lipschitz drift coefficients

In this section we collect some results concerning operators with globally Lipschitz drift coefficient and bounded potential term that will be used in the sequel. They are obtained in [12], assuming $c=0$, but the same results easily extend to this case, since c is bounded. Moreover, here we precise how the constants involved depend on the operator. Consider

$$B = \sum_{i,j=1}^{N} D_i(a_{ij}D_j) - \sum_{i=1}^{N} b_i D_i - c \tag{3.1}$$

and assume that

(i) $a_{ij} = a_{ji} \in C_b^1(\mathbb{R}^N)$, $\quad \sum_{i,j=1}^N a_{ij}\xi_i\xi_j \geq \nu|\xi|^2$,
(ii) $b = (b_1, ..., b_N)$ is globally Lipschitz continuous in $\mathbb{R}^N$,
(iii) $c \in L^\infty(\mathbb{R}^N)$,
(iv) $\sup_{x\in\mathbb{R}^N} |\langle Da_{ij}(x), b(x)\rangle| < +\infty\,, \quad i,j = 1, ..., N.$

Theorem 3.1. *There exists a constant C depending on p, N, ν, $\|a_{ij}\|_\infty$, $\|Da_{ij}\|_\infty$, $\|\langle Da_{ij}, b\rangle\|_\infty$, $\|c\|_\infty$ and the Lipschitz constant of b, $[b]_1$, such that for all $u \in C_c^\infty(\mathbb{R}^N)$*

$$\int_{\mathbb{R}^N} |D^2u|^p \, dx \leq C \int_{\mathbb{R}^N} (|Bu|^p + |u|^p) \, dx \,. \tag{3.2}$$

Proposition 3.2. *If $\lambda > \lambda_p$, $\lambda_p := \sup_{x\in\mathbb{R}^N} \{(1/p)\operatorname{div} b(x) - c(x)\}$, then, given $f \in L^p(\mathbb{R}^N)$, there exists a unique solution $u \in \mathcal{D} = \{u \in W^{2,p}(\mathbb{R}^N) : \langle b, Du\rangle \in L^p(\mathbb{R}^N)\}$ of $\lambda u - Bu = f$ which satisfies $\|u\|_p \leq (\lambda - \lambda_p)^{-1}\|f\|_p$.*

4. Estimates of Vu and Du

From now on, for clarity of exposition, we assume that $c_\alpha = c_\beta = c_\gamma = c_\tau = 0$ in conditions (H1), (H2), (H3). This is always possible, keeping the same constants $\alpha, \beta, \gamma, \tau$, just replacing V with $V + \lambda$ and choosing λ large enough (this implies possibly different constants in the statements).
In this section we provide, as a preliminary step, some a priori estimates for the solutions of the elliptic equation $\lambda u - Au = f$. Precisely, via integrations by parts we prove that for all $u \in C_c^\infty(\mathbb{R}^N)$, the L^p-norms of Vu and Du may be estimated by the L^p-norms of Au and u itself, with constants independent of u. Since (H4) implies that $C_c^\infty(\mathbb{R}^N)$ is dense in $(\mathcal{D}_p, \|\cdot\|_{\mathcal{D}_p})$ (see Lemma 4.1 in [4]), then a density argument enables us to obtain analogous estimates in the general case $u \in \mathcal{D}_p$.

Lemma 4.1. *Suppose that $\operatorname{div} F \leq pV$. Then $(A, C_c^\infty(\mathbb{R}^N))$ is dissipative in $L^p(\mathbb{R}^N)$, i.e. for all $\lambda > 0$ and $u \in C_c^\infty(\mathbb{R}^N)$*

$$\|u\|_p \leq \frac{1}{\lambda}\,\|\lambda u - Au\|_p \,. \tag{4.1}$$

Proof. Fix $\lambda > 0$ and $u \in C_c^\infty(\mathbb{R}^N)$ and set $u^* = u|u|^{p-2}$. Notice that

$$D(u^*) = (p-1)|u|^{p-2}Du \quad , \quad D(|u|^p) = pu^*Du \,. \tag{4.2}$$

Define $f = \lambda u - Au$. Multiplying both sides of this equation by u^* and integrating by parts, we obtain

$$\lambda \int_{\mathbb{R}^N} |u|^p + (p-1)\int_{\mathbb{R}^N} \langle QDu, Du\rangle |u|^{p-2}\, dx - \frac{1}{p}\int_{\mathbb{R}^N} \operatorname{div} F|u|^p \, dx + \int_{\mathbb{R}^N} V|u|^p \, dx =$$

$$= \int_{\mathbb{R}^N} fu^* \, dx \,.$$

By (2.1) we get

$$(p-1)\int_{\mathbb{R}^N}\langle QDu,Du\rangle|u|^{p-2}\,dx\geq(p-1)\nu\int_{\mathbb{R}^N}|Du|^2|u|^{p-2}\,dx\geq 0$$

and (4.1) easily follows thanks to the assumption.

□

Corollary 4.2. *If* $\operatorname{div}F\leq pV$, $1<p\leq 2$ *and* $u\in C_c^\infty(\mathbb{R}^N)$ *then*

$$\int_{\mathbb{R}^N}|Du|^p\leq c\int_{\mathbb{R}^N}(|Au|^p+|u|^p)\,dx\,,$$

where $c=c(\nu,p)>0$.

Proof. From the proof of Lemma 4.1, with $\lambda=1$, we deduce that

$$\int_{\mathbb{R}^N}|Du|^2|u|^{p-2}\,dx\leq c\int_{\mathbb{R}^N}(|Au|^p+|u|^p)\,dx \tag{4.3}$$

where $c=c(\nu,p)>0$. If $p=2$, we are done. If $1<p<2$, Young's inequality with exponent $2/p$ yields

$$\int_{\{u\neq 0\}}|Du|^p\,dx=\int_{\{u\neq 0\}}\left(|Du|^p|u|^{p(p-2)/2}\right)|u|^{-p(p-2)/2}\,dx\leq$$

$$\leq c_p\int_{\{u\neq 0\}}(|Du|^2|u|^{p-2}+|u|^p)\,dx$$

and the thesis follows from (4.3).

□

Using similar arguments, in the following lemma we state the estimate of the L^p-norm of Vu.

Lemma 4.3. *Let* $1<p<+\infty$. *Assume that* (H1), (H3) *and* $\operatorname{div}F\leq\beta V$ *hold with*

$$\frac{M}{4}(p-1)\alpha^2+\frac{\beta}{p}+\gamma\,\frac{p-1}{p}<1\,, \tag{4.4}$$

where $M:=\sup_{x\in\mathbb{R}^N}\max_{|\xi|=1}\langle Q(x)\xi,\xi\rangle$. *Then there is* $c>0$ *such that for every* $u\in C_c^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N}|Vu|^p\,dx\leq c\int_{\mathbb{R}^N}(|Au|^p+|u|^p)\,dx\,.$$

The proof goes as follows: multiply the equation $Au=f$ by $V^{p-1}u^*$, integrate by parts and use the assumptions to obtain suitable estimates of the different terms. We refer to [4] for the details.

Now we turn to prove an estimate for the L^p-norm of Du. Due to Corollary 4.2, it suffices to consider the case $p\geq 2$.

Proposition 4.4. *Assume that* $p\geq 2$ *and that* (H1), (H2), (H3), (4.4) *hold. Moreover, suppose* $\beta<p(1-\tau)$. *Then there exists a positive constant c such that, for any* $u\in C_c^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N}|Du|^p\leq c\int_{\mathbb{R}^N}(|Au|^p+|u|^p)\,dx\,. \tag{4.5}$$

Proof. We assume that the coefficients (q_{ij}) are of class $C^2(\mathbb{R}^N)\cap C^1_b(\mathbb{R}^N)$; the general case can easily be proved using a standard approximation argument.
Let $u\in C_c^\infty(\mathbb{R}^N)$ and define $f=\lambda u-Au$, with $\lambda>0$ to be chosen later. Differentiating the equation with respect to x_k, $k\in\{1,...,N\}$ being fixed, multiplying both the sides by $D_k u|Du|^{p-2}$, summing over $k=1,...,N$ and integrating on $\mathbb{R}^N$, we get

$$\lambda\int_{\mathbb{R}^N}|Du|^p\,dx+I_1+I_2+I_3+I_4+I_5+\int_{\mathbb{R}^N}V|Du|^p\,dx= \tag{4.6}$$

$$=\int_{\mathbb{R}^N}\langle Df,Du\rangle|Du|^{p-2}\,dx\ ,$$

where

$$I_1=-\int_{\mathbb{R}^N}\sum_{i,j,k}D_i(D_kq_{ij}D_ju)D_ku|Du|^{p-2}\,dx\ ,\ I_4=\int_{\mathbb{R}^N}\sum_{i,k}F_iD_{ik}uD_ku|Du|^{p-2}\,dx\,,$$

$$I_2=-\int_{\mathbb{R}^N}\sum_{i,j,k}D_i(q_{ij}D_{jk}u)D_ku|Du|^{p-2}\,dx\ ,\ I_5=\int_{\mathbb{R}^N}\langle DV,Du\rangle u|Du|^{p-2}\,dx\ ,$$

$$I_3=\int_{\mathbb{R}^N}\sum_{i,k}D_kF_i\ D_iu\ D_ku|Du|^{p-2}\,dx\ .$$

In order to estimate the integrals above, we integrate by parts and apply the assumptions together with Hölder's and Young's inequalities, obtaining

$$I_1\geq-\frac{c_1}{\varepsilon}\int_{\mathbb{R}^N}|Du|^p\,dx-c_1\,\varepsilon\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx\ ,$$

$$I_2\geq\nu\int_{\mathbb{R}^N}|D^2u|^2|Du|^{p-2}\,dx\quad,\quad I_3\geq-\tau\int_{\mathbb{R}^N}V|Du|^p\,dx\ .$$

$$I_4\geq-\frac{\beta}{p}\int_{\mathbb{R}^N}V|Du|^p\,dx$$

$$I_5\geq-c_2\int_{\mathbb{R}^N}|Vu|^p\,dx-c_2\int_{\mathbb{R}^N}|Du|^p\,dx-\varepsilon\alpha\int_{\mathbb{R}^N}V|Du|^p\,dx$$

where $c_1=c_1\,(p,N,\|Dq_{ij}\|_\infty)$, $c_2=c_2(\varepsilon,p,\alpha)$ and $\varepsilon>0$ is arbitrary.
The estimate of the integral in the right hand side in (4.6) can be handled similarly and produces a new constant $c_3=c_3(p,N,\varepsilon)$. Collecting all these estimates we find

$$\Big(\lambda-\frac{c_1}{\varepsilon}-c_2-c_3\Big)\int_{\mathbb{R}^N}|Du|^p\,dx+\Big(\nu-(c_1+p-1)\varepsilon\Big)\int_{\mathbb{R}^N}|Du|^{p-2}|D^2u|^2\,dx+$$

$$+\left(1-\frac{\beta}{p}-\tau-\varepsilon\alpha\right)\int_{\mathbb{R}^N}V|Du|^p\,dx\leq c_2\int_{\mathbb{R}^N}|Vu|^p\,dx+c_3\int_{\mathbb{R}^N}|f|^p\,dx\ .$$

Thus, the thesis follows choosing first ε small and then λ large.

□

5. Proof of Theorem 2.2

In this section, we prove the generation result Theorem 2.2, which holds for any $p > 1$. Since the variational method fails to estimate the L^p-norm of the second order derivatives of a solution $u \in \mathcal{D}_p$ of $Au = f$, $f \in L^p(\mathbb{R}^N)$, we employ a different technique, which works under the assumptions (H1') and (H4'), more restrictive than (H1) and (H4), respectively. As noticed in Section 2, these two assumptions imply (2.7). Moreover, (H5) is assumed.

The estimate of the second order derivatives is proved in Proposition 5.4 and represents the main result of the section. The idea is to get first local estimates. To this aim we change variables and localize the equation $Au = f$ in certain balls $B(x_0, r(x_0))$. The new operator A_{x_0} produced by this technique has a globally Lipschitz drift term and a bounded potential. Then we may apply Theorem 3.1 to A_{x_0} to obtain local estimates of the L^p-norm of the second order derivatives of u. In order to get global estimates, we use a covering argument based on Besicovitch's Covering Theorem (see Proposition 5.1 below).

In order to carry this program out, we have first to ensure that the radius $r(x_0)$ grows at most linearly with respect to $|x_0|$, namely, roughly speaking, that $r(x_0) \leq 1 + |x_0|$. Secondly, $V(x)$ has to be "close" to $V(x_0)$ if $|x - x_0| < r(x_0)$. These facts are exactly guaranteed by assumptions (H4') and (H1') (see Lemma 5.3). Finally, the transformed drift coefficient has to be Lipschitz continuous. This follows from (H2').

We note that the transformed operators $\{A_{x_0}\}$ turn out to be uniformly elliptic if and only if we require that $|F| \leq \theta V^{1/2}$, which is the case of [14].

In Proposition 5.6, we deal with the surjectivity of the operator $\lambda - A$, which is shown by means of an approximation procedure. We skip the proof and refer the reader to [4].

As in the previous section and without loss of generality, we assume $c_\beta = c_\kappa = 0$, see (H2') and (H5).

Proposition 5.1. *Let $\mathcal{F} = \{B(x, \rho(x))\}_{x \in \mathbb{R}^N}$ be a collection of balls such that*

$$|\rho(x) - \rho(y)| \leq L|x - y| \quad , \quad x, y \in \mathbb{R}^N , \tag{5.1}$$

with $L < 1/2$. Then there exist a countable subcovering $\{B(x_n, \rho(x_n))\}$ and a natural number $\zeta = \zeta(N, L)$ such that at most ζ among the doubled balls $\{B(x_n, 2\rho(x_n))\}$ overlap.

The above proposition relies on the following version of the Besicovitch covering theorem, (see e.g. [1, Theorem 2.18]).

Proposition 5.2. *There exists a natural number $\xi(N)$ satisfying the following property. If $\Omega \subset \mathbb{R}^N$ is a bounded set and $\rho : \Omega \to (0, +\infty)$, then there is a set $S \subset \Omega$, at most countable, such that $\Omega \subset \bigcup_{x \in S} B(x, \rho(x))$ and every point of $\mathbb{R}^N$ belongs at most to $\xi(N)$ balls $B(x, \rho(x))$ centered at points of S.*

We turn now to the proof of Proposition 5.1.

Proof of Proposition 5.1. If $L = 0$ then the radii are constant and the statement easily follows.

If $L > 0$, we consider the sets

$$\Omega_n := B\Big(0, 2\rho(0)(1 + L)^n\Big) \setminus B\Big(0, 2\rho(0)(1 + L)^{n-1}\Big) \quad , \quad n \geq 1$$

$$\Omega_0 := B(0, 2\rho(0)) .$$

Applying Proposition 5.2 we have that for all $n \in \mathbb{N}_0$ there exists an at most countable subset $S_n \subset \Omega_n$, such that $\Omega_n \subset \bigcup_{x\in S_n} B(x, \rho(x)) =: C_n$. Since (5.1) implies $\rho(x) \le \rho(0) + L|x|$, it is easy to prove that

$$C_n \subset B\Big(0, \rho(0)(2(1+L)^{n+1}+1)\Big) \setminus B\Big(0, \rho(0)(2(1-L)(1+L)^{n-1}-1)\Big) \ , \ \ n \ge 1 .$$

Note that $2(1+L)^{n-1}(1-L)-1 > 0$ for all $n \ge 1$ because $L < 1/2$. Since $1+L > 1$, there exists $k = k(L) \in \mathbb{N}$ such that for all $n \ge k$

$$2(1-L)(1+L)^{n-1} - 1 > 2(1+L)^{n-k+1} + 1 ,$$

which implies that $C_n \cap C_{n-k} = \emptyset$. Hence the intersection of at most k among the sets C_n can be nonempty. Moreover, at most $\xi(N)$ among the balls centered at points of S_n overlap. It turns out that $\mathcal{F}' = \{B(x, \rho(x)) \ : \ x \in S_n, \ n \in \mathbb{N}_0\} =: \{B(x_j, \rho_j)\}$ is a countable subcovering of $\mathbb{R}^N$ and if $\xi' = k\,\xi(N)$ then at most ξ' balls of $\mathcal{F}'$ overlap.

To estimate the number of overlapping doubled balls $\{B(x_j, 2\rho_j)\}$ we proceed as in [14, Lemma 2.2]. Let $B(x_i, \rho_i) \in \mathcal{F}'$ be fixed and set $J(i) = \{j \in \mathbb{N} \ : \ B(x_i, 2\rho_i) \cap B(x_j, 2\rho_j) \neq \emptyset\}$. If $j \in J(i)$ it turns out that $|\rho_i - \rho_j| \le 2L(\rho_i + \rho_j)$, because $|x_i - x_j| \le 2(\rho_i + \rho_j)$, yielding $((1-2L)/(1+2L))\ \rho_i \le \rho_j \le ((1+2L)(1-2L))\ \rho_i$. Thus, the balls $B(x_j, \rho_j)$, $j \in J(i)$, are contained in $B\left(x_i, ((5+2L)(1-2L))\ \rho_i\right)$. Since at most ξ' of the balls $B(x_j, \rho_j)$ overlap, we obtain

$$\left(\frac{1-2L}{1+2L}\right)^N \rho_i^N \operatorname{card} J(i) \le \sum_{j\in J(i)} \rho_j^N \le \xi' \left(\frac{5+2L}{1-2L}\right)^N \rho_i^N ,$$

which implies $\operatorname{card} J(i) \le \xi'\big((5+2L)(1+2L)/(1-2L)^2\big)^N$, so that the number of overlapping doubled balls is an integer ζ, with $\zeta \le 1 + \xi'\big((5+2L)(1+2L)/(1-2L)^2\big)^N$. □

The following simple lemma is a straightforward consequence of assumption (H1'), so we refer to [4] for a proof.

Lemma 5.3. *Assume that* (H1') *holds. Then there exist* $\varepsilon > 0$ *and two constants* $a, b > 0$*, depending on* α, σ, μ*, such that for all* $x_0 \in \mathbb{R}^N$

$$aV(x) \le V(x_0) \le bV(x) \quad ,\textit{for every } x \in B(x_0, 3\varepsilon r(x_0)) ,$$

with $r(x_0) := (1+|x_0|^2)^{\mu/2} V^{\sigma-1}(x_0)$.

Proposition 5.4. *Assume* (H1'), (H2'), (H4'), (H5) *with constants satisfying* (2.6)*. If* $u \in C_c^\infty(\mathbb{R}^N)$ *then*

$$\int_{\mathbb{R}^N} (|\langle F, Du\rangle|^p + |D^2u|^p)\, dx \le c \int_{\mathbb{R}^N} (|Au|^p + |u|^p)\, dx , \tag{5.2}$$

with c *depending only on* N, p, ν, M, $\|q_{ij}\|_\infty$, $\|Dq_{ij}\|_\infty$ *and the constants in* (H1'), (H2'), (H4') *and* (H5).

Proof. We just consider the second order derivatives of u; then, by difference, we obtain the estimate of $\langle F, Du\rangle$.

For every $x_0 \in \mathbb{R}^N$, let ε and $r = r(x_0)$ be as in Lemma 5.3. We point out that ε is independent of x_0.

Define y_0 equal to λx_0, with $\lambda := V^{1/2}(x_0)$. We consider two cut-off functions η and ϕ in $C_c^\infty(\mathbb{R}^N)$, $0 \le \eta, \phi \le 1$, satisfying the following conditions

$$\eta \equiv 1 \quad \text{in } B(y_0, \varepsilon\lambda r) \quad , \quad \operatorname{spt}\eta \subset B(y_0, 2\varepsilon\lambda r) \ ,$$

$$\phi \equiv 1 \quad \text{in } B(y_0, 2\varepsilon\lambda r) \quad , \quad \operatorname{spt}\phi \subset B(y_0, 3\varepsilon\lambda r) \ ,$$

$$|D\eta|^2 + |D^2\eta| + |D\phi|^2 + |D^2\phi| \le \frac{L}{\lambda^2 r^2} \ , \tag{5.3}$$

for some $L > 0$, depending on ε, but neither on x_0 nor on y_0. For every $x \in \mathbb{R}^N$, let $y = \lambda x$ and define $v(y) = u(y/\lambda)$ and then $w(y) = \eta(y)v(y)$. Being $\operatorname{spt} w \subset B(y_0, 2\varepsilon\lambda r)$, we deduce that

$$\sum_{i,j=1}^{N} D_{y_i}(\tilde{q}_{ij}(y) D_{y_j} w(y)) - \frac{1}{\lambda}\phi(y)\langle \tilde{F}(y), D_y w(y)\rangle - \tag{5.4}$$

$$-\frac{1}{\lambda^2}\phi(y)\tilde{V}(y)w(y) = g(y) \ , \ \ y \in \mathbb{R}^N \ ,$$

with $\tilde{q}_{ij}(y) = q_{ij}(y/\lambda)$, $\tilde{F}(y) = F(y/\lambda)$, $\tilde{V}(y) = V(y/\lambda)$ and $\tilde{f}(y) = f(y/\lambda)$ and with g defined as follows

$$g(y) := \frac{1}{\lambda^2}\eta(y)\tilde{f}(y) + 2\langle \tilde{q}(y) D\eta(y), Dv(y)\rangle + \operatorname{div}(\tilde{q}D\eta)(y)v(y) - \tag{5.5}$$

$$-\frac{1}{\lambda}\langle \tilde{F}(y), D\eta(y)\rangle v(y) \quad , \quad y \in \mathbb{R}^N \ .$$

Now, define the new operator $\tilde{A} = \sum_{i,j=1}^N D_{y_i}(\tilde{q}_{ij} D_{y_j}) - (1/\lambda)\,\phi\,\langle \tilde{F}, D_y\rangle - (1/\lambda^2)\,\phi\,\tilde{V}$. One can prove the following claims.

Claim 1. $(1/\lambda^2)\,\phi\,\tilde{V}$ and $\left|\langle (1/\lambda)\,\phi\tilde{F}, D\tilde{q}_{ij}\rangle\right|$ are bounded in $\mathbb{R}^N$ and $(1/\lambda)\,\phi\,\tilde{F}$ is globally Lipschitz in $\mathbb{R}^N$ with $\left\|(1/\lambda^2)\,\phi\,\tilde{V}\right\|_\infty$, $\left\|\langle (1/\lambda)\,\phi\tilde{F}, D\tilde{q}_{ij}\rangle\right\|_\infty$ and the Lipschitz constant of $(1/\lambda)\,\phi\,\tilde{F}$ independent of x_0.

Claim 2. The function g in (5.5) satisfies the estimate

$$\int_{\mathbb{R}^N} |g(y)|^p \, dy \le \tag{5.6}$$

$$\le \frac{C}{\lambda^{2p-N}} \int_{B(x_0, 2\varepsilon r)} \Big(|u(x)|^p + |f(x)|^p + |V(x)u(x)|^p + |V^{1/2}(x)Du(x)|^p \Big)\, dx,$$

for some C depending on ε, but not on x_0.

Let us now prove (5.2). Applying Theorem 3.1 with B replaced by $\tilde{A}$, and using also Lemma 5.3, we get K_1 independent of x_0 such that

$$\int_{B(x_0,\varepsilon r)} |D^2 u|^p \, dx \le K_1 \int_{B(x_0, 2\varepsilon r)} \Big(|u|^p + |f|^p + |Vu|^p + |V^{1/2}Du|^p \Big)\, dx \ . \tag{5.7}$$

From (H1') it can be proved that the radius $\varepsilon\, r$ is Lipschitz continuous with respect to x_0, with Lipschitz constant $\varepsilon(1 + (1-\sigma)\alpha)$, which is less than 1/2, for small ε. Let $\{B(x_j, \varepsilon r_j)\}$ be the covering of $\mathbb{R}^N$ yielded by Proposition 5.1. Applying (5.7) to each x_j and summing over j, we get

$$\int_{\mathbb{R}^N} |D^2 u|^p \, dx \le \zeta\, K_2 \int_{\mathbb{R}^N} \Big(|u|^p + |f|^p + |Vu|^p + |V^{1/2}Du|^p \Big)\, dx \ ,$$

where ζ is given by Proposition 5.1. Now, in [14] it is proved that for small γ

$$\|V^{1/2}Du\|_p \le \gamma\|D^2u\|_p + \frac{c}{\gamma}\,\|Vu\|_p\,.$$

Thus, Lemma 4.3 (whose assumptions are satisfied) gives us the estimate of the second order derivatives. By difference we get the estimate for $\langle F, Du\rangle$.

□

Collecting the results obtained here and in the previous section and using the density of $C_c^\infty(\mathbb{R}^N)$ in $\mathcal{D}_p$ we have the following result.

Lemma 5.5. *Suppose that* (H1'), (H2'), (H4') *and* (H5) *hold, with constants satisfying* (2.6). *Then* $(A, \mathcal{D}_p)$ *is closed in* $L^p(\mathbb{R}^N)$. *Moreover,* $(A, \mathcal{D}_p)$ *is dissipative.*

Now we turn to the surjectivity of the operator $\lambda - A$. As already pointed out at the beginning of the section, the idea to get a solution to the equation $\lambda u - Au = f$ in $\mathcal{D}_p$, is to proceed by approximation, introducing a family of new operators which fulfill the assumptions of section 3. Hence, Proposition 3.2 produces approximating solutions u_ε, that actually satisfy all the a-priori estimates of the previous sections, uniformly with respect to ε. A standard weak compactness result allows to pass to the limit, as $\varepsilon \to 0$.

Proposition 5.6. *Suppose that* (H1'), (H2'), (H4') *and* (H5) *hold, with constants satisfying* (2.6). *Then for every* $f \in L^p(\mathbb{R}^N)$ *and for every* $\lambda > 0$ *there exists a unique solution* $u \in \mathcal{D}_p$ *of* $\lambda u - Au = f$ *in* $\mathbb{R}^N$. *Moreover,* $\|u\|_p \le \lambda^{-1}\|f\|_p$.

Finally Theorem 2.2 follows from all the results so far proved by an application of the Hille-Yosida Theorem.

6. Comments and consequences

In this final section we establish some further properties of the semigroup $T_p(\cdot)$ generated by $(A, \mathcal{D}_p)$ on $L^p(\mathbb{R}^N)$. We point out that the semigroups given by Theorem 2.2 are not analytic, in general. An example is the Ornstein-Uhlenbeck semigroup (see e.g. [10, Example 4.4]).

In the following proposition we prove the consistency and the positivity of $T_p(\cdot)$.

Proposition 6.1. *The following assertions hold true.*

(i) *Assume that the assumptions of Theorem 2.2 hold for some* $1 < p, q < +\infty$. *If* $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, *then* $T_p(t)f = T_q(t)f$, *for all* $t \ge 0$;

(ii) $T_p(\cdot)$ *is positive.*

Proof. Concerning (i), by [9, Corollary III.5.5] it suffices to prove that the resolvent operators of $(A, \mathcal{D}_p)$, $(A, \mathcal{D}_q)$ are consistent, for λ large, i.e. that for every $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ there exists $u \in \mathcal{D}_p \cap \mathcal{D}_q$ such that $\lambda u - Au = f$. This follows from the proofs of Proposition 5.6 and [12, Theorem 2.2] since the same property holds for uniformly elliptic operators. Concerning (ii), we observe that the positivity of T_p is equivalent to the positivity of the resolvent $(\lambda - A)^{-1}$ for all λ sufficiently large. By the proof of Proposition 5.6 this last property turns out to be true once each approximating operator A_ε is shown to have a positive resolvent. From [12, Theorem 2.2] this holds because the operators A_ε can be approximated by uniformly elliptic operators.

□

In the following proposition we show the compactness of the resolvent of $(A, \mathcal{D}_p)$ assuming that the potential V tends to infinity as $|x| \to +\infty$. This result is similar to [14, Proposition 6.4] and hence we skip the proof.

Proposition 6.2. *If $\lim_{|x|\to+\infty} V(x) = +\infty$ then the resolvent of $(A, \mathcal{D}_p)$ is compact.*

Finally, as a corollary of the estimates proved in the previous sections we have an interpolatory estimate for the functions in $\mathcal{D}_p$.

Corollary 6.3. *For every $u \in \mathcal{D}_p$ the following estimate*

$$\|Du\|_p \leq c\|u\|_p^{1/2}\|\lambda u - Au\|_p^{1/2}$$

holds for every λ sufficiently large.

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