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A Steiner formula in the Heisenberg group for Carnot-Charathéodory balls

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Abstract[1]. We sum up some results related to a volume formula for a class of sets with intrinsic positive reach, see [13] and [14]. In particular we introduce new geometric invariants associated with the notion of intrinsic curvature and imaginary curvature of the boundary in the Heisenberg group $\mathbb{H}$, in a class of bounded set Ω.

1. Introduction

In this note we are going to introduce some results concerning the notion of curvature in the Heisenberg group presented in [13] and [14].

In particular we are interested to the following question.

What does it mean to state that the imbedded surface Σ in the Heisenberg group $\mathbb{H}$ is curved?

Several authors interested in the properties of the intrinsic *minimal surfaces*, see [6] for detailed references, have studied the *intrinsic mean curvature* of a surface in the Heisenberg group. Namely they deal with the following operator

$$\mathrm{div}_{\mathbb{H}}\nu ,$$

where ν is the unit intrinsic normal to the surface Σ, that is by definition the intrinsic mean curvature of the surface Σ.

Indeed $\mathrm{div}_{\mathbb{H}}\nu$ has a geometric meaning. More precisely, $\mathrm{div}_{\mathbb{H}}\nu$ valued in the non-charachteristic point $P \in \Sigma$ is the Euclidean curvature at the point P^* of the path $\tilde{\gamma} \subset \{t = 0\}$, $P^* \in \tilde{\gamma}$, obtained projecting orthogonally the horizontal path γ, laying on the surface Σ and passing through P, onto $\{t = 0\}$, see [7], [5], [30], [31], [2]. The analogy with the operator of the mean curvature of a surface in $\mathbb{R}^3$ is quite evident. Nevertheless, in the Euclidean case, there exist two principal curvatures. So, while in the Euclidean frame it is natural to deal with the mean curvature, namely the mean of the curvatures, and the Gauss curvature, i.e. the product of the two principal curvatures, in the Heisenberg group such parallel notions are not well understood. Indeed, in the Heisenberg group the intrinsic mean curvature has the geometric meaning described above, but it is not clear whether $\mathrm{div}_{\mathbb{H}}\nu$ is related to the mean of some intrinsic geometric curvatures. More precisely in the

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Heisenberg group, to each non-characteristic point $P \in \Sigma$, there exists just only one horizontal direction that belongs to the tangent space at the point P, see [2]. So, as a consequence, the intrinsic Weingarten map is defined on such subspace of dimension one spanned by the unique horizontal tangent direction at P. The unique eigenvalue associated with such map is precisely the intrinsic mean curvature, see [2].

Since such description of non-characteristic surfaces is not satisfactory, we are interested in the existence of other intrinsic geometric invariants in the Heisenberg group. Further motivations of this research are connected with the problem of giving a more satisfactory notion of convexity in the Heisenberg group, see [9] and [21]. Moreover in the applications we recall that in recent models of the vision, see e.g. [8], some geometric invariants like the intrinsic mean curvature seem to play a key role.

For such reasons in [2] we introduce the notion of *imaginary curvature* of a surface. In spite of the name, the imaginary curvature of a surface in a point P has an evident geometric meaning. Precisely it is, times a positive constant, the inverse of the radius of the largest Carnot-Charathéodory ball locally touching the surface at P.

Beside to this approach, in this note, we introduce another point of view concerning geometric invariants of a smooth non-charachteristic surface, see [13] and [14] for the details. More precisely our argument is associated with the existence of a Steiner type formula for smooth sets of (intrinsic) positive reach in the Heisenberg group $\mathbb{H}$. Indeed associated with the coefficients of such formula there are some second order objects invariant with respect to the group law in $\mathbb{H}$. For this reason we can consider them as geometric invariants in the Heisenberg group. In particular we obtain a sort of intrinsic Steiner formula using the key notion of *metric normal* introduced in [1].

To clarify our approach let us recall that in the Euclidean case, for any positive number ϵ, we get, denoting by $B_E(0, 1+\epsilon)$ the usual Euclidean ball of radius $1+\epsilon$ in $\mathbb{R}^3$:

$$|B_E(0,1+\epsilon)| = (1+\epsilon)^3\,|B_E(0,1)| = \sum_{i=0}^{3} b_k \epsilon^k \ , \tag{1}$$

where for $i = 0, \ldots, 3$, $b_i \in \mathbb{R}$. This kind of expantion, called Steiner formula, holds for any set of positive reach in the Euclidean space $\mathbb{R}^3$, see [11]. If we consider in the Euclidean space a convex set Ω, the coefficients of the Steiner formula are called in literature *quermassintegrals.*

In general, for any bounded C^2 set Ω in $\mathbb{R}^3$, the integral of the first and the second elementary symmetric functions of the principal (Euclidean) boundary curvatures against the Hausdorff measure $\mathcal{H}_E^{(2)}$ are respectively proportional to the third and fourth coefficient of the parallel Steiner formula. If Ω in $\mathbb{R}^3$ is convex, then such coefficients are rispectively the third and fourth quermassintegral. Let us remind that the first and the second symmetric elementary function of the principal curvatures in a point $\sigma \in \partial\Omega$, of the smooth surface $\partial\Omega$, are respectively proportional to the mean curvature $\mathcal{M}_E(\partial\Omega)(\sigma)$ and the Gauss curvature $\mathcal{G}_E(\partial\Omega)(\sigma)$ of $\partial\Omega$ at σ. The first and the second coefficient are respectively the Lebesgue measure of the set Ω and the Hausdorff measure of the boundary of Ω. See also [34] and [12],

[15] for further recent results in the Euclidean setting and [32] for a comprehensive overview concerning the subject.

Now, recalling the homogeneity property of the Carnot-Charathéodory balls with respect to the dylation group at the origin, we get the following formula in the Heisenberg group:

$$|B(0,1+\epsilon)| = (1+\epsilon)^4 \, |B(0,1)| = \sum_{i=0}^{4} a_k \epsilon^k \ . \tag{2}$$

Thus, we have five geometric invariant objects: the coefficients $a_k \in \mathbb{R}$, $k = 0, \ldots, 4$. See [14] for other details related to the meaning of such coefficients. So, comparing formulae (1) and (2), we notice that in the first case we have four coefficients while in the second one there are five coefficients.

Previous argument, even if applied to a non-smooth set (notice that Carnot-Carathéodory balls are not sets of positive reach), suggests that the intrinsic mean curvature in the Heisenberg group could be inadequate to characterize the volume expansion $|B(0, 1 + \epsilon)|$ by a Steiner type formula. Moreover such remark introduces the problem of understanding the relationship between such coefficients and some new intrinsic objects that we could consider as related to *measures of curvatures*, see [11].

Actually, are the coefficients of a formula like (2) related to some functions depending on some intrinsic curvatures in the Heisenberg group as in the Euclidean case?

Moreover, which are the involved curvatures?

We give some partial answers to these questions in [13] and [14]. We resume them in the next Section 3: see respectively Theorem 3.1, Theorem 3.2 and Theorem 3.3. As a consequence we think that the functions arising in the expantion of the volume can be naturally associated with new intrinsic curvatures of the boundary of Ω in the Heisenberg group.

The existence of invariant objects in sub-Riemannian structures has been studied in [23], [24], [25]. We also recall the contributions in [20], [10] and [22] concerning the CR structures.

We complete such Introduction just recalling the plan of this note. In Section 2 we introduce the main notation, in Section 3 we collect the main results of the papers [13] and [14], in Section 4 we give a rough description of the proofs of the main results of Section 3 and eventually, in Section 5, we make some comments about such research.

In the sequel we shall write $d(\cdot,\cdot)$, or analogously $d_{CC}(\cdot,\cdot)$, and $\mathcal{H}^{(3)}$, or analogously $\mathcal{H}^{(3)}_{CC}$, to denote respectively the Carnot-Charathéodory distance and the 3-Hausdorff measure with respect to the Carnot-Charatheodory distance. Any time we write an Euclidean quantity we stress this fact with an E. For example $d_E(\cdot,\cdot)$ is the usual Euclidean distance.

2. Notation

Before going into the detail of the subject, let us recall the main notation. For every $P, Q \in \mathbb{R}^3$, $P = (x_1, y_1, t_1), Q = (x_2, y_2, t_2)$, we consider in $\mathbb{R}^3$ the following non-commutative law :

$$P \cdot Q = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_2 y_1 - x_1 y_1)) \ .$$

We define $\mathbb{H} = (\mathbb{R}^3, \cdot)$ as the Heisenberg group, see [33]. For every positive number λ and for every $(x, y, t) \in \mathbb{H}$, let δ_λ be the dilation group on $\mathbb{H}$ such that

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t) \ ,$$

see [33] . Moreover at each point $P = (x, y, t)$ let $\mathcal{H}_P$ be the vector space spanned by X_P and Y_P, where $X_P = (1, 0, 2y)$ and $Y_P = (1, 0, -2x)$. We define on $\mathcal{H}_P$ the sub-Riemannian metric $\langle \cdot, \cdot \rangle$ that makes pointwise orthonormal X_P and Y_P. We endow $\mathbb{H}$ with the *Carnot-Charathéodory* distance, see [26], starting from previous sub-Riemannian metric $\langle \cdot, \cdot \rangle$. Indeed the distance between two points P and Q in $\mathbb{H}$, the Carnot-Charatheódory distance, is by definition:

$$d(P, Q) = \inf_{\gamma \in \mathrm{Hor}(P,Q)} l_{\mathbb{H}}(\gamma) \ ,$$

where

$$\mathrm{Hor}(P, Q) = \{\gamma \in \mathcal{AC}([\alpha, \beta]) \to \mathbb{H} \ : \ \dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)} \ , \ \text{ a.e. in } \ [a, b]\} \ ,$$

$\mathcal{AC}([\alpha, \beta])$ denotes the set of absolutely continuous paths, parametrized on $[\alpha, \beta]$,

$$l_{\mathbb{H}}(\gamma) = \int (a(t)^2 + b(t)^2)^{1/2} \ dt$$

and $\dot{\gamma}(t) = a(t)X_{\gamma(t)} + b(t)Y_{\gamma(t)}$ (see also [27] for other equivalent definition of the Carnot-Charathéodory distance and [3] and [17] for further details). Let X_P and Y_P be the left invariant vector fields

$$X_P = \partial_x + 2y\partial_t, \ Y_P = \partial_y - 2x\partial_t \ .$$

As usual we use the same notation for the vector fields in the algebra and the related horizontal vectors X_P and Y_P.

Notice that the vector fields X_P and Y_P do not commute: $[X_P, Y_P] = -4\partial_t$. For any $P \in A \subset \mathbb{H}$, let $\nabla_{\mathbb{H}} f(P) = X_P f X_P + Y_P Y_P f (= X f(P) X_P + Y_P Y f(P))$ be the *intrinsic gradient vector* of the smooth real valued function $f : A \to \mathbb{R}$, defined in the open set A, see [30], [31]. With ν_P we denote the intrinsic unit normal to a non-characteristic surface in the point P. In particular if Σ is a level surface of f, we get $\nu_P = \nabla_{\mathbb{H}} f(P) / |\nabla_{\mathbb{H}} f(P)|$, see [29], [16]. In the sequel, for the sake of simplicity, we often shall not write the dependence on the point P. Let

$$H_{\mathbb{H}} f = \begin{bmatrix} XXf & YXf \\ \\ XYf & YYf \end{bmatrix}$$

be the *horizontal Hessian* matrix of f, and let $(XY)^* f = (XYf + YXf)/2$.

In the Heisenberg group the Lebesgue measure $|\cdot|$ is invariant with respect to the group law. In particular, if $B(0, 1)$ is the unitary Canoth-Charathéodory ball centered in 0, then

$$\delta_\lambda(B(0, 1)) = B(0, \lambda) \ \ , \ \text{ and } \ |B(0, \lambda)| = \lambda^4 \, |B(0, 1)| \ .$$

Let us just remark that in the Heisenberg group the homogeneous dimension is 4, see [35], while in $\mathbb{R}^3$ is 3.

Definition 2.1. Let Q be a point in $S \subset \mathbb{H}$. The *metric normal* to S at Q, denoted by $\mathcal{N}_Q S$, is the set of the points P such that

$$d_S(P) = d(P, Q) \ .$$

Let S be an oriented surface and f a C^2 function in an open set A. The oriented surface S induces, in a natural way, an orientation on $\mathcal{N}_P S$. We denote with the metric normal $\mathcal{N}_P^+ S$ the oriented metric normal.

Proposition 2.1. *Let $S = \{t = g\}$ be a C^3 surface. The equation of $\mathcal{N}_P^+ S$ can be written in terms of g's partial derivatives. $\mathcal{N}_P^+ S = P \cdot \eta$ (left translation by P), where $\eta = (u, v, s)$ and*

$$\eta(\sigma) = \begin{cases} u(\sigma) = \dfrac{1}{4\partial_t g}\left\{Yg(P)\left(1 - \cos\left(\dfrac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right) + \right. \\ \qquad\qquad \left. + Xg(P)\sin\left(\dfrac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right\} \\ v(\sigma) = \dfrac{1}{4\partial_t g}\left\{-Xg(P)\left(1 - \cos\left(\dfrac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right) + \right. \\ \qquad\qquad \left. + Yg(P)\sin\left(\dfrac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right\} \\ s(\sigma) = \dfrac{|\nabla_{\mathbb{H}} g(P)|^2}{8(\partial_t g(P))^2}\left\{\dfrac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|} - \sin\left(\dfrac{4\partial_t g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right\} \end{cases} \tag{3}$$

Observe that $\mathcal{N}_P^+ S$ points upwards if $\partial_t g(P) > 0$ and downwards if $\partial_t g(P) < 0$. If $\partial_t g(P) = 0$, then $\Pi_P S$ has characteristic point C at infinity, $d(P, C) = \infty$, and (3) *becomes*

$$\eta(\sigma) = \left(\frac{Xg(P)}{|\nabla_{\mathbb{H}} g(P)|}\,\sigma\,,\ \frac{Yg(P)}{|\nabla_{\mathbb{H}} g(P)|}\,\sigma\,,\ 0\right) \tag{4}$$

These equations show that, for some smooth function Φ,

$$\mathcal{N}_P^+ S(\sigma) = \Phi(\sigma, P, \nabla_{\mathbb{H}} g(P), [X, Y]g(P))\ .$$

Let us introduce the *exponential map* F, see [1]. Our exponential map acts as follows. For every point $P \in S$ and for every positive number $\sigma < \sigma_0$ there exists just only a point P' on $\mathcal{N}_P S$ on the positive part such that $d(P, P') = \sigma$ and P' lies on the positive part of $\mathbb{H}$ determined by the orientation of S :

$$(P, \sigma) \to F(P, \sigma) = P'\ .$$

In local coordinates, if $S = \{t = f(u, v)\}$, $F : (u, v, \sigma) \mapsto F(u, v, \sigma)$, is defined from an open subset of $\mathbb{R}^2 \times \mathbb{R}$ with values in $\mathbb{H}$,

$$F(u, v, \sigma) = \mathcal{N}^+_{(u,v,f(u,v))} S(\sigma)\ . \tag{5}$$

We are going to use the coordinates

$$(x, y, t) = F(u, v, \sigma) = (u, v, f(u, v)) \circ (x', y', t')$$

where $(x', y', t') = \gamma_{u,v}(\sigma)$ and $\gamma_{u,v}$ is the metric normal's left translate by P^{-1}.

Other significant details on the property of the metric normal are described in [1] and [2]. Let us introduce some other notation and definitions. We denote by

$$p \equiv p_P = \frac{4g_t(P)}{|\nabla_{\mathbb{H}} g(P)|}\ ,$$

the *imaginary curvature* of the smooth surface $S = \{g(x,y,t) = 0\}$ at the point $P \in S$, where $g_t(P) \equiv \partial g(P)/\partial t$.

3. Main results

Let us recall that, see [1], any C^3 surface S, without characteristic points in the Heisenberg group, is endowed by the metric normal described in Proposition 2.1.

Moreover, for every point $P \in S$, denoting $r \equiv r_P = |\nabla_{\mathbb{H}} g(P)| / g_t(P)$ and $v_P \equiv v_P S = (-Yg(P), Xg(P))/ |\nabla_{\mathbb{H}} g(P)|$, we define

$$S_0 = \frac{r^2}{4} \quad , \quad S_1 = \frac{r^3}{32\,|\nabla_{\mathbb{H}} g|} \left(\Delta_{\mathbb{H}} g - r\langle \nabla_{\mathbb{H}} g_t, \nu\rangle\right) \ ,$$

$$S_2 = -\frac{r^3}{16\,|\nabla_{\mathbb{H}} g|} \left(\langle H_{\mathbb{H}} g \nu, \nu\rangle + r\langle \nabla_{\mathbb{H}} g_t, \nu\rangle\right) \ ,$$

$$S_3 = -\frac{1}{8} r^2 \quad , \quad S_4 = \frac{r^3}{16\,|\nabla_{\mathbb{H}} g|} \left(\langle H_{\mathbb{H}} g v_P, \nu\rangle + r\langle \nabla_{\mathbb{H}} g_t, \nu\rangle\right) \ ,$$

$$S_5 = \frac{r^3}{32\,|\nabla_{\mathbb{H}} g|} \left(\det H_{\mathbb{H}} g + r\langle H_{\mathbb{H}} g v_P, \nu\rangle\right) \ .$$

For $i = 1, \ldots, 6$ we denote also:

$$l_1(s) = -\cos(s) + \frac{1}{2}\sin^2(s) + 1 \quad , \quad l_2 = s\sin(s) + \cos(s) - 1 \ ,$$

$$l_3(s) = \frac{1}{2}(s - \cos(s)\sin(s)) \quad , \quad l_4(s) = (-s\cos(s) + \sin(s)) \ ,$$

$$l_5(s) = (s - \frac{3}{2}\sin(s) + \frac{1}{2}s\cos(s)) \quad , \quad l_6(s) = (-\cos(s) - \frac{1}{2}\sin^2(s) + 1) \ .$$

The main results of the paper [13] are the following.

Theorem 3.1. *Let $\Omega \subset \mathbb{H}$ be an open subset with C^3 boundary with positive reach. For every set $\Sigma \subset \partial\Omega$, let $T_\epsilon = F(\Sigma, \epsilon)$. Then there exists $\epsilon_0 > 0$ such that, for every positive $\epsilon < \epsilon_0$,*

$$|\Omega_\epsilon \cap T_\epsilon| = |\Omega \cap T_\epsilon| + \sum_{k=0}^{6} \int_\Sigma \frac{pS_k}{4}\, l_k(p\epsilon)\sqrt{\langle X, n\rangle^2 + \langle Y, n\rangle^2}\; d\mathcal{H}_E^{(2)}(\sigma) \ .$$

Moreover

Corollary 3.1. *Let $\Omega \subset \mathbb{H}$ be an open subset with C^3 boundary with positive reach. Then there exists $\epsilon_0 > 0$ such that, for every positive $\epsilon < \epsilon_0$,*

$$|\Omega_\epsilon| = |\Omega| + \sum_{k=0}^{6} \int_{\partial\Omega} \frac{pS_k}{4}\, l_k(p\epsilon)\; d\mathcal{H}_{CC}^{(3)} \ . \tag{6}$$

Let us denote here

$$h_{\partial\Omega} = \mathrm{div}_{\mathbb{H}}(\nu) \ ,$$

where ν is the intrinsic normal vector to the surface $\partial\Omega$.

Theorem 3.2. *Let Ω be a C^3 non-characteristic set locally given by $\{g(u,v,t)=0\}$. Then there exist a sequence of functions $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$, $\mathcal{C}_k=\mathcal{C}_k(x)$ on $\partial\Omega$ and $\epsilon_0>0$, such that, for every positive $\epsilon<\epsilon_0$,*

$$|\Omega_\epsilon| = |\Omega| + \sum_{k=1}^{\infty}\left[\int_{\partial\Omega}\mathcal{C}_k\, d\mathcal{H}^{(3)}_{CC}\right]\epsilon^k \ , \tag{7}$$

where $\mathcal{C}_1=1$, $\mathcal{C}_2=(1/2)\,h_{\partial\Omega}$ and for, $j\in\mathbb{N}\setminus\{0\}$,

$$\mathcal{C}_{2j} = \frac{(-1)^j p^{2j+1}}{4(2j)!}\,(-2S_1+(1-2j)S_2)$$

and

$$\mathcal{C}_{2j+1} = \frac{(-1)^j p^{2j+2}}{4(2j+1)!}\,(-2^{2j-1}S_3-2jS_4+(j-1)S_5)\ .$$

The following result is proved in [14].

Theorem 3.3. *For every positive numbers r,ϵ the following formula holds:*

$$|B(0,r+\epsilon)| =$$
$$= (16\pi\beta+\frac{3}{2\pi^2})r^4 + 2(32\pi\beta+\frac{3}{\pi^2})r^3\epsilon + 2(48\pi\beta+\frac{9}{2\pi^2})r^2\epsilon^2 +$$
$$+2(32\pi\beta+\frac{3}{\pi^2})r\epsilon^3 + 2(8\pi\beta+\frac{3}{4\pi^2})\epsilon^4\ ,$$

where

$$\beta = \frac{4\mathrm{Si}(2\pi)\pi^3-9+2\pi^2}{96\pi^3}\ .$$

Moreover

$$a_0 = |B(0,1)| = \frac{2\mathrm{Si}(2\pi)\pi+1}{3} = (16\pi\beta+\frac{3}{2\pi^2})\ ,$$
$$a_1 = \mathcal{H}^3(\partial B(0,1)) = 4\,|B(0,1)| = 64\pi\beta+\frac{6}{\pi^2}$$
$$a_2 = 2(48\pi\beta+\frac{9}{2\pi^2}) \quad , \quad a_3 = 2(32\pi\beta+\frac{3}{\pi^2})$$

and

$$a_4 = 2(8\pi\beta+\frac{3}{4\pi^2})\ .$$

4. Sketches of the proofs of Theorem 3.1, Theorem 3.2 and Theorem 3.3

We shall argue miming the idea contained in the paper by Federer, [11], in the Euclidean spaces for sets of *positive reach.*

Federer simply considered the map $F:\mathbb{R}^+\times\partial\Omega\to\mathbb{R}^n$ defined as follows

$$F(t,P) = x + tn(P)\in\mathbb{R}^n\ ,$$

where $\Omega\subset\mathbb{R}^3$ is regular enough and n is the unit normal vector, pointing outward, at the point P of the boundary, see also [18]. Indeed such map is the metric normal associated with the set $\partial\Omega$ embedded in the Euclidean space $\mathbb{R}^3$. For every set $\Sigma\subset\partial\Omega$ we denote $F_\epsilon(\Sigma)=F(\Sigma,\epsilon)$.

Thus we know that, for $\Omega\subset\mathbb{R}^3$

$$\left|\{x\in\mathbb{R}^3\ :\ \mathrm{dist}_E(x,\Omega)<\epsilon\}\cap T^E_\epsilon\right| = \left|\Omega\cap T^E_\epsilon\right| + \int_{Q\times]0,\epsilon[}\left|\det JF_E(x,y,s)\right|\,dx\,dy\,ds\ ,$$

where

$$T_\epsilon^E = F_E(Q,\epsilon)\ ,$$

$$F_E\ :\ Q\times[-\epsilon,\epsilon]\to\mathbb{R}^3\quad ,\quad F_E(x,y,s)=\phi(x,y)+sn(\phi(x,y))$$

and $\phi : Q\to\Sigma$, $Q\subset\mathbb{R}^2$, is a parametrisation of Σ. Thus, developing $|\det JF_E(x,y,s)|$ we get, in the Euclidean case,

$$\int_{Q\times]0,\epsilon[}|\det JF_E(x,y,s)|\ dx\,dy\,ds=\sum_{i=k}^{3}\int_\Sigma C_i^E(\sigma)\ d\mathcal{H}_E^{(2)}(\sigma)\epsilon^k$$

where $C_1^E=1$, S_2^E is proportional to the (Euclidean) mean curvature of $\partial\Omega$ and S_3^E is proportional to the (Euclidean) Gauss curvature of Ω.

Sketch of proof of Theorem 3.1. Actually, in the Heisenberg group, we just consider bounded set Ω with smooth boundary $\partial\Omega$ without characteristic points. The key tool here is the *metric normal*, see [1], associated with the sub-Riemannian geometry of $\partial\Omega$ in the Heisenberg group.

To any given set Ω with smooth boundary $\partial\Omega$ we consider the set

$$\Omega_\epsilon=\{x\in\mathbb{H}\ :\ d_{CC}(x,\Omega)<\epsilon\}\ .$$

Let $\phi : Q\to\mathbb{R}^3$ be a parametrization of a piece Σ of the boundary such that

$$\phi(x,y)=(x,y,\psi(x,y))\ ,$$

where $\psi : Q\to\mathbb{R}$ is a C^3 function, and $Q\subset\mathbb{R}^2$ an open subset such that $\phi(Q)=\Sigma$. Assume that $\partial\Omega$ do not have characteristic points, then recalling [1], we set

$$F : Q\times[-\epsilon,\epsilon]\to\mathbb{H}\quad ,\quad F(x,y,s)=\phi(x,y)\circ\mathcal{N}\Sigma_{\phi(x,y)}(s)\ .$$

Let $T_\epsilon=F(Q\times]-\epsilon,\epsilon[)$. Then

$$|\Omega_\epsilon\cap T_\epsilon|=|\Omega\cap T_\epsilon|+\int_{Q\times]0,\epsilon[}|\det JF(x,y,s)|\ dx\,dy\,ds\ .$$

Now we have to compute $|\det JF(x,y,s)|$. Indeed the following result holds, see [13].

Lemma 4.1. *Let $\partial\Omega$ be a C^3 non-characteristic surface locally graph of a function. Assume that $\partial\Omega=\{g=0\}$. Then there exists a positive number t_0 such that for every $\tau\in(0,t_0]$:*

$$|\det J\exp_{\partial\Omega}(P(u,v),\tau)|=$$
$$=\frac{4g_t}{|\nabla_{\mathbb{H}}g|}\ \{-(S_3-S_5)-S_3\cos^2\alpha+2S_1\sin\alpha-$$
$$-S_5\cos\alpha+\alpha(S_2\cos\alpha+S_4\sin\alpha-\frac{1}{2}S_5\sin\alpha)\}\ ,$$

where locally $\partial\Omega$ is represented by the C^3 function $P : Q\subset\mathbb{R}^2\to\mathbb{H}$, i.e. $\partial\Omega=P(Q)$.

Now by changing the order of integration and collecting the addends we get the formula of the Theorem 3.1, see [13].

Sketch of the proof of the Theorem 3.2. As far as Theorem 3.2 concerns, it is enough to develop the trigonometric polynomial in the formula proved in Theorem 3.1. Indeed there are not characteristic points on the surface. So by collecting the new coefficients we get the result, see [13].

Sketch of the proof of the Theorem 3.3. This proof requires to split in two parts the computation of the volume of $B(0, r + \epsilon) \setminus B(0, r)$. Indeed the boundary of the Carnot-Charathéodory ball is not smooth. In particular, in the computation, we always use the properties of the metric normal. Nevertheless we have to consider the effects of the cut locus of the Carnot-Charathéodory ball. Fortunately a symmetric argument helps us and we can achieve the result stated in Theorem 3.3, see [14].

5. Remarks

Formula (6) in Theorem 3.2 is very different to the classic Steiner formula. Indeed it depends on ϵ as a trigonometric polynomial. It is not an algebraic polynomial, as in the classic Steiner representation. Nevertheless formula (7) is interesting because the first and the second addend are respectively the volume of Ω and the intrinsic Hausdorff measure of $\partial\Omega$. Moreover the third addend is proportional to the integral of the mean curvature against the intrinsic Hausdorff measure $\mathcal{H}^{(3)}$, what we could call, inappropriately, the sub-quermassintegral associated with the mean sub-curvature $h_{\partial\Omega}/2$. Eventually we have other new invariant objects, respectively

$$\int_{\partial\Omega} \mathcal{C}_3 \, d\mathcal{H}^{(3)}_{CC}$$

and

$$\int_{\partial\Omega} \mathcal{C}_4 \, d\mathcal{H}^{(3)}_{CC} \, .$$

The density of the previous integrals can be related with higher curvatures. Indeed e.g. in the Euclidean frame, the fourth integral density is proportional to the Gauss curvature of $\partial\Omega$.

Could we consider $\mathcal{C}_4$ a sort of intrinsic Gauss curvature of $\partial\Omega$ in the Heisenberg group?

Remark 5.1. Notice, in particular, that

$$\mathcal{C}_3 = -\frac{p^2}{6} + \frac{1}{3}\,\frac{p}{|\nabla_{\mathbb{H}} g|}\langle H_{\mathbb{H}} g \nu, v_P\rangle - \frac{2}{3}\,\langle \frac{\nabla_{\mathbb{H}} g_t}{|\nabla_{\mathbb{H}} g|}, v_P\rangle \, .$$

and

$$\mathcal{C}_4 = \frac{-1}{6} p \left\{ p\,\frac{\Delta_{\mathbb{H}} g}{4\,|\nabla_{\mathbb{H}} g|} - 3\,\frac{\langle H_{\mathbb{H}} \nu, \nu, \rangle}{|\nabla_{\mathbb{H}} g|^2} + 2\,\langle \frac{\nabla_{\mathbb{H}} g_t}{|\nabla_{\mathbb{H}} g|}, \nu\rangle \right\} \, .$$

As a consequence the following result concerning an approximated Steiner formula holds.

Corollary 5.1. *Let Ω be a C^3 non-characteristic set. Then*

$$|\Omega_\epsilon| = |\Omega| + \sum_{k=1}^{4} \left[\int_{\partial\Omega} \mathcal{C}_k \, d\mathcal{H}^{(3)}_{CC} \right] \epsilon^k + o(\epsilon^4) \, , \tag{8}$$

as $\epsilon \to 0$.

We remark that a classic algebraic Steiner formula in the Heisenberg group, at least for Carnot-Charathéodory balls, exists; see formula (2). Unfortunately Carnot-Charatheodory balls are not sets of positive (intrinsic) reach, so we need more careful computation to obtain some further information on the five coefficients a_i, $i = 0, \ldots, 4$. In particular we would like to know if it is possible to represent such coefficients with respect to the integrals of intrinsic curvatures, like those of previous formula, against intrinsic Hausdorff measure $\mathcal{H}^{(3)}$. Such subject has been studied in [14]. In conclusion, we think that such geometric approach could give some useful information about a more general and accurate notion of intrinsic curvature. We are concerned, for example, with the *measures of curvatures* introduced by Federer in the Euclidean setting, see [11]. Indeed, in a forthcoming paper, we shall investigate such notion in the Heisenberg group, and possibly in a more general sub-Riemannian structure.

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