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Kolmogorov equations arising in finance:
direct and inverse problems

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Abstract[1]. Recent results about linear partial differential equations of Kolmogorov type are reviewed. They are examined in the context of financial mathematics: specifically, applications to arbitrage valuation, model calibration and estimation of stochastic processes are discussed.

1. Introduction

We consider a class of the differential equations of Kolmogorov type of the form

$$Lu := \sum_{i,j=1}^{p_0} a_{ij}(z)\partial_{x_i x_j} u + \sum_{i=1}^{p_0} a_i(z)\partial_{x_i} u + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u + c(z)u - \partial_t u = 0 , \tag{1.1}$$

where $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $1 \le p_0 \le N$. By convenience, hereafter the term "Kolmogorov equation" will be shortened to KE. We assume the following hypotheses:

[H.1] the matrix $A_0 = (a_{ij})_{i,j=1,\cdots,p_0}$ is symmetric and uniformly positive definite in $\mathbb{R}^{p_0}$: there exists a positive constant Λ such that

$$\frac{|\eta|^2}{\Lambda} \le \sum_{i,j=1}^{p_0} a_{ij}(z)\eta_i\eta_j \le \Lambda|\eta|^2 \quad , \quad \eta \in \mathbb{R}^{p_0},\ z \in \mathbb{R}^{N+1} ; \tag{1.2}$$

[H.2] the matrix $B := (b_{ij})$ has constant real entries and takes the following block from:

$$\begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_r \\ * & * & * & \cdots & * \end{pmatrix} \tag{1.3}$$

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where B_j is a $p_{j-1} \times p_j$ matrix of rank p_j, with

$$p_0 \geq p_1 \geq \cdots \geq p_r \geq 1 \quad , \quad p_0 + p_1 + \cdots + p_r = N \ ,$$

and the $*-$blocks are arbitrary.

The regularity hypotheses on the coefficients a_{ij}, a_i, c will be specified later: roughly speaking, we assume the Hölder continuity with respect to some homogeneous norm naturally induced by the equation.

The prototype of (1.1) is the following equation

$$\partial_{x_1 x_1} u + x_1 \partial_{x_2} u - \partial_t u = 0 \quad , \quad (x_1, x_2, t) \in \mathbb{R}^3 \ , \tag{1.4}$$

whose fundamental solution was explicitly constructed by Kolmogorov [23]. In his celebrated paper [20], Hörmander generalized this result to *constant coefficients KEs*, i.e. equations of the form (1.1), with constant a_{ij} and $a_i = c \equiv 0$ for $i = 1, \cdots, p_0$, satisfying the following condition:

(1.5) Ker(A) *does not contain non-trivial subspaces which are invariant for* B .

In (1.5), A denotes the $N \times N$ matrix

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} . \tag{1.6}$$

For constant coefficients equations, condition (1.5) is equivalent to the structural assumptions [H.1]-[H.2] which in turn are equivalent to the classical Hörmander condition:

$$\text{rank Lie}\,(X_1, \cdots, X_{p_0}, Y) = N + 1 \ , \tag{1.7}$$

at any point of $\mathbb{R}^{N+1}$. In (1.7), Lie$\,(X_1, \cdots, X_{p_0}, Y)$ denotes the Lie algebra generated by the vector fields

$$X_i = \sum_{j=1}^{p_0} a_{ij} \partial_{x_j} \quad , \quad i = 1, \cdots, p_0, \quad \text{and} \quad Y = \langle x, BD \rangle - \partial_t \ , \tag{1.8}$$

where $\langle \cdot, \cdot \rangle$ and D respectively denote the inner product and the gradient in $\mathbb{R}^N$. A proof of the equivalence of these conditions is given by Kupcov in [24], Theorem 3 and by Lanconelli and Polidoro in [26], Proposition A.1.

Linear KEs naturally arise in mathematical finance in some generalization of the celebrated Black&Scholes model [10]. Consider a "stock" whose price S_t is modeled as the solution to the stochastic differential equation

$$d\,S_t = \mu S_t\,dt + \sigma S_t\,dW_t \ , \tag{1.9}$$

where μ and σ are positive constants and W_t is a Wiener process. Also consider a "bond" whose price B_t only depends on a constant interest rate r:

$$B_t = B_0 e^{rt} \ .$$

Finally, consider an "European option" which is a contract which gives the *right* (but not the *obligation*) to buy the stock at a given "strike price" E and at a given "expiry date" T. The problem studied in [10] is to find a fair price of the option contract. Under some assumptions on the financial market, Black&Scholes show that the price $V(t, S_t)$ of the option is the solution of the following parabolic equation

$$-rV + \frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} = 0$$

in the domain $(S,t) \in \mathbb{R}^+ \times]0,T[$, subject to the *final condition*

$$V(T,S_T) = \max(S_T - E, 0) .$$

More simply, the forward price of the option $u = e^{r(T-t)}V$ as a function of the time to maturity $\tau = T - t$ and of the forward log-prices $x_\tau = \log(e^{r\tau}S_{T-\tau})$, satisfies the PDE

$$\frac{1}{2}\,\sigma^2(\partial_{xx}u - \partial_x u) - \partial_\tau u = 0 ,$$

on $(x,\tau) \in \mathbb{R} \times]0,T]$ with initial condition

$$u(0,x) = \max(e^x - E, 0) .$$

In the last decades the Black&Scholes theory has been developed by many authors and mathematical models involving KEs have appeared in the study of the so-called path-dependent contingent claims (see, for instance, [6], [7], [8] and [34]). *Asian options* are derivatives whose exercise price is not fixed as a given constant E, but depends on some average of the history of the stock price. In this case, the value of the option at the expiry time T is (for a a geometric average option):

$$V(S_T,M_T) = \max\left(S_T - e^{M_T/T}, 0\right) \quad , \quad M_t = \int_0^t \log(S_\tau)\, d\tau .$$

If we suppose by simplicity that the interest rate is $r = 0$, the Black&Scholes method leads to the following degenerate equation

$$\frac{1}{2}\,\sigma^2 S^2 \partial_{SS}V + (\log S)\partial_M V + \partial_t V = 0 \quad , \quad S,t > 0,\ M \in \mathbb{R} ,$$

which, by considering the option price $V = u(\tau,x,M)$ in terms of log-prices $x = \log(S)$ and time-to-maturity $\tau = T - t$, reduces to the KE (cf. [9])

$$\frac{1}{2}\,\sigma^2(\partial_{xx}u - \partial_x u) + x\partial_M u - \partial_\tau u = 0 \quad , \quad t > 0 \text{ and } x, M \in \mathbb{R} . \tag{1.10}$$

Based on the approximation scheme introduced by Polidoro and Mogavero in [31], a numerical study of the solution of the Cauchy problem related to (1.10) is also proposed in [9].

A recent motivation in finance comes from the model by Hobson&Rogers [19]. In the Black&Scholes theory the hypothesis that the volatility σ in the stochastic differential equation (1.9) is constant contrasts with the empirical observations. Aiming to overcome this problem, many authors proposed alternative theories which generally lead to incomplete market models. On the contrary, the model proposed by Hobson and Rogers assumes that the volatility only depends on the difference between the present stock price and an average of past prices. This simple model seems to capture the features observed in the market and avoid the problems related to the use of many sources of randomness.

The path dependent volatility model proposed in [18] generalizes the Hobson& Rogers model by defining the average factor M_t as

$$M_t = \frac{1}{\Phi(t)} \int_{-\infty}^t \varphi(s) Z_s\, ds , \tag{1.11}$$

where $Z_t = \log(S_t)$ are the forward log-prices, the average weight φ is a non-negative, piecewise continuous function, integrable on $]-\infty, T]$, strictly positive on $[0,T]$ and $\Phi(t) = \int_{-\infty}^{t} \varphi(s)\ ds$. The process with components Z_t and M_t is Markovian and defined by the SDEs

$$dZ_t = \mu(Z_t - M_t)dt + \sigma(Z_t - M_t)dW_t\ , \tag{1.12a}$$

$$dM_t = g(t)(Z_t - M_t)dt\ , \tag{1.12b}$$

where $g(t) = \varphi(t)/\Phi(t)$. The original Hobson&Rogers model corresponds to $\varphi(t) = e^{\lambda t}$, $g(t) = \lambda$.

As in the study of Asian options, in the Hobson&Rogers model for European options the value of the option $V(t, Z_t, M_t)$ which depends on time t, on the log-price of the stock Z_t, on the average M_t must satisfy the KE with non-constant coefficients

$$\frac{1}{2}\,\sigma^2(Z-M)\,(\partial_{ZZ}V - \partial_Z V) + g(t)(Z-M)\partial_M V + \partial_t V = 0\ . \tag{1.13}$$

The next section reports some results on Kolmogorov equations with constant coefficients and their extensions to the non-constant and non-homogeneous case (cf. [33, 21, 32, 28, 29, 15]).

The third section reviews numerical approximation studies on the Cauchy problems related to (1.10) and (1.13) which appeared in [9, 14, 13]. The numerical schemes proposed in these papers rely on the approximation of the directional derivative Y by the finite difference

$$-\frac{u(x,y,t) - u(x, y+\delta x, t-\delta)}{\delta}\ .$$

This method, which is respectful of the non-Euclidean geometry of the Lie group, seems to provide a good approximation of the solution.

The fourth section considers the inverse problem of determining the coefficient functions $\sigma^2(S,M)$ and $g(S,M)$ in (1.13) knowing the function V at a finite set of points (cf. [2, 11, 17]).

2. Kolmogorov equations

We call *constant coefficients* KE any equation of the form (1.1), with constant a_{ij}, null a_i and satisfying hypothesis [H.1]-[H.2]. These KEs have the remarkable property of being invariant with respect to the left translations in the law defined by

$$(x,t)\circ(\xi,\tau) = (\xi + E(\tau)x, t+\tau)\quad ,\quad (x,t),(\xi,\tau)\in\mathbb{R}^N\times\mathbb{R}\ , \tag{2.1}$$

where

$$E(t) = e^{-tB^T}\ . \tag{2.2}$$

Moreover, let us consider the family of dilations $(D(\lambda))_{\lambda>0}$ on $\mathbb{R}^{N+1}$ defined by

$$D(\lambda) := \left(D_0(\lambda), \lambda^2\right) = \operatorname{diag}\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, \cdots, \lambda^{2r+1} I_{p_r}, \lambda^2\right)\ , \tag{2.3}$$

where I_{p_j} denotes the $p_j \times p_j$ identity matrix. It is known that if (and only if) all the $*$-blocks in (1.3) are zero matrices, then L is also homogeneous of degree two with respect to $(D(\lambda))$ in the sense that

$$L\circ D(\lambda) = \lambda^2 (D(\lambda)\circ L)\quad ,\quad \lambda > 0\ .$$

We remark explicitly that $\mathcal{G}_B := \left(\mathbb{R}^{N+1}, \circ, D(\lambda)\right)$ is a homogeneous Lie group only determined by the matrix B.

In some particular cases, variable coefficients KEs were first studied by Weber [33], Il'in [21] and Sonin [32] who used the parametrix method to construct a fundamental solution. Yet in these papers unnecessary restrictive conditions on the regularity of the coefficients are required. Assuming that the KE in (1.1) satisfies the hypotheses [H.1]-[H.2] and that the $*$-blocks in (1.3) are zero matrices, the previous results were considerably generalized in a series of papers by Polidoro [28], [29], [30], by assuming a notion of regularity modeled on the homogeneous Lie group $\mathcal{G}_B$ (see Definitions 2.2 and 2.3 below). We also refer to [25] for a survey of the most recent results about KEs. The results, in the form here reported, in Theorem 2.4 are due to Di Francesco and Pascucci [15].

In order to state our main results, we recall the definition of homogeneous norm and B-Hölder continuity given by Polidoro [28].

Definition 2.1. Given a constant matrix B of the form (1.3) and $(D(\lambda))_{\lambda>0}$ defined as in (2.3), let $(q_j)_{j=1,\cdots,N}$ be such that

$$D(\lambda) = \mathrm{diag}(\lambda^{q_1}, \lambda^{q_2}, \cdots, \lambda^{q_N}, \lambda^2) .$$

For every $z = (x,t) \in \mathbb{R}^{N+1}$, we set

$$|x|_B = \sum_{j=1}^{N} |x_j|^{1/q_j} \quad \text{and} \quad \|z\|_B = |x|_B + |t|^{1/2} . \tag{2.4}$$

Clearly $||\cdot||_B$ is a norm on $\mathbb{R}^{N+1}$ homogeneous of degree one with respect to the dilations $(D(\lambda))$.

Definition 2.2. We say that a function f is *B-Hölder continuous of order* $\alpha \in]0,1]$ on a domain Ω of $\mathbb{R}^{N+1}$, and we write $f \in C^{\alpha}_{B}(\Omega)$, if there exists a constant C such that

$$|f(z) - f(\zeta)| \le C||\zeta^{-1} \circ z||^{\alpha}_{B} \quad , \quad z, \zeta \in \Omega . \tag{2.5}$$

In (2.5), ζ^{-1} denotes the inverse of ζ in the law "$\circ$" in (2.1).

Next, we give the definition of solution to equation $Lu = f$.

Definition 2.3. We say that a function u is a *solution* to the equation $Lu = f$ in a domain Ω of $\mathbb{R}^{N+1}$, if there exist the Euclidean derivatives $\partial_{x_i}u, \partial_{x_i x_j}u \in C(\Omega)$ for $i, j = 1, \cdots, p_0$, the Lie[2] derivative $Yu \in C(\Omega)$ and equation

$$\sum_{i,j=1}^{p_0} a_{ij}(z)\partial_{x_i x_j}u(z) + \sum_{i=1}^{p_0} a_i(z)\partial_{x_i}u(z) + Yu(z) + c(z)u(z) = f(z)$$

is satisfied at any $z \in \Omega$.

We are now in position to state the following

[2]A function u is Lie differentiable w.r.t. the vector field Y in (1.8), at the point $z = (x,t)$, if there exists

$$\lim_{\delta\to 0} \frac{u(\gamma(\delta)) - u(\gamma(0))}{\delta} =: Yu(z) ,$$

where γ denotes the integral curve of Y from z:

$$\gamma(\delta) = (E(-\delta)x, t - \delta) \quad , \quad \delta \in \mathbb{R} .$$

Clearly, if $u \in C^1$ then $Yu(x,t) = \langle x, BDu(x,t)\rangle - \partial_t u(x,t)$.

Theorem 2.4. *Assume that L in* (1.1) *verifies hypotheses [H.1]-[H.2] and that the coefficients $a_{ij}, a_i, c \in C_B^\alpha(\mathbb{R}^{N+1})$ are bounded functions. Then there exists a fundamental solution Γ to L with the following properties:*

1. $\Gamma(\cdot,\zeta) \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1} \setminus \{\zeta\})$ *for every* $\zeta \in \mathbb{R}^{N+1}$;
2. $\Gamma(\cdot,\zeta)$ *is a solution to* $Lu = 0$ *in* $\mathbb{R}^{N+1} \setminus \{\zeta\}$ *for every* $\zeta \in \mathbb{R}^{N+1}$ (*in the sense of Definition 2.3*);
3. *let* $g \in C(\mathbb{R}^N)$ *such that*
$$|g(x)| \le C_0 e^{C_0|x|^2} \quad , \quad x \in \mathbb{R}^N \, , \tag{2.6}$$
for some positive constant C_0. *Then there exists*
$$\lim_{t \to \tau^+} \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau) g(\xi)\, d\xi = g(x) \quad , \quad x \in \mathbb{R}^N, \ \tau \in \mathbb{R} \, ;$$
4. *let* $g \in C(\mathbb{R}^N)$ *verifying* (2.6) *and* f *be a continuous function in the strip* $S_{T_0,T_1} = \mathbb{R}^N \times]T_0, T_1[$, *such that*
$$|f(x,t)| \le C_1 e^{C_1|x|^2} \quad , \quad (x,t) \in S_{T_0,T_1} \tag{2.7}$$
and for any compact subset M *of* $\mathbb{R}^N$ *there exists a positive constant* C *such that*
$$|f(x,t) - f(y,t)| \le C|x-y|_B^\beta \quad , \quad x,y \in M, \ t \in]T_0,T_1[\, ,$$
for some $\beta \in]0,1[$. *Then there exists* $T \in]T_0, T_1]$ *such that the function*
$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t,\xi,T_0) g(\xi)\, d\xi - \int_{T_0}^t \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \tag{2.8}$$
is a solution to the Cauchy problem
$$\begin{cases} Lu = f & \text{in } S_{T_0,T} \, , \\ u(\cdot,T_0) = g & \text{in } \mathbb{R}^N \, ; \end{cases} \tag{2.9}$$
5. *if u is a solution to the Cauchy problem* (2.9) *with null f and g, and verifies estimate* (2.7), *then* $u \equiv 0$ (*see also Theorem 2.6 below*). *In particular, the function in* (2.8) *is the unique solution to problem* (2.9) *verifying estimate* (2.7);
6. *the reproduction property holds:*
$$\Gamma(x,t,\xi,\tau) = \int_{\mathbb{R}^N} \Gamma(x,t,y,s) \Gamma(y,s,\xi,\tau)\, dy \quad , \quad x,\xi \in \mathbb{R}^N, \ \tau < s < t \, ;$$
7. *if* $c(z) \equiv c$ *is constant then*
$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\, d\xi = e^{-c(t-\tau)} \quad , \quad x \in \mathbb{R}^N, \ \tau < t \, ;$$
8. *let* Γ^ε *denote the fundamental solution to the constant coefficients KE*
$$L^\varepsilon = (\Lambda + \varepsilon)\Delta_{\mathbb{R}^{p_0}} + \langle x, B\nabla \rangle - \partial_t$$

where $\varepsilon > 0$, Λ is as in (1.2) *and $\Delta_{\mathbb{R}^{p_0}}$ denotes the Laplacian in the variables $x_1, \cdots, x_{p_0}$. Then for every positive ε and T, there exists a constant C, only dependent on Λ, B, ε and T, such that*

$$\Gamma(z,\zeta) \le C\Gamma^\varepsilon(z,\zeta) \; ,$$

$$|\partial_{x_i}\Gamma(z,\zeta)| \le \frac{C}{\sqrt{t-\tau}}\, \Gamma^\varepsilon(z,\zeta) \; ,$$

$$\left|\partial_{x_i x_j}\Gamma(z,\zeta)\right| \le \frac{C}{t-\tau}\, \Gamma^\varepsilon(z,\zeta) \quad , \quad |Y\Gamma(z,\zeta)| \le \frac{C}{t-\tau}\, \Gamma^\varepsilon(z,\zeta) \; ,$$

for any $i, j = 1, \cdots, p_0$ and $z, \zeta \in \mathbb{R}^{N+1}$ with $0 < t - \tau < T$.

Under the further hypothesis

[H.3] for every $i, j = 1, \cdots, p_0$, there exist the derivatives $\partial_{x_i} a_{ij}, \partial_{x_i x_j} a_{ij}, \partial_{x_i} a_i \in C^\alpha_B(\mathbb{R}^{N+1})$ and are bounded functions,

we define as usual the adjoint operator L^* of L:

$$L^* v = \sum_{i,j=1}^{p_0} a_{ij}\partial_{x_i x_j} v + \sum_{i=1}^{p_0} a_i^* \partial_{x_i} v - \langle x, B\nabla v\rangle + c^* v + \partial_t v$$

where

$$a_i^* = -a_i + 2\sum_{j=1}^{p_0} \partial_{x_i} a_{ij} \quad , \quad c^* = c + \sum_{i,j=1}^{p_0} \partial_{x_i x_j} a_{ij} - \sum_{i=1}^{p_0} \partial_{x_i} a_i - \mathrm{tr}(B) \; .$$

Theorem 2.5. *There exists a fundamental solution Γ^* of L^* verifying the dual properties in the statement of Theorem 2.4. Moreover it holds*

$$\Gamma^*(z,\zeta) = \Gamma(\zeta, z) \quad , \quad z, \zeta \in \mathbb{R}^{N+1}, \; z \ne \zeta \; .$$

We close this section by stating a uniqueness result.

Theorem 2.6. *Assume that L in* (1.1) *verifies the hypotheses* [*H.1*]-[*H.2*]-[*H.3*] *and that the coefficients $a_{ij}, a_i, c \in C^\alpha_B(\mathbb{R}^{N+1})$ are bounded functions. If u is a solution to the Cauchy problem* (2.9) *with null f and g, such that*

$$\int_{T_0}^{T}\int_{\mathbb{R}^N} |u(x,t)| e^{-C|x|^2}\, dx\, dt < +\infty$$

for some positive constant C, then $u \equiv 0$.

3. Numerical approximations

We investigate the numerical solution of the equation (1.13). More generally, we consider the following KE

$$Lu := \frac{1}{2}\,\sigma^2(x,y)(\partial_{xx}u - \partial_x u) + g(t)(x-y)\partial_y u - u_t = 0 \; , \tag{3.1}$$

where $g \in C^1$ is strictly positive on $[0,T]$.

In the numerical approximation the best results are obtained by considering the main directional derivatives of L which are given by the operators ∂_x and Y defined as

$$Yu = g(t)(x-y)\partial_y u - \partial_t u \; . \tag{3.2}$$

We consider the approximation of (3.1) on the uniform grid

$$G := \{(i\Delta_x, j\Delta_y, n\Delta_t) \mid i, j, n \in \mathbb{Z},\ n \geq 0\}\ , \tag{3.3}$$

we approximate the derivatives $\partial_{xx}u$ and $\partial_x u$ by the three point and centered two point schemes, respectively:

$$\partial_{xx}u \ \sim\ D^2_{\Delta_x}u(x,y,t) := \frac{u(x+\Delta_x,y,t) - 2u(x,y,t) + u(x-\Delta_x,y,t)}{\Delta_x^2} \tag{3.4}$$

and

$$\partial_x u \ \sim\ D_{\Delta_x}u(x,y,t) := \frac{u(x+\Delta_x,y,t) - u(x-\Delta_x,y,t)}{2\Delta_x}\ . \tag{3.5}$$

Thus, the approximation of $\partial_{xx}u - \partial_x u$ is of the order of Δ_x^2.

The second main directional derivative Y is approximated by

$$Yu(x,y,t) \ \sim\ Y_{\Delta_t}u(x,y,t) := \frac{\widetilde{u}(x, y + g(t)(x-y)\Delta_t, t-\Delta_t) - u(x,y,t)}{\Delta_t} \tag{3.6}$$

where $\widetilde{u}(x,y,t)$ denotes the linear interpolation of $u(x,y,t)$ based on the two nearest grid points. The function $\widetilde{u}(x,y,t)$ approximates $u(x,y,t)$ with an error of the order of Δ_y^2 which depends on the L^∞ norm of $\partial_{yy}u$ on the domain. Then approximation (3.6) is of the order of $\Delta_t + \Delta_y^2/\Delta_t$. We remark that interpolation in (3.6) is necessary because (x,y,t) and $(x, y+g(t)(x-y)\Delta_t, t-\Delta_t)$ cannot both belong to the same uniform grid. In specific cases, like the standard Hobson&Rogers model [13] and that of Asian options [9] specifically designed changes of variable allow for both the point to belong to the same grid. These approaches have the advantage of having an approximation error that does not depends on the regularity on the ∂_y direction, but at the cost of imposing the grid size condition $\Delta_y = \Delta_x\Delta_t$.

The discrete operator, defined by

$$L_G u = \frac{1}{2}\,\sigma^2(D^2_{\Delta_x}u - D_{\Delta_x}u) + Y_{\Delta_t}u\ , \tag{3.7}$$

approximates L in the sense that

$$\|L - L_G u\|_{L^\infty} \leq C(\Delta_x^2 + \Delta_t + \Delta_y^2/\Delta_t) \tag{3.8}$$

for some positive constant C depending on the L^∞-norms of σ, g', $\partial_{xxx}u$, $\partial_{yy}u$, $\partial_{xxxx}u$, Y^2u, $\partial_{xx}Yu$ and $\partial_{xxyy}u$ on the domain. The resulting scheme is of an implicit type and it is unconditionally stable (cf. [13, 14, 17]).

4. Inverse problems in finance

In option pricing the coefficients σ and g typically are not known or directly observable. Thus practitioners face the problem of inferring them either from historical prices of the underlying asset or from observation of related option prices.

4.1. **Estimation of diffusions.** Consider as input data a finite number of points $(Z_i, M_i) = (Z_{t_i}, M_{t_i})$, for $i = 0, \cdots, I$, in one trajectory of (Z_t, M_t) solution to (1.12). One way to estimate the coefficients functions μ, σ and g from these observations is given by the maximum-likelihood approach. In that estimation technique, roughly speaking, we want to select the model which has the highest probability of having generated the observed data. Hereafter we assume that $g = \lambda\lambda$ is constant. Denote by $U(z, m, t, z_1, m_1, t+\Delta; \mu, \sigma, g)$ the conditional probability density function of $(z, m) = (Z_t, M_t)$ conditioned on $(z_1, m_1) = (Z_{t+\Delta}, M_{t+\Delta})$.

In a time-homogeneous Markovian settings U does not depend on absolute time, that is $U(z, m, t, z_1, m_1, t + \Delta; \mu, \sigma, \lambda) = U(z, m, 0, z_1, m_1, \Delta, \mu, \sigma, \lambda)$. Under that hypothesis, Bayes' rule implies that likelihood function, that is the density of observed data, is given by

$$\mathcal{L}(\mu, \sigma, \lambda) = \prod_{i=0}^{I-1} U(Z_i, M_i, Z_{i+1}, M_{i+1}\,; \mu, \sigma, \lambda), \tag{4.1}$$

apart for a scaling constant and a factor depending on the density of the initial observation, which becomes irrelevant for I suitably large.

In a maximum-likelihood approach we want to maximize $\mathcal{L}$ for μ, σ and λ in a given functional space. Now, by the Feynman-Kac formula, U is the fundamental solution of the backward KE

$$Ku := \mu\,(z-m)\partial_z u + \lambda(z-m)\partial_m u + \frac{1}{2}\,\sigma(z-m)^2\partial_{zz}u + \partial_t u = 0\,. \tag{4.2}$$

When using a descent method for the numerical solution of this optimization problem one needs to address the following problems. Firstly, one should derive the differential of the functional U w.r.t. μ, σ and the parameter λ. In a parametric setting, the existence of the derivatives of U has been studied by Di Francesco and Pascucci in [16]. The second and more important problem is that at each step of these algorithms we need to compute the log-likelihood and its differentials. However, since the coefficients of the PDE (4.2) are not constant, the solution cannot be translated and each term in the likelihood function requires the computation of the solution of a different Cauchy problem. This becomes immediately prohibitive if finite-differences or -elements methods are adopted for the discretization.

A more promising approach consists in approximating the fundamental solution U by means of a truncated parametrix series (cf. Corielli and Pascucci [27]). Other approaches have been proposed by Aït Sahalia in [3, 4], where an Hermite polynomial expansion have been used, and by Aït Sahalia and Yu [5], who consider a saddle point approximation [12]. Alternatively like simulation methods can be used which are generally more computationally expensive or less precise (see Jensen and Poulsen for a survey [22]).

4.2. **Calibration.** We consider the second inverse problem known by financial practitioners as calibration. In this problem, we observe at time $t = 0$ a set of option prices c_i, along with their log-moneyness x_i and maturity τ_i, for $i = 1, \cdots, I$. To be consistent with the market, a model should at least be able to replicate the quoted option prices. For the path dependent volatility model, this means that the observed prices should be equal to the corresponding values $u(x, m, \tau)$ of the solution of the Cauchy problem

$$\begin{aligned} Lu := {} & \frac{1}{2}\,\sigma^2(x-m)\,(\partial_{xx}u - \partial_x u) + \\ & + g(\tau)(x-m)\partial_m u - \partial_\tau u = 0 \quad , \quad (x, m) \in \mathbb{R}^2,\ \tau > 0\,, \end{aligned} \tag{4.3a}$$

$$u(x, m, 0) = (e^x - 1)^+ \quad , \quad (x, m) \in \mathbb{R}^2\,. \tag{4.3b}$$

The functions σ and g should be chosen to reduce the calibration error defined as $c_i - u(x_i, M_0, \tau_i)$, where M_0 is the initial value of the average process M_t.

The calibration problem reduces to the constrained optimization problem of the form

$$\sup_{\sigma,g} \psi(\sigma, g) = \sum_{i=1}^{I} (c_i - u(x_i, M_0, \tau_i))^2 \tag{4.4}$$

subject to (4.3). This calibration problem is considered in [17] and [18], where a parametric approach is used. The optimization algorithm is a quasi-Newton method which requires the computation of the derivatives of u w.r.t. the parameters. These derivatives are computed by solving several Cauchy problems similar to that in (4.3), but with an additional non-homogeneous term (cf. Di Francesco and Pascucci [16]).

Here we illustrate a more efficient approach, proposed by Achdou and Pironneau in [1, 2] in the context of parabolic PDEs, for computing the derivatives of ψ. We illustrate how, instead of solving a set of Cauchy problems as in [17, 18], we can solve only one Cauchy problem based on the adjoint of the KE (4.3).

Consider an admissible "variation" $\delta\sigma^2$ and δg of the functions σ^2 and g respectively. It follows that

$$\delta\psi := \psi(\sigma^2 + \delta\sigma^2, g + \delta g) - \psi(\sigma, g) = \tag{4.5}$$

$$= 2\sum_{i=1}^{I}(c_i - u(x_i, M_0, \tau_i))\delta u(x_i, M_0, \tau_i) + O(\|\delta\sigma^2\|^2 + \|\delta g\|^2) ,$$

where δu is the solution of the Cauchy problem

$$L\,\delta u = v \quad , \quad x, m \in \mathbb{R} \text{ and } \tau > 0 , \tag{4.6a}$$

$$\delta u(x, m, 0) = 0 \quad , \quad x, m \in \mathbb{R} , \tag{4.6b}$$

where

$$v = \delta\sigma^2(\partial_{xx}u - \partial_x u) + \delta g(\tau)(x-m)\partial_m u .$$

Under conditions of Theorems 2.4-2.6, we denote by L^* the adjoint of L and consider the adjoint KE

$$L^* p = 2\sum_i (c_i - u(x_i, M_0, \tau_i))\delta_{x_i, M_0, \tau_i} , \tag{4.7}$$

with terminal condition

$$p(x, m, T) = 0 , \tag{4.8}$$

where $T \geq \tau_i$ $(1 \leq i \leq I)$ and $\delta_{x_i, M_0, \tau_i}$ denotes the Dirac's delta.

Formally (4.7) defines p as

$$p(x, m, \tau) = 2\sum_{\{i | t_i \leq t\}} (c_i - u(x_i, M_0, \tau_i))\Gamma^*(x, m, \tau; x_i, M_0, \tau_i) , \tag{4.9}$$

where Γ^* denotes the fundamental solution of the adjoint operator L^*. Now, multiplying (4.7) by δu and integrating on $\Omega = \mathbb{R}^2 \times [0, T]$, we get $\delta\psi$ on the right and side and thus

$$\delta\psi = \int_\Omega (L^* p)\ \delta u = \int_\Omega p\,(Lu) = \int_\Omega p\,v = \int_\Omega p\left(\delta\sigma^2\,(\partial_{xx}u - \partial_x u) + \delta g\,(x-m)\partial_m u\right) ,$$

up to an $O(\|\delta\sigma\|^2 + \|\delta g\|^2)$ term.

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