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Lower bounds for solutions of degenerate parabolic equations

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Abstract[1]. We show how pointwise lower bounds for positive weak solutions of degenerate parabolic equations can be derived from the intrinsic Harnack inequality they satisfy. This generalizes a result proved by Moser for linear parabolic equations with bounded and measurable coefficients.

1. Introduction

In his seminal paper [7], relying on the Harnack inequality for positive solutions, Moser proves that for linear parabolic equations with bounded and measurable coefficients, the following holds

Theorem 1.1. *There exist two positive constants A and a such that, for any x and y in $\mathbb{R}^N$, for any $0 < s < t < T$ and for any non-negative solution of*

$$u_t - \operatorname{div}(a_{ij}(x,t)Du) = 0$$

defined in $\mathbb{R}^N \times (0,T)$, we have

$$u(t,y) \geq u(s,x)\left(\frac{s}{t}\right)^a \exp\left(-A\left(1+\frac{|x-y|^2}{t-s}\right)\right) . \tag{1.1}$$

Let us remark that in the x-variable we have the well-known exponential behavior of the fundamental solution, whereas in the t-variable we have a power-like decay, which is not the optimal one. More details on this kind of results can be found also in [8]. The proof of Theorem 1.1 has been adapted by several authors to a wide class of linear degenerate parabolic operators (see [5], [6], and [9]). A lower bound for Hörmander operators in the form $\sum_{j=1}^m X_j^2 u + X_0 u = \partial_t u$ has been given in [2] by using optimal control techniques.

It is a natural question to investigate whether the same method can be used in the study of degenerate parabolic equations of p-Laplacian or porous medium type. The aim of this note is precisely to prove such a result.

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2. Lower bounds for solutions to the p-Laplacian

Let E be an open set in $\mathbb{R}^N$ and for $T>0$ let E_T denote the cylindrical domain $E\times(0,T]$. Consider quasi-linear, parabolic differential equations of the form

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = 0 \quad \text{weakly in } E_T \tag{2.1}$$

where the function $\mathbf{A}: E_T\times\mathbb{R}^{N+1}\to\mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du)\cdot Du \ge C_o|Du|^p \\ |\mathbf{A}(x,t,u,Du)| \le C_1|Du|^{p-1} \end{cases} \quad \text{a.e. in } E_T \tag{2.2}$$

where $p\ge 2$ and C_o and C_1 are given positive constants.

A function

$$u \in C_{\text{loc}}\left(0,T;L^2_{\text{loc}}(E)\right)\cap L^p_{\text{loc}}\Big(0,T;W^{1,p}_{\text{loc}}(E)\Big) \tag{2.3}$$

is a local, weak solution to (2.1)-(2.2) if for every compact set $K\subset E$ and every sub-interval $[t_1,t_2]\subset(0,T]$

$$\int_K u\varphi\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \left[-u\varphi_t + \mathbf{A}(x,t,u,Du)\cdot D\varphi\right]\,dx\,dt = 0 \tag{2.4}$$

for all bounded testing functions

$$\varphi \in W^{1,2}_{\text{loc}}\left(0,T;L^2(K)\right)\cap L^p_{\text{loc}}\left(0,T;W^{1,p}_o(K)\right)\ . \tag{2.5}$$

In [4] the following results is proved.

Theorem 2.1 (Intrinsic Harnack Inequality)**.** *Let u be a continuous, non-negative, weak solution to* (2.1)-(2.2)*. There exist positive constants C and M depending only upon the data, such that for all cylinders $K_{4\rho}(x_o)\times(t_o-4\theta\rho^p, t_o+4\theta\rho^p)$, contained in E_T,*

$$u(x_o,t_o) \le M \inf_{K_\rho(x_o)} u(x, t_o+\theta\rho^p)\ , \tag{2.6}$$

where

$$\theta = \frac{C}{u(x_o,t_o)^{p-2}}\ . \tag{2.7}$$

In the following, using (2.6)-(2.7), and some Control Theory tools, we prove a lower pointwise bound for positive local weak solutions to (2.1)-(2.2).

Theorem 2.2. *Let u be a positive weak solution of* (2.1)-(2.2) *defined in $\mathbb{R}^N$. Then for any (x,t), $(y,s)\in\mathbb{R}^N$ with $t>s$ we have*

$$u(x,t) \ge \left(\frac{s}{t}\right)^a u(y,s)\times \tag{2.8}$$

$$\times\left(1 - C\left(\frac{t}{s}\right)^{a(p-2)/(p-1)} \frac{M^{(p-2)/(p-1)}-1}{M^{(p-2)/(p-1)}u(y,s)^{(p-2)/(p-1)}}\, \frac{|x-y|^{p/(p-1)}}{(t-s)^{1/(p-1)}}\right)_+^{(p-1)/(p-2)}$$

where a is a positive parameter that depends only on the data.

Proof. We consider solutions defined in the whole $\mathbb{R}^N$, mainly for the sake of simplicity. It is important to remark that the Harnack inequality holds, provided the space-time cylinder $K_{4\rho}(x_o) \times [t_o - 4^p\theta\rho^p, t_o + 4^p\theta\rho^p] \subset \mathbb{R}^N \times]0, +\infty[$.

Let us now fix (x_o, t_o) e (y, T), assuming for the moment that $x_o \neq y$, and let us denote with $\gamma(s)$ the parametrization of the path that joins ($\gamma(0) = x_o$, $\gamma(T - t_o) = y$). Using some typical Control Theory notation, in the following we will denote $\dot\gamma$ with ω.

Let $\rho > 0$ and $t = t_o + s = t_o + \theta\rho^p$. We have trivially

$$\rho = \left(\frac{s}{\theta}\right)^{1/p} = C_1\, s^{1/p}\, u(x_o, t_o)^{(p-2)/p} \; .$$

Moreover

$$|\gamma(s) - x_o| \le \rho \quad \Rightarrow \quad |\gamma(s) - x_o| \le C_1\, s^{1/p}\, u(x_o, t_o)^{(p-2)/p}$$

and also

$$|\gamma(s) - x_o| \le \int_0^s |\omega(\tau)|\, d\tau \le s^{1/p} \left(\int_0^s |\omega(\tau)|^{p/(p-1)}\, d\tau\right)^{(p-1)/p} \; .$$

If all the quantities are upper bounded by ρ, we get

$$\int_0^s |\omega(\tau)|^{p/(p-1)}\, d\tau \le C_2 u(x_o, t_o)^{(p-2)/(p-1)} \; . \tag{2.9}$$

Let us now choose t_1, t_2, $\dots$, $t_k = T$, with $x_j = \gamma(t_j)$ and $j = 1, \dots, k$, in such a way that

$$\int_0^{t_1 - t_o} |\omega(\tau)|^{p/(p-1)}\, d\tau = C_2 u(x_o, t_o)^{(p-2)/(p-1)} \quad \Rightarrow \quad u(x_1, t_1) \ge \frac{u(x_o, t_o)}{M} \; ,$$

$$\vdots$$

$$\int_{t_j - t_o}^{t_{j+1} - t_o} |\omega(\tau)|^{p/(p-1)}\, d\tau = C_2 \left(\frac{u(x_o, t_o)}{M^j}\right)^{(p-2)/(p-1)} \le C_2 u(x_j, t_j)^{(p-2)/(p-1)}$$

$$\text{this ensures that } |\gamma(s) - x_j| \le \rho \quad \Rightarrow \quad u(x_{j+1}, t_{j+1}) \ge \frac{u(x_o, t_o)}{M^{j+1}} \; .$$

We have to determine the number k of steps we need, in order to go from (x_o, t_o) to (y, T). From the previous calculations, we get

$$k = \min\left\{ j \in \mathbb{N} \; : \; \int_0^{T - t_o} |\omega(\tau)|^{p/(p-1)}\, d\tau \le \le C_2 u(x_o, t_o)^{(p-2)/(p-1)} \left(1 + m + \dots + m^j\right) \right\} \tag{2.10}$$

where we have set

$$m = \frac{1}{M^{(p-2)/(p-1)}} \; \in]0, 1[\; .$$

Notice that $k = 0$ is not ruled out. We easily obtain

$$u(y, T) \ge M^{-k-1} u(x_o, t_o) \; . \tag{2.11}$$

Now we have to evaluate

$$\min \int_0^{T-t_o} |\omega(\tau)|^{p/(p-1)}\, d\tau \equiv \min \int_0^{T-t_o} |\dot\gamma(\tau)|^{p/(p-1)}\, d\tau$$

with the limit conditions $\gamma(0) = x_o$, $\gamma(T-t_o) = y$. From the Euler-Lagrange equation we get

$$\dot\gamma(\tau) = cost \quad\Rightarrow\quad \gamma(\tau) = \frac{y-x_o}{T-t_o}\,\tau + x_o\ .$$

Correspondingly

$$\min \int_0^{T-t_o} |\omega(\tau)|^{p/(p-1)}\, d\tau = \frac{|y-x_o|^{p/(p-1)}}{(T-t_o)^{1/(p-1)}}\ . \tag{2.12}$$

Moreover form (2.10) we have

$$1+m+\cdots+m^{k-1} = \frac{1-m^k}{1-m} \le \frac{|y-x_o|^{p/(p-1)}}{C_2 u(x_o,t_o)^{(p-2)/(p-1)}(T-t_o)^{1/(p-1)}} \le$$

$$\le \frac{1-m^{k+1}}{1-m} = 1+\cdots+m^k\ ,$$

$$m^{k+1} \le 1 - \frac{1-m}{C_2}\ \frac{|y-x_o|^{p/(p-1)}}{C_2 u(x_o,t_o)^{(p-2)/(p-1)}(T-t_o)^{1/(p-1)}} < m^k\ . \tag{2.13}$$

It is obvious that (2.13) holds true only when the central quantity is strictly positive, i.e. if $u(x_o,t_o)$ is not too small; otherwise we can only state that $u(x,t)\ge 0$.

Recalling the definition of m, combining (2.11) with (2.13), we finally have

$$u(y,T) \ge u(x_o,t_o)\times \tag{2.14}$$

$$\times\left(1 - \frac{M^{(p-2)/(p-1)}-1}{C_2 M^{(p-2)/(p-1)} u(x_o,t_o)^{(p-2)/(p-1)}}\ \frac{|y-x_o|^{p/(p-1)}}{(T-t_o)^{1/(p-1)}}\right)_+^{(p-1)/(p-2)}\ ,$$

that is, setting (x,t), instead of (y,T),

$$u(x,t) \ge u(x_o,t_o)\left(1 - \frac{C_3}{u(x_o,t_o)^{(p-2)/(p-1)}}\ \frac{|x-x_o|^{p/(p-1)}}{(t-t_o)^{1/(p-1)}}\right)_+^{(p-1)/(p-2)}\ , \tag{2.15}$$

where C_3 depends only upon the data.

The previous result shows no explicit decay in time and moreover was obtained assuming $|x-x_o|\ne 0$. Now we want to take these two aspects into account. Let us consider $(0,s)$ and (x,t) with $0<s<t$ and suppose for the moment that $t>4s$. Let k be the integer such that $2^{k+1}s < t \le 2^{k+2}s$ and set $\tau = 2^k s$. From (2.15) we get

$$u(x,t) \ge u(0,\tau)\times \tag{2.16}$$

$$\times\left(1 - C\ \frac{M^{(p-2)/(p-1)}-1}{M^{(p-2)/(p-1)}\ u(0,\tau)^{(p-2)/(p-1)}}\ \frac{|x|^{p/(p-1)}}{(t-\tau)^{1/(p-1)}}\right)_+^{(p-1)/(p-2)}\ .$$

Let us now set $u_0 \equiv u(0,s)$ and define

$$t_0 = s\ ,\ \ t_1 = 2s\ ,\ \ \cdots\ ,\ \ t_k = 2^k s = \tau\ .$$

We can repeatedly apply the Harnack inequality and conclude that

$$u(0,s) \le M^k\, u(0,\tau)\ ,$$

that is

$$u(0,\tau) \geq \left(\frac{s}{\tau}\right)^a u(0,s) \geq \left(\frac{s}{t}\right)^a u(0,s) \ , \tag{2.17}$$

where a is a positive parameter that depends only on M.

Due to the definition, it is immediate to verify that

$$t-\tau \geq \frac{t}{2} \geq \frac{t-s}{2} \ .$$

We can then combine (2.16) and (2.17), and we can then conclude that, if $t > 4s$,

$$u(x,t) \geq \left(\frac{s}{t}\right)^a u(y,s) \times \tag{2.18}$$

$$\times \left(1 - C\left(\frac{t}{s}\right)^{a(p-2)/(p-1)} \frac{M^{(p-2)/(p-1)}-1}{M^{(p-2)/(p-1)}\, u(y,s)^{(p-2)/(p-1)}} \, \frac{|x-y|^{p/(p-1)}}{(t-s)^{1/(p-1)}}\right)_+^{(p-1)/(p-2)} .$$

On the other hand, if $s < t \leq 4s$, it is apparent that $1 < t/s \leq 4$ and therefore we can conclude that (2.18) holds a lower bound on u for any $0 < s < t$.

□

Remark 2.1. If we deal with porous media like equations, the Harnack inequality becomes

$$u(x_o,t_o) \leq M \inf_{K_\rho(x_o)} u(x, t_o + \theta\rho^2) \quad , \quad \theta = \frac{C}{u(x_o,t_o)^{m-1}} \tag{2.19}$$

and again M, C are constants that depend only on the data. If we repeat the same procedure as before, we get

$$u(x,t) \geq \left(\frac{t_o}{t}\right)^a u(x_o,t_o) \times \tag{2.20}$$

$$\times \left(1 - C\left(\frac{t}{t_o}\right)^{a(m-1)} \frac{M^{m-1}-1}{M^{m-1} u(x_o,t_o)^{m-1}} \, \frac{|x-x_o|^2}{t-t_o}\right)_+^{1/(m-1)} .$$

Remark 2.2. As already remarked in Section 1 for linear parabolic equations, also in the degenerate context, both in (2.18) and (2.20), we have a Barenblatt-like dependance with respect to the x-variable, whereas the time decay, albeit power-like, is not the optimal one. On the other hand, this seems unavoidable, as all our estimates depend in a critical way on the value of M. We will comment on this in the next Section.

Remark 2.3. In (2.6) and (2.19), all the constants are stable as $p \to 2$ or $m \to 1$. It is therefore a natural question to investigate the behavior of (2.18) or (2.20) as $p \to 2$ or $m \to 1$.

It is a matter of simple calculus to check that in both cases at the limit we obtain

$$u(t,y) \geq u(s,x) \left(\frac{s}{t}\right)^a \exp\left(-A\left(1 + \frac{|x-y|^2}{t-s}\right)\right) .$$

which is exactly the behavior given by Moser in the linear case. The constants a and A depend only on the limit value of the parameters $C(p)$, $C(m)$, $M(p)$, $M(m)$.

3. Comparison with Auchmuty and Bao's results

Here we limit ourselves to the porous medium equation, but a similar remark can be repeated for the p-Laplacian.

Starting from a Hamilton-Jacobi differential inequality, with Control Theory arguments, in [1] for non-negative solutions to

$$u_t - \Delta u^m = 0 \quad , \quad m > 1 \, ,$$

defined in the whole $\mathbb{R}^N$, Auchmuty e Bao prove the following lower bound

$$u(x_2, t_2) \geq \left(\frac{t_1}{t_2}\right)^k u(x_1, t_1) \times \times \left[1 - \frac{\delta}{4} \, \frac{|x_2 - x_1|^2}{t_2^\delta - t_1^\delta} \, \frac{m-1}{m[u(x_1,t_1)]^{m-1}} \, \frac{1}{t_1^\mu}\right]^{1/(m-1)} \, , \tag{3.1}$$

where $k = N/[N(m-1)+2]$, $\mu = (m-1)k$, $\delta = 1 - \mu$.

Other than (2.20), (3.1) has the right Barenblatt-like decay, both in time and in space. Unfortunately the proof of (3.1) depends in a fundamental way on the Aronson-Bénilan estimate,

$$\Delta f \geq -\frac{k}{t} \quad , \quad f := \left(\frac{m}{m-1}\right) u^{m-1}$$

which does not seem to be extendable to equations with the fully quasi-linear structure, as dealt with here, and in [4] (see also [3]). It remains an open problem to see if some kind of "full" Barenblatt-like lower bounds can be proved for positive weak solutions to such equations.

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