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Semilinear subelliptic equations on rational Carnot groups

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Abstract[1]. We analyze the notion of periodicity on Carnot groups and we study the semilinear subelliptic equation

$$-\Delta_{\mathbb{G}} u + u = f(\xi, u) \ \ , \ \ \xi \in \mathbb{G}$$

where $\Delta_{\mathbb{G}}$ is a sublaplacian on a Carnot group $\mathbb{G}$ with rational structure and the function f is assumed to be periodic with respect to a discrete co-compact subgroup of $\mathbb{G}$. The results here announced are proved in [9].

1. Introduction

In this note, we study semilinear subelliptic equations with periodic nonlinearity in the context of Carnot groups. In particular, we consider those Carnot groups which admit lattice subgroups and therefore provide a natural notion of periodicity.

We recall that a *lattice subgroup* (or simply *lattice*) Γ of a nilpotent Lie group N is a discrete co-compact subgroup of N.

According to a classical result by *Mal'cev* [11], a nilpotent Lie group contains a lattice if and only if it possesses a rational structure, i.e. a basis of the Lie algebra with rational structure constants (see Section 2 for definitions). Therefore, we shall confine our analysis to Carnot groups endowed with such a structure, briefly called *rational*.

Let $\mathbb{G}$ be a rational Carnot group and Γ a lattice in $\mathbb{G}$. We say that a function $f : \mathbb{G} \to \mathbb{R}$ is *periodic under* Γ (or simply Γ-*periodic*) if, for any $\xi \in \mathbb{G}$,

$$f(\eta \circ \xi) = f(\xi) \quad , \quad \forall \eta \in \Gamma \ .$$

In this context, denoted by Q the homogeneous dimension of $\mathbb{G}$ and by $2^* = 2Q/(Q-2)$ the critical exponent for the Folland-Stein embedding, we study the following class of semilinear subcritical equations

$$-\Delta_{\mathbb{G}} u + u = f(\xi, u) \quad , \quad \xi \in \mathbb{G} \tag{1.1}$$

where $\Delta_{\mathbb{G}}$ denotes a fixed sublaplacian on $\mathbb{G}$ and $f \in C^2(\mathbb{G} \times \mathbb{R}, \mathbb{R})$ satisfies:

(f_1) f is Γ-periodic in the first variable, i.e. $f(\eta \circ \xi, u) = f(\xi, u)$ for all $\eta \in \Gamma$, $\xi \in \mathbb{G}$ and $u \in \mathbb{R}$;

(f_2) $f(\xi, 0) = f_u(\xi, 0) = 0$;

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(f_3) There exist $a_1, a_2 > 0$ and $1 < s < (Q+2)/(Q-2)$ such that

$$|f_u(\xi, u)| \le a_1 + a_2|u|^{s-1}$$

for all $\xi \in \mathbb{G}$ and $u \in \mathbb{R}$;

(f_4) There exists $\mu > 2$ such that $0 < \mu F(\xi, u) := \mu \int_0^u f(\xi, t)\, \mathrm{d}t \le f(\xi, u)u$ for all $\xi \in \mathbb{G}$ and $u \neq 0$.

We also require that Γ is the largest subgroup of $\mathbb{G}$ such that (f_1) holds.

Note that hypothesis (f_4) implies that there exists $a_3 > 0$ such that $F(\xi, u) \ge a_3|u|^\mu$ for large $|u|$, i.e. $F(\xi, u)$ grows more than quadratically for large $|u|$.

A simple example of a function f satisfying the above conditions is

$$f(\xi, u) = a(\xi)|u|^{s-2}u \ ,$$

where $a \in C^2(\mathbb{G}, \mathbb{R})$ is Γ-periodic and $2 < s < 2^*$.

The functional corresponding to the semilinear subelliptic equation (1.1) is

$$J(u) = \frac{1}{2}\|u\|^2_{S^2_1} - \int_{\mathbb{R}^N} F(\xi, u)\, \mathrm{d}\xi \ , \tag{1.2}$$

where the norm $\|u\|_{S^2_1}$ is given by

$$\|u\|^2_{S^2_1} = \int_{\mathbb{R}^N} (|\nabla_{\mathbb{G}} u|^2 + |u|^2)\, \mathrm{d}\xi$$

and one seeks solutions of equation (1.1) in the Folland-Stein space $E \equiv S^2_1(\mathbb{G})$, defined as the completion of C_0^∞ with respect to the above norm. From now on, we will simply denote by $\|\cdot\|$ the S^2_1-norm.

For $\eta \in \Gamma$, let us denote

$$\tau_\eta u(\xi) = u(\eta \circ \xi) \quad , \quad \forall \xi \in \mathbb{G} \ .$$

Observe that the set of critical points of J, i.e. $\mathcal{K} = \{u \in S^2_1(\mathbb{G}) \,|\, J'(u) = 0\}$, is invariant under the action τ_η. Moreover, $J(\tau_\eta u) = J(u)$ for all $\eta \in \Gamma$. Two critical points u_1 and u_2 will be considered equivalent if there exists $\eta \in \Gamma$ such that $u_1 = \tau_\eta u_2$. Critical points which are not equivalent will be called *geometrically distinct*.

Our main result is the following:

Theorem 1.1. *Suppose that f satisfies $(f_1) - (f_4)$. Then, the functional J as defined by* (1.2) *has infinitely many geometrically distinct critical points and the corresponding equation* (1.1) *has infinitely many geometrically distinct solutions.*

Let us introduce some notations. For $a, b \in \mathbb{R}$, let $J^a = \{u \in E \,|\, J(u) \le a\}$, $J_a = \{u \in E \,|\, J(u) \ge a\}$, $J^b_a = J_a \cap J^b$ and, denoted by $\mathcal{K}$ the set of critical points u of J in E, let $\mathcal{K}^b = J^b \cap \mathcal{K}$, $\mathcal{K}^b_a = J^b_a \cap \mathcal{K}$, $\mathcal{K}(a) = \mathcal{K}^a_a$. Moreover, let

$$\Lambda = \{g \in C([0,1], E) \,|\, g(0) = 0,\ J(g(1)) < 0\}$$

and

$$c = \inf_{g \in \Lambda} \max_{t \in [0,1]} J(g(t)) \ . \tag{1.3}$$

A refined version of Theorem 1.1, whose proof can be found in [9], gives a more precise description of the location of the infinitely many critical points of J. Indeed, supposing that the following condition holds

($*$) there exists $\alpha > 0$ such that $\mathcal{K}^{c+\alpha}/\Gamma$ contains only finitely many critical points of J, it is possible to prove that c is a critical value for J and that $\mathcal{K}_{kc-\alpha}^{kc+\alpha}/\Gamma$ contains infinitely many critical points of J, for all $k \in \mathbb{N} \setminus \{1\}$. Note that it is not easy to verify condition $(*)$, but of course if $(*)$ fails, then J already has infinitely many distinct critical points in $J^{c+\alpha}/\Gamma$.

We point out that our results generalize to the Carnot context the Euclidean results by Coti-Zelati and Rabinowitz in [2], [3]. Analogous results in the particular case of the Heisenberg group $\mathbb{H}^n$, the simplest non-abelian Carnot group, have been proved by Maad in [10]. In that case, the function f in the right hand side of (1.1) was assumed to be invariant, in the first variable, with respect to the discrete subgroup $\mathbb{H}^n_{\mathbb{Z}}$ consisting of the points of $\mathbb{H}^n$ with integer coordinates.

Finally, we want to observe that our results also hold if the operator $-\Delta_{\mathbb{G}} + I$ in (1.1) is replaced by a more general divergence structure operator on $\mathbb{G}$ of the form

$$Lu = -\sum_{i,j=1}^{N_1} X_i(a_{ij}(\xi)X_j u) + b(\xi)u \ ,$$

where $X = (X_1, \ldots, X_{N_1})$ denotes any basis of the first layer of the Lie algebra of $\mathbb{G}$ and the coefficients a_{ij} and b satisfy the following conditions:

(i) $a_{ij} = a_{ji} \in C^2(\mathbb{R}^N)$, $b \in C^1(\mathbb{R}^N)$ and a_{ij}, b are Γ-periodic;

(ii) there exists a constant $C > 0$ such that $\sum a_{ij}(\xi)\eta_i\eta_j \geq C|\eta|^2$, $\forall \xi, \eta \in \mathbb{R}^N$;

(iii) $b(\xi) > 0$, $\forall\, \xi \in \mathbb{R}^N$.

In fact, the proofs are essentially the same with only a few technical modifications.

2. Lattices and periodicity on Carnot groups

2.1. **Carnot groups.** According to the classical definition, a Carnot group $\mathbb{G}$ is a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification, namely a decomposition $\mathfrak{g} = \bigoplus_{j=1}^r V_j$, where $\dim V_j = N_j$ and $\sum_{j=1}^r N_j = N$, such that $[V_1, V_j] = V_{j+1}$ for $1 \leq j < r$, and $[V_1, V_r] = \{0\}$. Up to a canonical isomorphism, a Carnot group $\mathbb{G}$ can be identified with a Lie group $(\mathbb{R}^N, \circ) = (\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \ldots \times \mathbb{R}^{N_r}, \circ)$, endowed with a homogeneous structure by means of a family $\{\delta_\lambda\}_{\lambda>0}$ of group automorphisms (called *dilations*) of the following form

$$\delta_\lambda(\xi) = \delta_\lambda(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(r)}) = (\lambda\, \xi^{(1)}, \lambda^2\, \xi^{(2)}, \cdots, \lambda^r\, \xi^{(r)}) \ ,$$

where $\xi^{(j)} \in \mathbb{R}^{N_j}$ for $j = 1, \ldots, r$. For $i = 1, \ldots, N_1$, let $X_i \in \mathfrak{g}$ be the left invariant vector field which agrees at the origin with the partial derivative $\partial/\partial\xi_i^{(1)}$. We require that the Lie algebra generated by $X_1, \ldots X_{N_1}$ is the whole $\mathfrak{g}$. Under these assumptions, we call $\mathbb{G} = (\mathbb{R}^N, \circ)$ a *Carnot group.*

The natural number $Q = \sum_{j=1}^r j\, N_j$, naturally attached to the dilations $\{\delta_\lambda\}_{\lambda>0}$ is called the *homogeneous dimension* of $\mathbb{G}$. The Lebesgue measure is invariant w. r. t. left and right translations on $\mathbb{G}$ and the volumes scale as λ^Q, i.e. $|\delta_\lambda(E)| = \lambda^Q\, |E|$ for any measurable set $E \subset \mathbb{R}^N$. In the sequel, we shall assume $Q \geq 3$.

Now, if $\{Y_1, \ldots, Y_{N_1}\}$ is any basis of $\mathrm{span}\{X_1, \ldots X_{N_1}\}$, the second order differential operator

$$\Delta_{\mathbb{G}} = \sum_{i=1}^{N_1} Y_i^2$$

is called a *sublaplacian* on $\mathbb{G}$. We shall denote by $\nabla_{\mathbb{G}} = (Y_1, \dots, Y_{N_1})$ the related *subelliptic gradient*. $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$ are left-translation invariant w.r.t. the group action and δ_λ-homogeneous, respectively of degree one and two. In other words, $\nabla_{\mathbb{G}}(u \circ \tau_\xi) = \nabla_{\mathbb{G}} u \circ \tau_\xi$, $\nabla_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda \nabla_{\mathbb{G}} u \circ \delta_\lambda$, $\Delta_{\mathbb{G}}(u \circ \tau_\xi) = \Delta_{\mathbb{G}} u \circ \tau_\xi$, $\Delta_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda^2 \Delta_{\mathbb{G}} u \circ \delta_\lambda$. Moreover, since $X_1, \dots, X_{N_1}$ generate the whole $\mathfrak{g}$, any sublaplacian on $\mathbb{G}$ satisfies Hörmander's hypoellipticity condition.

2.2. **Lattice subgroups and periodicity.** This section is devoted to recall the definitions of lattice, tiling, periodic tiling and periodic function w.r.t. a lattice in the general context of nilpotent Lie groups (see e.g. [1], [14], [12], [8]).

A *lattice* Γ in a nilpotent Lie group N is a discrete co-compact subgroup, i.e. a discrete subgroup of N such that the quotient N/Γ is compact.

A *tiling* (or *pavage*) of N is a locally finite family $\mathcal{T}$ of non-empty subsets of N such that

(i) $\bigcup_{A \in \mathcal{T}} A = N$; (covering condition)

(ii) $\mathrm{int(A)} \cap \mathrm{int(B)} = \emptyset$, $\forall A, B \in \mathcal{T}$, $A \neq B$ (non-overlap condition).

Let $\Gamma \subset N$ a lattice. A tiling $\mathcal{T}$ is said to be *periodic under* Γ (or simply a Γ-*tiling*) if there exists $Q \in \mathcal{T}$ such that

(iii) $\mathcal{T} = \{\eta(Q) \mid \eta \in \Gamma\}$.

In this case, the tile Q is called a *prototile* or a *fundamental domain* for $\mathcal{T}$.

Furthermore, assume that $\phi : N \to N$ is an expanding map (i.e. stretching all distances by at least a factor $c > 1$), such that $\phi \Gamma \phi^{-1} \subset \Gamma$. A Γ-tiling $\mathcal{T}$ is called *self-similar* if

(iv) $\phi(Q) = \bigcup_{i=1}^{k} \eta_i(Q)$, for certain $\eta_1, \dots \eta_k \in \Gamma$.

Now, let Γ be a lattice in N. We say that a function $f : N \to \mathbb{R}$ is *periodic under* Γ (or simply Γ-*periodic*) if for any $\xi \in N$

$$f(\eta \circ \xi) = f(\xi) \qquad \forall \eta \in \Gamma \, .$$

In the sequel, we shall use this definition in the context of Carnot groups possessing lattice subgroups.

We point out that another notion of periodicity, different from the lattice invariance above, has been introduced by Franchi-Gutierrez and Nguyen [7] to study homogenization on Carnot groups. This notion is based on the possibility to construct a natural tiling on any Carnot group, as follows. Let $\mathbf{Y} := \{\xi \in \mathbb{G} \mid -1/2 < \xi_i \leq 1/2, \ i = 1, \dots, N\}$ denote the unit cube in $\mathbb{G}$ centered at the origin and let $Q_\eta := \eta \circ \mathbf{Y}$, for $\eta \in \mathbb{Z}^N$. It can be proved that the family $\{Q_\eta\}_{\eta \in \mathbb{Z}^N}$ is a tiling of $\mathbb{G}$. Now, according to the definition in [7], a function $f : \mathbb{G} \to \mathbb{R}$ is said to be $\mathbf{Y}$-periodic with respect to the reference tiling $\{Q_\eta\}$ if

$$f(\eta \circ \xi) = f(\xi) \qquad \forall \, \xi \in \mathbf{Y} \ , \ \forall \, \eta \in \mathbb{Z}^N \, .$$

Unfortunately, this notion of periodicity, which does not require the existence of any lattice and can be given in any Carnot group, does not work for our purpose.

2.3. **The existence of lattices in nilpotent Lie groups.** The problem of the existence of lattices in nilpotent Lie groups is a classical topic. We recall here a remarkable criterion, known as *Mal'cev criterion*, that enables us to decide when a simply connected nilpotent Lie group admits a lattice.

Theorem 2.1 (A.I. Mal'cev, [11])**.** *A simply connected nilpotent Lie group N contains a lattice if and only if its Lie algebra $\mathfrak{n}$ admits a rational structure, i.e. if $\mathfrak{n}$ possesses a basis with respect to which the structure constants are rational.*

(We recall, here, that, if $X = \{X_1, \ldots, X_N\}$ is a basis of the Lie algebra $\mathfrak{n}$, the structure constants of $\mathfrak{n}$ with respect to the basis X are the real numbers c_{ij}^k such that $[X_i, X_j] = \sum_{k=1}^N c_{ij}^k X_k$, for all $i, j = 1, \ldots N$).

Nilpotent Lie groups endowed with a rational structure are briefly called *rational.*

Observe that a nilpotent Lie algebra does not always have such a structure. Only for dimensions up to 6, all nilpotent Lie algebras have $\mathbb{Q}$-structures (see e.g. [12]). On the other hand, a Lie algebra can admit different rational structures. For example, the Heisenberg algebra has uncountably many different $\mathbb{Q}$-structures (see [1, Remark p. 195]).

2.4. **Examples.**

Example 1. The simplest example of rational Carnot group is, of course, the Euclidean space $(\mathbb{R}^N, +)$, with the lattice subgroup $\Gamma = \mathbb{Z}^N$.

Example 2. The simplest non abelian example is the Heisenberg group $(\mathbb{H}^n, \circ)$. Choosing the so-called *polarized coordinates* on $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, the composition law is given by

$$(p, q, s) \circ (p', q', s') = (p + p', q + q', s + s' + p \cdot q') \ .$$

(Recall that complex coordinates are related to polarized coordinates by the relations $z = q + ip$, $t = 4s - 2p \cdot q$). The simplest example of lattice subgroup of $\mathbb{H}^n$ is the subgroup $\mathbb{H}^n_{\mathbb{Z}}$ consisting of all elements of $\mathbb{H}^n$ with integer coordinates. More generally, it is possible to characterize the lattice subgroups of $\mathbb{H}^n$. Let $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ s.t. l_j divides l_{j+1} for $j = 1, \ldots, n-1$. Let

$$\Gamma_l = \{(p, q, s) \in \mathbb{H}^n \ | \ p, q \in \mathbb{Z}^n \ , \ s \in \mathbb{Z} \ , \text{ s. t. } l_j \text{ divides } q_j \ , \text{ for } j = 1, \ldots, n\} \ .$$

It is easy to check that Γ_l is a subgroup of $\mathbb{H}^n$ and that the rectangle

$$Q = [0, 1)^n \times \left(\prod_1^n [0, l_j) \right) \times [0, 1)$$

is a fundamental domain for Γ_l. The manifold $M_l = \mathbb{H}^n / \Gamma_l$ is homeomorphic to the closure of Q with the faces of the boundary glued to each other in a suitable way, so M_l is compact and Γ_l is a lattice subgroup. As recalled by Folland in [6], the following holds:

Proposition 2.2. *If Γ is any lattice of $\mathbb{H}^n$, there exists a unique $l \in \mathbb{Z}^n$ and $\phi \in Aut(\mathbb{H}^n)$ s.t. $\phi(\Gamma) = \Gamma_l$.*

Example 3. The so-called H-type groups are rational groups. Indeed, as proved by Crandall and Dodziuk in [4], the following holds:

Proposition 2.3. *For any H-type Lie algebra $\mathfrak{g} = U_m \oplus V_n$ there exist orthonormal bases $\{X_1, \ldots, X_m\}$ and $\{Y_1, \ldots, Y_n\}$ of U_m and V_n respectively, so that the structure constants c_{ij}^k of $\mathfrak{g}$ w.r.t. the basis $\{X_1, \ldots, X_m, Y_1, \ldots, Y_n\}$ are integers and in fact take the values $0, 1, -1$.*

Example 4. We show here how a *non rational* two-step group can be constructed following the procedure in [1, Example 5.1.13]. Let m, n be positive integers with $(m-1)mn > 2(m^2+n^2)$; for instance, we could take $m = 6$, $n = 4$. Choose real numbers c_{ij}^k, $1 \le i, j \le m$ and $1 \le k \le n$, such that

(i) $c_{ij}^k = -c_{ji}^k$ for $1 \le i, j \le m$;

(ii) the c_{ij}^k's are algebraically independent if $i < j$.

Define $\mathfrak{g}$ as the Lie algebra spanned by $X_1, \dots, X_m, Y_1, \dots, Y_n$, where $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k Y_k$ and the Y_j's are central. It can be verified that $\mathfrak{g}$ does not possess a rational structure. Indeed, supposing by contradiction that $\mathfrak{g}$ admits a basis with rational structure constants, one gets that the c_{ij}^k's can be expressed in the following form

$$c_{ij}^k = \sum_{p=1}^m \sum_{q=1}^m \sum_{l=1}^n \alpha_{ip}\alpha_{jq} b_{lk} \tilde{d}_{pql} \quad , \quad \alpha_{ip} \ , \ b_{lk} \in \mathbb{R} \ , \ \tilde{d}_{pql} \in \mathbb{Q} \, .$$

Thus, the c_{ij}^k's are $(1/2)nm(m-1)$ numbers which are algebraically independent and belong to $\mathbb{Q}(\alpha_{ip}, b_{lk})$, a field with transcendence degree $\le m^2 + n^2 < (1/2)nm(m-1)$, that is a contradiction.

To conclude this overview, following the analysis of Strichartz [15], let us see how a lattice can be constructed in any rational Carnot group, starting from the group law. Recall that the composition law on a Carnot group $\mathbb{G}$ has the following form

$$(\xi \circ \eta)^{(i)} = \xi^{(i)} + \eta^{(i)} + H^{(i)}(\xi, \eta) \qquad 1 \le i \le r \ , \ \text{ for all } \xi, \eta \in \mathbb{G} \, , \tag{2.1}$$

where $H^{(1)} \equiv 0$ and the components of the functions $H^{(i)}$, $1 < i \le r$, are mixed polynomials in ξ and η; moreover $H^{(i)}$ depends only on $\xi^{(1)}, \dots, \xi^{(i-1)}$, $\eta^{(1)}, \dots, \eta^{(i-1)}$ and $H^{(i)}(\delta_\lambda \xi, \delta_\lambda \eta) = \lambda^i H^{(i)}(\xi, \eta)$. Now, in the case of a rational group, up to a change of coordinates, the polynomial functions $H^{(i)}$ have rational coefficients. Let $k_1, k_2, \dots, k_r$ be positive integers and define $\Gamma \subset \mathbb{G}$ by letting $\xi = \xi_j^{(i)} \in \Gamma$ if and only if $k_i \xi_j^{(i)} \in \mathbb{Z}$. Due to the rationality of the coefficients of the polynomials $H^{(i)}$, it is always possible to choose the integers $k_1, k_2, \dots, k_r$ in such a way that Γ is a subgroup of $\mathbb{G}$. Of course, Γ is co-compact.

Strichartz shows that a periodic tiling with respect to the lattice Γ can be constructed by Γ-translating the fundamental domain

$$Q = \{\xi \in \mathbb{G} \ | \ 0 \le \xi_j^{(i)} - A_j^{(i)}(\xi) < 1/k_i\} \, , \tag{2.2}$$

where $A_j^{(i)}$ are bounded measurable functions of $\xi^{(1)}, \dots, \xi^{(i-1)}$ (in particular, $A_j^{(1)} \equiv 0$). It is easy to see that any choice of the functions $A_j^{(i)}$ yields a prototile. A tiling so constructed is called a *stacked tiling*.

Observe that it is possible to choose suitable functions $A_j^{(i)}$ so that the stacked tiling defined by the fundamental domain (2.2) is self-similar with respect to the dilation δ_2, in the sense that there exists a finite subset $\Gamma_0 \subset \Gamma$ such that $\delta_2 Q = \bigcup_{\eta \in \Gamma_0} \eta(Q)$ (see [15, Theorem 3.3]).

3. Proof of Theorem 1.1

We only sketch the main ideas of the proof, referring the reader to the paper [9], where all the details are given.

One of the main ingredients is the minimax characterization of the values kc, for all $k \in \mathbb{N} \setminus \{1\}$, where c is the minimax value (1.3). Indeed, denoted by Λ_k the set of functions

$$\Lambda_k = \{G = g_1 + \ldots + g_k \ \mid \ g_i \text{ satisfies } (G_1) - (G_3)\ ,\ 1 \leq i \leq\ k\}\ ,$$

where

(G_1) $g_i \in C([0,1]^k, E)$ for $1 \leq i \leq\ k$;

(G_2) $g_i(\theta_1, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_k) = 0$ and $J(g_i(\theta_1, \ldots, \theta_{i-1}, 1, \theta_{i+1}, \ldots, \theta_k)) < 0$ for $1 \leq i \leq\ k$ and for all $\theta \in [0,1]^k$;

(G_3) there exist open sets $\mathcal{O}_i$, $1 \leq i \leq\ k$, with $\overline{\mathcal{O}}_i \cap \overline{\mathcal{O}}_j = \emptyset$ for $i \neq j$, such that $\operatorname{supp} g_i(\theta) \subset \mathcal{O}_i$ for all $\theta \in [0,1]^k$,

we prove that

$$k\,c = \inf_{G \in \Lambda_k} \max_{\theta \in [0,1]^k} J(G(\theta))\ . \tag{3.1}$$

Now, we can sketch the proof of Theorem 1.1.

Outline of the proof. We give here an outline of the proof, which is organized in four steps. The first two steps mainly involve approximation and critical point theory arguments which can be easily adapted to the Carnot setting; step 3 and 4 just need a bit more care, since they are based on regularity results, maximum principle and Sobolev-type embeddings which are more delicate in the subelliptic context under consideration.

Step 1. Let $\mathcal{K}$ denote the set of critical points of J and let $\mathcal{F}$ be a set of representatives of $\mathcal{K}$ under the action of Γ. Suppose, by contradiction, that $\mathcal{F}$ is finite. Then, choose $k \in \mathbb{N}$ so large that $J'(u) \neq 0$ for every $u \in J^{-1}[kc-1, \infty)$.

For an appropriate $\varepsilon > 0$, we construct a function $G \in \Lambda_k$, whose support (for any θ) consists of k components $S_1, \ldots, S_i = \tau_{\eta_i} S_1$, $\eta_i \in \Gamma$, whose relative distance is measured by a parameter β. Moreover, G satisfies

$$\max_{\theta \in [0,1]^k} J(G(\theta)) \leq k\,c + \varepsilon\ .$$

Step 2. By means of the flow corresponding to an appropriate pseudo-gradient vector field, the function G is deformed to $\overline{G}$ such that

$$\max_{\theta \in [0,1]^k} J(\overline{G}(\theta)) \leq k\,c - \varepsilon\,.$$

If $\overline{G} \in \Lambda_k$, we would have a contradiction at this point. Unfortunately, we only know that $\overline{G}$ is small in the L^∞-sense outside of the sets S_i.

Step 3. By multiplying $\overline{G}$ by a suitable cut-off function, we obtain a smooth function $\widehat{G}$ with compact support contained in a gauge ball $B^d_R(0)$, such that

$$\max_{\theta \in [0,1]^k} J(\widehat{G}(\theta)) \leq k\,c - \varepsilon/2\ .$$

Thus, we replace $\widehat{G}$ outside of the sets S_i by a function v which equals $\widehat{G}$ on ∂S_i and can be made exponentially small in some annular regions $\mathcal{A}_i$ separating the sets S_i. Letting

$$S = B_R^d(0) \setminus \bigcup_{i=1}^{k} S_i \ ,$$

the function $v = v(\theta)$ is obtained as the minimum of the functional

$$\Psi(v) = \int_S \left(\frac{1}{2}(|\nabla_{\mathbb{G}} v|^2 + v^2) + F(\xi, v) \right) \mathrm{d}\xi$$

over the class of functions which equal $\widehat{G}(\theta)$ on ∂S and are appropriately small in the S_1^2-sense.

Step 4. Denoted by U the function obtained in Step 3, i.e.

$$U(\theta)(\xi) = \begin{cases} \widehat{G}(\theta)(\xi) & \xi \notin S \\ \\ v(\theta)(\xi) & \xi \in S \ , \end{cases}$$

we appropriately truncate U in the regions $\mathcal{A}_i$, getting a function $H \in \Lambda_k$ such that

$$\max_{\theta \in [0,1]^k} J(H(\theta)) < k\, c \ ,$$

that is a contradiction to (3.1).

□

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