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On positive solutions of hypoelliptic second order differential equations

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Abstract[1]. In this note we survey some results obtained in cooperation with Sergio Polidoro. We point out the study of a Harnack inequality invariant for positive solutions of the hypoelliptic equation

$$L_0\,u \,+\, V\,u \,=\, 0\ ,$$

and some results related for the Cauchy-Dirichlet problem associated to it, where L_0 is the Kolmogorov operator and the function V belong to the Stummel Kato class.

1. Introduction

The purpose of this note is to present some results obtained in cooperation with Sergio Polidoro. We obtain a Harnack inequality and study regularity properties of the Cauchy-Dirichlet problem associated to hypoelliptic equations of the type

$$L_0 u + V u = 0\ , \tag{1.1}$$

where L_0 is the Kolmogorov operator in $\mathbb{R}^{n+1}$:

$$L_0 u \equiv div\ (ADu) + \langle x, BDu\rangle - \partial_t u \tag{1.2}$$

and V belongs to a class of functions of Stummel-Kato type that we will define. We suppose that the $n \times n$ matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ have real constant entries, $z = (x,t) = (x_1, ..., x_n, t) \in \mathbb{R}^{n+1}$, $D = (\partial_{x_1}, ..., \partial_{x_n})$, div and $\langle \cdot\ ,\cdot\rangle$ are, respectively, the gradient, the divergence and the inner product in $\mathbb{R}^n$. We also assume that the matrices have the following form

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{1.3}$$

where A_0 is the identity $m_0 \times m_0$ matrix and every B_j is a $m_{j-1} \times m_j$ block matrix with rank m_j, where $j = 1, 2, ..., r$, $m_0 \geq m_1 \geq \cdots \geq m_r \geq 1$ and $m_0 +$

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$m_1 + \cdots + m_r = n$. A careful analysis of operators similar to L_0 having non-constant coefficients has been made in the notes [20], [22] and [19]. We point out our attention in a Harnack inequality invariant for positive solutions of (1.1). This problem has been considered, for a significant range of classes of operators L_0, by many authors in the last years. As a matter of fact we recall that Stampacchia in [27] considers L_0 uniformly elliptic and shows that if $V \in L^p, p > n/2$ then the solutions of $L_0 u + V u = 0$ are Hölder continuous and it is true a uniform Harnack inequality. Ladyženskaja and Ural'ceva in [17] prove that this is the best possible condition in the L^p spaces, but could be weakened by Aizeman and Simon (see [1]) who prove a Harnack inequality in the case that L_0 is the Laplace operator and $V \in SK$ class of locally summable functions in an open set $\Omega \subset \mathbb{R}^n$ such that

$$\lim_{r\to 0}\left(\sup_{x\in\Omega}\int_{|x-y|<r}\frac{|V(y)|}{|x-y|^{n-2}}\,dy\right) = 0\ . \tag{1.4}$$

We observe that Aizeman and Simon's result improves Stampacchia's result because $L^p(\Omega) \subset SK(\Omega)$, $\forall\, p > n/2$. The proof by Aizeman and Simon is based on probabilistic techniques, but in the note by Chiarenza, Fabes and Garofalo [6] the same inequality is extended to uniformly elliptic operators using analytic techniques. Similar results are later obtained for istance by Hinz and Kalf in [12] and by Simader in [26]. Techniques and results by Chiarenza, Fabes and Garofalo are later semplified and extended by many authors and generalized in different directions: Gutierrez in [11] studies elliptic operators with degeneration of weight type, Dal Maso and Mosco in [7] consider a class of relaxed Dirichlet problems having solutions that can be characterized as $\gamma-$limits of classic Dirichlet problems. In this family of problems figure a Schrödinger type equation where the potential V is replaced with a Borel measure μ. More recently the regularity problem of solutions of $L_0 u + V u = 0$ has been considered for a class of uniformly parabolic operators, precisely Sturm in [28] suppose that the coefficients of the second order derivatives are continuous and Zhang in [29] considers only measurable coefficients. In both papers is defined a class of parabolic Stummel-Kato class individualized by the fundamental solution of the heat equation, Γ_0, as follows:

$$\sup_{(x,t)\in\mathbb{R}^{n+1}}\int_{(y,s)\in\Omega,\, t-r^2<s<t}\Gamma_0(x,t,y,s)|V(y,s)|\,dy\,ds\ , \tag{1.5}$$

$$\sup_{(y,s)\in\mathbb{R}^{n+1}}\int_{(x,t)\in\Omega,\, s<t<s+r^2}\Gamma_0(x,t,y,s)|V(x,t)|\,dx\,dt\ , \tag{1.6}$$

tend to 0, as $r \to 0$.

In 1967 Hörmander (see [13]) studied the remarkable class of operators:

$$L_0 \equiv \sum_{j=0}^{m} X_j^2\ , \tag{1.7}$$

where $X_0, X_1, \dots, X_m$ are vector fields defined in an open set $\Omega \subset \mathbb{R}^n$ of the form

$$X_j = \sum_{i=1}^{n} a_{ij}(x)\partial_{x_i} \quad , \quad j = 0, 1, \dots m \tag{1.8}$$

and $a_{ij} \in C^\infty(\Omega)$ verify hypoellipticity Hörmander condition. After the paper by Hörmander begin a systematic study to establish the principle properties of operators of type (1.7). The most important results of general character are existence and local estimates of the fundamental solution, see for istance the study made by Folland in [10], Nagel, Stein and Wainger in [21], Sánchez-Calle in [25] and Jerison, Sánchez-Calle [14]. Based on these instruments has been built a regularity theory for solutions of the equation $L_0 u = f$, in Sobolev spaces and in spaces of Hölder functions, similar to that one related to equations of elliptic and parabolic type (see the study made by Rotshild and Stein in [24]).

Regularity problem for solutions of Schrödinger equation $L_0 u + V u = 0$ has been studied by Citti, Garofalo and Lanconelli in [5] where the authors suppose that the potential V in (1.1) is only measurable. The last mentioned note has been generalized in some directions, for example in [18] where Lu considers degeneration of type A_2 weight or by Biroli who studies subelliptic p-Laplace operators as

$$\sum_{j=1}^{m} X_j^* \left(|Xu|^{p-2} X_j u\right) + V|u|^{p-2}u = 0 \,, \tag{1.9}$$

where $Xu = (X_0 u, ..., X_m u)$ and the fields X_j verify Hörmander condition (see [2] and [3]). These two papers can be considered as a part of the study made by Capogna, Danielli and Garofalo (see e. g. [4]). We explicity observe that the above operator (1.9) is nonlinear then it is not possible to define a Kato class using the fundamental solution, for this reason Biroli in [2] use the following definition

$$\lim_{h\to 0} \left(\sup_{x\in\Omega} \int_0^r \left(\frac{1}{mis(B(x,s))} \left(|V(y)| \, dy\right) s^p \right) \frac{ds}{s} \right) = 0$$

being $B(x, s)$ a sphere centered at x with radius s and mis is the Lebesgue measure.

In this note the study made by the authors is focus in a more general class that we will consider is the sequel:

$$L_0 = \sum_{j=1}^{m} X_j^2 + Y \tag{1.10}$$

where it is a sum of square and also a field of first order Y. The operator L_0 in (1.2) can be express as in (1.10) if we set

$$Y = \langle x, BD\rangle - \partial_t \quad , \quad X_j = \partial_{x_j} \,, \;\; j = 1, \cdots, m_0 \,.$$

We wish to call the attention of the reader on the fact that Citti, Garofalo and Lanconelli in their above mentioned note used techniques considered by Lanconelli and Polidoro [16] while we handle some ideas by Zhang, contained in [8] and [9], related to the homogeneous parabolic equation $L_0 u = 0$.

2. Preliminary tools and main results

To begin with let us define a Stummel-Kato class more general of that one used before in (1.5) and (1.6).

Definition 2.1. Let Ω be an open bounded open set in $\mathbb{R}^{n+1}$ and Γ_0 the fundamental solution related to the Kolmogorov operator L_0 defined in (1.2). If $V \in L^1(\Omega)$

we set $\eta_V(h)$ and $\eta_V^*(h)$

(2.1) $$\sup_{(x,t)\in\Omega} \int_{(y,s)\in\Omega, t-h^2<s<t} \Gamma_0(x,t,y,s)|V(y,s)|\,dy\,ds\ ,$$

$$\sup_{(y,s)\in\Omega} \int_{(x,t)\in\Omega, s<t<s+h^2} \Gamma_0(x,t,y,s)|V(x,t)|\,dx\,dt\ ,$$

we say that V *belongs to* $SK(\Omega)$ *related to* L_0 if

$$\lim_{h\to 0}\eta_V(h)=0 \quad , \quad \lim_{h\to 0}\eta_V^*(h)=0\ .$$

If $A=I_n$, $B=0$ the above operator L_0 is the heat operator and (2.1) gives back the definition by Zhang.

Because of our aim is to obtain a Harnack inequality invariant for positive solutions of $L_0u+Vu=0$, and some results of the Cauchy-Dirichlet problem associated to it, let us preliminary give the definition of weak solution of the above operator.

Definition 2.2. Let $\Omega\subset\mathbb{R}^{n+1}$ a bounded open set and $L=L_0+V$, where L_0 the Kolmogorov operator and $V\in SK(\Omega)$. We say that u is a *weak solution* of

$$Lu=0$$

if

$$\exists\, p>1\ :\ u,\partial_1 u,\ldots,\partial_{m_0}u\in L^p_{loc}(\Omega)\ ,$$
$$Vu\in L^1_{loc}(\Omega)\ ,$$

and

$$-\int_\Omega \langle ADu, D\varphi\rangle + \int_\Omega uY^*\varphi + \int_\Omega uV\varphi = 0 \quad , \quad \forall\,\varphi\in C_0^\infty(\Omega)$$

where $Y=\langle x, BD\rangle-\partial_t$ and $Y^*=-Y$.

The operator L_0 is invariant with respect to a homogeneous Lie group being the product of the group defined by:

$$(x,t)\cdot(\xi,\tau)=(\xi+E(\tau)x, t+\tau)$$

while the dilations by :

$$(\mathcal{D}(\lambda),\lambda^2)_{\lambda>0}\ ,$$

where $E(t)=\exp(-tB^T)$, $\mathcal{D}(\lambda)=diag\,(\lambda I_{m_0},\lambda^3 I_{m_1},\cdots,\lambda^{2r+1}I_{m_r})$ and B is the matrix introduced before.

The integer $N=m_0+3m_1+\cdots+(2r+1)m_r+2$ is the homogeneous dimension of the group. In the sequel will be also useful the following definitions

$$Q_R(\xi,\tau,h)=\left\{(x,t)\in\mathbb{R}^{n+1}\ :\ \tau<t<\tau+h\ ;\ \left|\mathcal{D}\left(\frac{1}{R}\right)(E(-t)x-E(-\tau)\xi)\right|<1\right\}\ ,$$

$$S_R(\xi,\tau)=\left\{(x,t)\in\mathbb{R}^{n+1}\ :\ t=\tau\ ;\ \left|\mathcal{D}\left(\frac{1}{R}\right)E(-\tau)(x-\xi)\right|<1\right\}\ ,$$

$$S_R(\xi,\tau,h)=\left\{(x,t)\in\mathbb{R}^{n+1}\ :\ t=\tau+h\ ;\ \left|\mathcal{D}\left(\frac{1}{R}\right)(E(-t)x-E(-\tau)\xi)\right|<1\right\}\ ,$$

$$M_R(\xi,\tau,h)=\left\{(x,t)\in\mathbb{R}^{n+1}\,:\;\tau<t<\tau+h\;;\;\left|\mathcal{D}\left(\frac{1}{R}\right)(E(-t)x-E(-\tau)\xi)\right|=1\right\}\;,$$

respectively the open set, its basis, its section at the height $t=\tau+h$ and its "lateral boundary".

Definition 2.3. Let us set u solution of the Cauchy-Dirichlet problem

$$\begin{cases} Lu=0 & \text{in } Q_R(\xi,\tau,h) \\ u=0 & \text{in } M_R(\xi,\tau,h) \\ u=f & \text{in } S_R(\xi,\tau)\;, \end{cases}$$

where $f\in C_0(S_R(\xi,\tau))$, a *weak solution* of $Lu=0$ in $Q_R(\xi,\tau,h)$ such that satisfy the following conditions

$$\lim_{(x,t)\to(y,s)} u(x,t)=0 \qquad \forall\,(y,s)\in M_R(\xi,\tau,h)$$

$$\lim_{(x,t)\to(y,\tau)} u(x,t)=f(y,\tau) \qquad \forall\,(y,\tau)\in S_R(\xi,\tau)\;,$$

where $(x,t)\in Q_R(\xi,\tau,h)$.

Definition 2.4. We say that G is the *Green function* for Q_R if, for every $f\in C_0(S_R)$, the function

$$u(x,t)=\int_{S_R} G(x,t,\xi,\tau)f(\xi)\,d\xi$$

is solution of the Cauchy-Dirichlet problem.

Let us consider the cylinder $Q_R(\xi,\tau,R^2)$ and, for every $\alpha,\beta,\gamma,\delta\in(0,1):\alpha<\beta<\gamma$, let us set

$$Q^-=\left\{(x,t)\in Q_{\delta R}(\xi,\tau,R^2):\tau+\alpha R^2\le t\le\tau+\beta R^2\right\}\;,$$

$$Q^+=\left\{(x,t)\in Q_{\delta R}(\xi,\tau,R^2):\tau+\gamma R^2\le t\right\}\;.$$

We are now ready to state the main result proved by the authors in the paper [23].

Theorem 2.5. . *Let $u\ge 0$ be a weak solution of $Lu=0$ in $Q_R(\xi,\tau,R^2)$ and $V\in SK(Q_R(\xi,\tau,R^2))$. Then there exists $M>0$, dependent on V and on the constants $\alpha,\beta,\gamma,\delta$, such that*

$$\max_{Q^-} u\le M\min_{Q^+} u\;.$$

Proof. To obtain the requested inequality the first step is to prove an estimate of the modulus of continuity of a weak solution of $Lu=0$, making the additional assumption that the solution u is bounded. Then let us use the following representation formula for a weak solution v of the equation $L_0v=f$.

If v is zero outside a compact set and $f\in L^1(\mathbb{R}^{n+1})$, we have

$$v(z)=-\int_{\mathbb{R}^{n+1}}\Gamma_0(z,\zeta)f(\zeta)\,d\zeta \quad,\quad \forall\,z\in\mathbb{R}^{n+1}\;.$$

Let $z_0 \in \Omega$, $r \in (0,1)$, $B_{4r}(z_0) \subset \Omega$, $z \in B_{r^2}(z_0)$, $\rho = 2\sqrt{d(z,z_0)}$, $\phi \in C_0^\infty(B_{2\rho}(z_0))$, $\phi \equiv 1$ in $B_\rho(z_0)$ and $v = \phi u$ defined on $B_{2\rho}(z_0)$ such that

$$L_0(\phi u) = u L_0 \phi + 2 \langle ADu; D\phi \rangle - \phi V u$$

for every $z \in B_\rho(z_0)$. We have

$$u(z) = -\int_{\mathbb{R}^{n+1}} \Gamma_0(z,\zeta) L_0 \phi(\zeta) u(\zeta)\, d\zeta -$$

$$-2\int_{\mathbb{R}^{n+1}} \Gamma_0(z,\zeta) \langle A(\zeta) Du(\zeta), D\phi(\zeta)\rangle\, d\zeta +$$

$$+\int_{\mathbb{R}^{n+1}} \Gamma_0(z,\zeta) V(\zeta) u(\zeta) \phi(\zeta)\, d\zeta =$$

$$= I_1(z) + I_2(z) + I_3(z)\ .$$

The quantity $|I_1(z) - I_1(z_0)|$ can be estimate using the inequality

$$|\Gamma_0(z_1,\zeta) - \Gamma_0(z_2,\zeta)| \le \tilde{C}\, \frac{d(z_1,z_2)}{d(z_1,\zeta)^{N-1}}\ ,$$

$\forall\, z_1, z_2, \zeta \in \mathbb{R}^{n+1}$ such that $d(z_1,\zeta) \ge 2d(z_1,z_2)$ (see [19]). The third term satisfy

$$|I_3(z)| \le \sup_{B_{2\rho}(z_0)} |u| \cdot \tilde{\eta}_V(c\sqrt{d(z,z_0)})\ .$$

Let us study the quantity I_2.

$$I_2 = 2\int_{\mathbb{R}^{n+1}} \Gamma_0(z,\zeta) \langle A(\zeta) Du(\zeta), D\phi(\zeta)\rangle\, d\zeta =$$

$$= 2\sum_{j=1}^{m_0} \int_{\mathbb{R}^{n+1}} \Gamma_0(z,\zeta) \partial^2_{x_j} \phi(\zeta) u(\zeta)\, d\zeta +$$

$$+2\sum_{j=1}^{m_0} \int_{\mathbb{R}^{n+1}} \partial x_j \Gamma_0(z,\zeta) \partial_{x_j} \phi(\zeta) u(\zeta)\, d\zeta\ .$$

The first term is estimate as I_1 while the second using the following property

$$|\partial_{x_j}\Gamma_0(z_1,\zeta) - \partial_{x_j}\Gamma_0(z_2,\zeta)| \le \tilde{C}\, \frac{d(z_1,z_2)}{d(z_1,\zeta)^{N}}\ ,$$

(see [19]). Then we have proved that

$$|u(z) - u(z_0)| \le$$

$$\le \Big(C d(z,z_0)^{1/2} + 2\eta_V(5 d(z,z_0)^{1/2}) \Big) \sup_{B_{4r}(z_0)} |u|\ .$$

The second step is to built a Green function for suitable bounded open sets using the parametrix method and show that the set of bounded functions is a class of functions such that the Cauchy-Dirichlet problem is well-posed in the sense that exists and is unique the solution. Finally we use a suitable density argument that allows us to remove the extra assumption of boundedness of u and prove the invariant Harnack inequality.

Remark 2.6. An improvement of the study of operators L_0 having $A = (a_{ij})$ real constant coefficients has been obtained by the authors proving the existence of a Green function, the uniqueness of the solution of the Cauchy-Dirichlet problem and a Harneck inequality for positive solutions of hypoelliptic equations of the type

$$L_0 u + V u = 0$$

where V is in the SK class and

$$L_0 = \sum_{j=1}^{m} X_j^2 + X_0 - \partial_t = \sum_{j=1}^{m} X_j^2 + Y$$

where $X_0, X_1, \ldots, X_m$ is a family of left invariant smooth vector fields in $\mathbb{R}^n$, $n \geq 2$ such that the Lie algebra generated by them equals n (Hörmander condition).
We assume $X_1, \ldots, X_m$ be vector fields of degree one with coefficients of C^∞ class and X_0 of degree two.
This study can be also view as an improvement of the note by Kogoj and Lanconelli [15] because they consider the class of hypoelliptic ultraparabolic equations $L_0 u = 0$, without the term Vu.

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