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Optimal regularity in lower dimensional obstacle problems

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Abstract[1]. We present some recent results on the optimal regularity of the solution and the free boundary in the so called "boundary obstacle" problem, of Signorini type. This kind of problem is strictly connected with the obstacle problem for the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$. Concerning this problem we give an outline of some current developments, also showing the connection with the pricing of perpetual American options in finance.

1. Introduction

The main aim of this paper is to present some recent results on the regularity of the solution and the free boundary in the so called "boundary obstacle" problems. The typical example we have in mind is a version of the Signorini problem. This type of problem arise in elasticity, when an elastic body is at rest, partially lying on a surface, or in optimal control of the temperature across a surface,([1],[9]), or in the modelling of semipermeable membranes, where some saline concentration can flow through a membrane only in one direction ([8]).

Another purpose is to show how boundary obstacle problems are strictly connected with obstacle problems for the pseudodifferential operators $(-\Delta)^s$. Many results concerning one of the two problems can be transferred to the other one, in some case with enormous gain in clarity and simplification of proofs, also of already known results (see [7]).

On the other hand, fractional Laplace operators are the infinitesimal generators of some pure jump Lévy processes, and the related obstacle problems arise in financial mathematics, in particular in pricing perpetual American options.

The plan of the paper is the following. In section 2 we give a formulation of the *thin obstacle problem*, a version of the Signorini problem, we shall deal with. In section 3 we show that the global version turns out to be equivalent to the obstacle problem for the pseudodifferential operator $(-\Delta)^{1/2}$. The fourth section is devoted to the application to financial math. In section 5 we go back to the thin obstacle problem and present some results on the optimal regularity of the solution while section 6 is devoted to the regularity of the free boundary. Finally, in section 7 we outline some current developments on the obstacle problem for the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$.

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2. Thin obstacle and Signorini problems

Let $B_1 = B_1(0)$ be the unit ball in $\mathbb{R}^n, n \geq 2$; we write points $x \in \mathbb{R}^n$ as $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and denote by Π the hyperplane $\{(x', x_n) \; : \; x_n = 0\}$.

Moreover, let $B_1' = B_1 \cap \Pi$ and $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth function (the *thin obstacle*), negative on $\partial B_1'$ and positive in some open set inside B_1'. We look at the unique minimizer of the Dirichlet integral

$$J(v) = \int_{B_1} |\nabla u|^2$$

over the closed convex set

$$\mathbb{K} = \{v \in H_0^1(B_1) \; , \; v(x', 0) \geq \varphi(x') \text{ on } B_1'\} \; .$$

The minimizer u can be constructed also as the least superharmonic function in $\mathbb{K}$. We denote by

$$\Lambda(u) = \{(x', 0) \; : \; u(x', 0) = \varphi(x')\}$$

the *coincidence set* and by $F(u)$ the *free boundary*, that is the boundary of $\Lambda(u)$ on $\Pi \cap B_1$. It is understood that $0 \in F(u)$.

Without loss of generality, we may assume φ symmetric with respect to Π. If not, let w be the solution of $\Delta w = 0$ in $B_{1-\varepsilon}$, $w = u$ on $\partial B_{1-\varepsilon}$. Then $w - u$ is symmetric with respect to Π and satisfies in $B_{1-\varepsilon}$ the same kind of problem with obstacle given by $w(x', 0) - \varphi(x')$.

Restricting ourselves to the upper half ball B_1^+ we can give another characterization of u. Namely, u is the least superharmonic in B_1' satisfying the following conditions:

(a) $u \geq 0$ on $\partial B_1 \cap \{x_n > 0\}$ and $u \geq \varphi$ on B_1'
(b) $u_{x_n}(x', 0) \leq 0$, $u_{x_n}(x', 0) = 0$ if $u(x', 0) > \varphi(x')$.

Conditions (a) and (b) are known as Signorini type conditions; thus, u is a solution of a Signorini problem.

Suitably modifying u, we can convert the above problem in the upper half ball into a problem in the upper half plane (see [11]). In fact let η be a smooth radially symmetric cutoff function supported in B_1, such that $\{\varphi > 0\} \subset\subset \{\eta = 1\}$. Extending it to zero outside B_1^+, the function ηu is defined in $\mathbb{R}_+^n$. Then, we have, on Π,

$$\eta u \geq \varphi \quad , \quad (\eta u)_{x_n} = \eta_{x_n} u + \eta u_{x_n} \leq 0$$

and $(\eta u)_{x_n} = 0$ when $\eta u > \varphi$. Moreover, $\Delta(\eta u)$ is a smooth function. Let now v be the solution of the Neumann problem

$$\Delta v = \Delta(\eta u) \qquad \text{in } \mathbb{R}_+^n$$

and

$$v_{x_n}(x', 0) = 0 \;\text{ in } \mathbb{R}^{n-1} \quad , \quad v \to 0 \;\text{ as }\; |x| \to \infty \; .$$

Then, the function $z = \eta u - v$ is a solution of the following global Signorini problem:

$$
(1) \qquad \begin{cases} \Delta z = 0 & \text{in } \mathbb{R}^n_+ \\ z(x',0) \geq \varphi(x') - v(x',0) \equiv \psi(x') & \text{in } \mathbb{R}^{n-1} \\ z_{x_n}(x',0) \leq 0 \ , \quad z_n(x',0)[z(x',0) - \psi(x')] = 0 \ , & \text{in } \mathbb{R}^{n-1} \\ z \to 0 & \text{as } |x| \to \infty \ . \end{cases}
$$

3. Obstacle problem for the fractional Laplacian

Let now explore the connection between the global Signorini problem (1) and the obstacle problem for the operator $(-\Delta)^{1/2}$.

Let $u_0 : \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth function with rapid decay at infinity and let u be the unique solution of the Dirichlet problem

$$
\begin{cases} \Delta u = 0 & \text{in } \ \mathbb{R}^n_+ \\ u(x',0) = u_0(x') & \text{in } \ \mathbb{R}^{n-1} \\ u \to 0 & \text{as } \ |x| \to \infty \ . \end{cases}
$$

Consider the operator $T : u_0(x') \to -u_{x_n}(x',0)$.

Claim: $T = (-\Delta)^{1/2}$. In fact, since $u_{x_n}(x',x_n)$ is also harmonic in $\mathbb{R}^n_+$, we can write

$$
T \circ T u_0 = -\partial_n(-\partial_n)u_0 = u_{x_n x_n}(x',0) = -\sum_{j=1}^{n-1} u_{x_j x_j}(x',0) = -\Delta u_0 \ .
$$

Moreover

$$
\begin{aligned}
(u_0, Tu_0) = & \int_{\mathbb{R}^{n-1}} -u(x',0)u_{x_n}(x',0) = \\
= & \int_{\mathbb{R}^n} |\nabla u|^2 \ dx + \int_{\mathbb{R}^{n-1}} -u\Delta u \ dx = \int_{\mathbb{R}^n} |\nabla u|^2 \ dx \ .
\end{aligned}
$$

Thus T is a positive operator and the claim is proved. Now, if u is a solution of the Signorini problem (1) and $u_0(x') = u(x',0)$, then

$$
-u_{x_n}(x',0) = (-\Delta)^{1/2} u_0 \geq 0
$$

in $\mathbb{R}^{n-1}$ and therefore u_0 is a solution of the following obstacle problem:

$$
(2) \qquad \begin{cases} u_0 \geq \psi & \text{in } \ \mathbb{R}^{n-1} \\ (-\Delta)^{1/2} u_0 \geq 0 \ , & \\ & \text{in } \ \mathbb{R}^{n-1} \\ (u_0 - \psi)(-\Delta)^{1/2} u_0 = 0 & \\ u_0 \to 0 & \text{as } \ |x| \to \infty \ . \end{cases}
$$

On the other hand, if u_0 solves (2), its harmonic extension in the upper half plane is a solution of (1). Thus, there is a complete equivalence between the two problems.

A solution of (2) can be obtained for instance as the least supersolution of $(-\Delta)^{1/2}$, greater than equal to ψ.

In general, the pseudo differential operator $(-\Delta)^\sigma$, for $\sigma \in (-n/2, 1]$, is defined on the Schwartz space $\mathcal{S}$ of smooth rapidly dacaying functions via the Fourier transform:

$$\widehat{(-\Delta)^\sigma} f(\xi) = |\xi|^{2\sigma} \hat{f}(\xi)$$

or via the singular integral

$$(-\Delta)^\sigma f(x) = c(n, \sigma) \text{ p.v.} \int_{\mathbb{R}^n} \frac{f(x+y) - f(y)}{|y|^{n+2\sigma}} \, dy \; .$$

For $s \in (0, 1]$, we can generalize (2) by replacing $(-\Delta)^{1/2}$ with $(-\Delta)^s$. Again, the solution is obtained as the least supersolution of $(-\Delta)^s$, greater than equal to ψ.

Incidentally, the minimizer can be characterized in another way. If $\dot{H}^s$ denotes the completion of $\mathcal{S}$ with respect to the the norm

$$\|u\|^2_{\dot{H}^s} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x+y) - f(y)}{|y|^{n+2\sigma}} \, dy \, dx = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi$$

then, the solution to (2) is the minimizer of $\|u\|^2_{\dot{H}^s}$ over the set $\{u \geq \psi\}$.

The same considerations hold for the obstacle problem for the operator $(-\Delta)^s + qu$, where q is a positive real number, replacing the norm $\|u\|_{\dot{H}^s}$ by the usual H^s norm

$$\|u\|^2_{H^s} = \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\hat{f}(\xi)|^2 \, d\xi \; .$$

4. Levi processes and mathematical finance

The fractional Laplace operator is the infinitesimal generator of a particular Lévy process. These are $\mathbb{R}^n$-valued stochastic processes $\{X_t\}_{t \geq 0}$ with the following properties.

1. The trajectories are right continuous with left limits, almost surely (a.s.).
2. $X_0 = 0$ a.s.
3. For $0 \leq t_0 < t_1 < \cdots < t_m$, the increments $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_m} - X_{t_{m-1}}$ are independent.
4. The distribution of $X_{t+s} - X_t$ is independent of s.
5. The trajectories are stochastically continuous:

$$\lim_{s \to 0} \text{Prob} \{|X_{t+s} - X_t| > \varepsilon\} = 0 \qquad \forall \varepsilon > 0 \; .$$

To every Lévy process is associated a *characteristic exponent* $\psi = \psi(\xi)$ such that (E denotes expectation)

$$E\left(e^{i\xi \cdot X_t}\right) = e^{-t\psi(\xi)} \qquad \xi \in \mathbb{R}^n \; , \; t > 0 \; .$$

The exponent ψ is given by the Lévy-Khinchin formula

$$\psi(\xi) = -\frac{1}{2} \xi \cdot A\xi + i\gamma \cdot \xi + \int_{\mathbb{R}^n} \left(e^{i\xi \cdot x} - 1 - i\xi \cdot x \chi_{\{|x|<1\}}\right) \nu(dx)$$

where A is a symmetric nonnegative matrix (*Gaussian covariance matrix*), $\gamma \in \mathbb{R}^n$ and ν is a measure (*Lévy measure*) satisfying the condition

$$\int_{\mathbb{R}^n} \min\left\{1, |x|^2\right\} \nu(dx) < \infty \; .$$

The first term represents the diffusive part, the second one is basically a drift term, while the integral part codifies the jump properties of the process. The triplet (A, γ, ν) is called the *generating triplet* the process.

The characteristic exponent

$$\psi(\xi) = |\xi|^\alpha \quad , \quad (0 < \alpha < 2)$$

corresponds to a $\alpha-$*stable symmetric Lévy process* and its infinitesimal generator is precisely $L = (-\Delta)^{\alpha/2}$.

Now, let $0 < s < 1$ and suppose X_t is a $2s-$stable Lévy process such that $X_0 = x$. Let $\mathcal{M}$ the class of finite stopping times τ , $0 \leq \tau$ and define

$$u(x) = \sup_{\tau \in \mathcal{M}} E^x \left[e^{-qt} g(X_\tau) \right]$$

where $q > 0$. Thus, u is a solution of an optimal stopping time with discounted terminal payoff $e^{-qt} g$ and represents the *rational price* of a *perpetual american option*, where the price of the underlying is modelled by $S_t = \exp X_t$.

It turns out that u is the solution of the obstacle problem

$$\begin{cases} u \geq g & \text{in } \mathbb{R}^n \\ q + (-\Delta)^s u \geq 0 \ , & \\ & \text{in } \mathbb{R}^n \\ (u - \psi)(q + (-\Delta)^s u) = 0 & \\ u \to 0 & \text{as } |x| \to \infty \ . \end{cases}$$

One of the main point of interest in this context is the so called *smooth pasting* condition, which amounts to have a C^1 solution of the obstacle problem, across the free boundary.

5. Thin obstacle: optimal regularity of the solution

We now go back to the thin obstacle problem. When studying the regularity of the solution we expect that the optimal regularity is $C^{1,1/2}$ up to the free boundary, from either sides of Π. This follows from the following observation of Hans Lewy in dimension 2, for the zero obstacle problem ($\varphi = 0$). The complex function $w = u_x - i\, u_y$ ($x_n = y$) is analytic outside $\Lambda(u)$, thus

$$u_x u_y = -\frac{1}{2}\, \mathcal{I}m(w^2)$$

is harmonic and vanishes on $y = 0$. Then $u_x u_y$ has a odd harmonic extension across $y = 0$ and w^2 has an analytic extension. Therefore w is $C^{1/2}$ and u is $C^{1,1/2}$.

Frehse in 1978 ([9]) shows that u is C^1, considering a more general operators than the Laplacian. In 1978 Richardson ([10]) proves optimal regularity in dimension 2. In 1979, Caffarelli ([6]) shows $C^{1,\alpha}$ regularity for any $\alpha < 1/2$. He also shows that u is globally Lipschitz continuous and tangentially semiconvex, that is:

$$\|u\|_{Lip(B_{1/2})} \leq C \|u\|_{L^2(B_1)}$$

and

(3) $$\inf_{B_{1/2}} u_{\tau\tau} \geq -C \|u\|_{L^2(B_1)}$$

for every direction $\tau \in \Pi$.

Finally, in 2005, Athanasopoulos and Caffarelli ([4]) managed to prove optimal regularity in any number of dimensions, in the case of zero obstacle. Moreover, they show that u^+ and u^- are subharmonic across $x_n = 0$ and

$$\|u\|_{C^{1,1/2}(B^{\pm}_{1/2})} \le C\|u\|_{L^2(B_1)} \ .$$

One of the key ingredients of their proof is the calculation of the first eigenvalue λ_0 of the problem

$$\begin{aligned} \Delta_\theta w =& \ \lambda w \\ w \in& \ H^{1/2}(\partial B_1^+) \\ w =& \ 0 \ \text{ on } \ (\partial B_1')^- \ , \end{aligned}$$

where Δ_θ is the Laplace-Beltrami operator and

$$(\partial B_1')^- = \{x' = (x'', x_{n-1}) \ : \ x_{n-1} < 0\} \ .$$

It turns out that (Lemma 3 in [4])

$$\lambda_0 = \frac{2n-3}{4}$$

with first eigenfunction given by

$$w_0 = \rho^{1/2} \cos\left(\frac{\omega}{2}\right)$$

where $\rho^2 = x_n^2 + x_{n-1}^2$ and $\tan\omega = x_n/x_{n-1}$. Thus, (∇_θ denotes the tangential gradient)

$$\int_{\partial B_1^+} |\nabla_\theta w|^2 \, d\sigma \ge \frac{2n-3}{4} \int_{\partial B_1^+} w^2 \, d\sigma \tag{4}$$

for every $w \in H^{1/2}(\partial B_1^+)$, $w = 0$ on $(\partial B_1')^-$, with equality when $w = w_0$.

The second main ingredient is the following monotonicity Lemma.

Lemma 1. *Let w be a continuous function in $\overline{B_r^+}$, harmonic in B_r^+, $w(0) = 0$, $w(x', 0) \le 0$, $w(x', 0)w_{x_n}(x', 0) = 0$ for every $x' \in B_r'$. Assume that the set*

$$\{x' \in B_r' \ : \ w(x', 0) < 0\}$$

is non empty and convex. Set

$$\varphi(r) = \frac{1}{r} \int_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-2}} \, dx \ .$$

Then $\varphi(r)$ is finite and increasing in $(0, 1)$.

The proof follows from (4), since

$$\varphi'(r) \ge -\frac{2n-3}{4r^{n+1}} \int_{\partial B_r^+} w^2 \, d\sigma + \frac{1}{r^{n+1}} \int_{\partial B_r^+} |\nabla_\theta w|^2 \, d\sigma \ge 0 \ .$$

To give an idea of how the monotonicity Lemma gives the optimal regularity of u, we consider the simplified case of global solutions, coming for instance as limit as $r \to 0$ of blow up sequences of the type

$$u_r(x) = \frac{u(r\,x)}{\sqrt{r}} \ .$$

Note that these global solutions are tangentially convex, thanks to (3). As a consequence, also the contact set $\Lambda(u)$ is a convex subset of Π.

It is enough to show that u_{x_n} is $C^{1/2}$ up to the coincidence set from either sides. Set $w = u_{x_n}$. Then w satisfies the hypotheses of Lemma 1. Therefore

$$\frac{1}{r^{n-1}} \int_{B_r^+} |\nabla w|^2 \, dx \leq \frac{1}{r} \int_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-2}} \, dx < C \, .$$

Since, by convexity, w vanishes on at least half of the ball B'_r, by Poincaré inequality we have

$$\frac{1}{|B_r|} \int_{B_r^+} w^2 \, dx \leq c \, r^2 \frac{1}{|B_r|} \int_{B_r^+} |\nabla w|^2 \, dx \leq C \, r \, .$$

Since w^2 is subharmonic across $x_n = 0$ we have

$$w^2(x) \leq c \, r \qquad \forall x \in B^+_{r/2}$$

from which the optimal regularity follows.

6. Regularity of the free boundary

In a 2006 paper Athanasopoulos, Caffarelli and Salsa ([5]) take up the question of the regularity of the free boundary for the zero obstacle problem. There is only one previous paper in this direction by Athanasopoulos and Caffarelli [3] in 1985. In this extremely nice paper they prove that if the free boundary is (locally) given by th graph of a Lipschitz function (of the type $x_{n-1} = f(x_1, \cdots, x_{n-2})$) then the free boundary is actually a $(n-2)$-dimensional $C^{1,\alpha}$ manifold.

The optimal regularity of the solution opened the way to study more deeply the properties of the interphase. The strategy, well established after Caffarelli's fundamental work on the classical obstacle problem, is to blow up around a free boundary point, to see which are the possible asymptotic profiles, and from this knowledge infer the corresponding order of regularity of $F(u)$.

In [5] the authors show that there is one basic non-degenerate profile after blow up, and that in a neighborhood of a point with that asymptotic profile, the free boundary is a $C^{1,\alpha}$ manifold.

On the other hand, simple examples show that singular free boundary points and degenerate profile are unavoidable. For instance, in dimension $n = 2$, there are global solutions like

$$\rho^{k+1/2} \cos((k+1/2)\theta) \quad , \quad k \in \mathbb{N} \, , \; k > 1$$

or

$$\rho^{2k} \cos(2k\theta) \quad , \quad k \in \mathbb{N} \, , \; k \geq 1$$

with higher order asymptotic behavior. Notice that $k > 1$ follows from optimal regularity and that these two dimensional solutions can be considered also n-dimensional solutions, constant with respect to the other $n-2$ variables. In correspondence to points with these asymptotic profiles, the free boundary can be very narrow or singular and nothing can be said.

A key tool in the analysis is the following version of the *frequency lemma* of Almgren (Lemma 1 in [5]).

Lemma 2. *Let u be continuous in $\bar{B}_r$, harmonic in $B_r \setminus \Lambda(u)$, with $u(0) = 0$ and $u(x',0)u_{x_n}(x',0) = 0$. Set*

$$D_r(u) = r \ \frac{\displaystyle\int_{B_r} |\nabla u|^2 \, dx}{\displaystyle\int_{\partial B_r} u^2 \, d\sigma} \ \equiv \ r \ \frac{V_r}{S_r} \ .$$

Then, for $0 < r \le 1/2$,

$$\frac{d}{dr} D_r(u) \ge 0 \ .$$

Let $\mu = \lim_{r \to 0} D_r(u)$. Then

$$\frac{d}{dr} D_r(u) \equiv 0 \qquad \text{in } (0, 1/2)$$

if and only if

$$u(x) = |x|^\mu g(\theta) \qquad \theta \in \partial B_1$$

and $\mu \ge 3/2$.

If we set

$$\varphi(r) = \frac{1}{|B_r|} \int_{\partial B_r} u^2 \, d\sigma \ ,$$

Lemma 2 implies that the function

$$r \longmapsto r^{-2\mu} \varphi(r)$$

is increasing. Moreover, let $0 < r < R \le 1$. Then, given $\varepsilon > 0$, for $r < r_0(\varepsilon)$ we have

$$\varphi(R) \le \left(\frac{R}{r} \right)^{2(\mu + \varepsilon)} \varphi(r) \ .$$

This inequality implies that the correct rescaling to get non trivial asymptotic profiles is

$$v_r(x) = \frac{u(r\,x)}{\left(\dfrac{1}{|B_r|} \displaystyle\int_{\partial B_r} u^2 \, d\sigma \right)^{1/2}} \ .$$

In fact, observe that

$$\|v_r\|_{L^2(\partial B_1)} = 1$$

and

$$\|v_r\|_{L^2(\partial B_R)} = R^{(\mu+\varepsilon)}$$

for every $R > 1$ and every small r such that $r\,R \le r_0(\varepsilon)$.

Thus, for $3/2 \le \mu \le 2$, there exists a subsequence v_{r_j} converging in L^2 and uniformly on each compact set in $\mathbb{R}^n$ as $r_j \to 0$ to a non trivial global solution v_0. The following theorem classifies such asymptotic limits.

Theorem 3. (*a*) *Assume $3/2 \le \mu < 2$. Then, up to a multiplicative constant, in a suitable system of coordinates,*

$$v_0(x) = \rho^{3/2} \cos \left(\frac{3}{2} \, \omega \right)$$

where $\rho^2 = x_n^2 + x_{n-1}^2$ and $\tan \omega = x_n / x_{n-1}$.

(*b*) *If* $\mu = 2$, *then* v_0 *is a quadratic polynomial:*

$$v_0(x) = \sum_{i<n} a_i x_i^2 - c\, x_n^2 \quad , \quad a_i \geq 0 \; .$$

When $3/2 \leq \mu < 2$, it turns out that the free boundary is a Lipschitz graph around the origin. Thus, the result in [4] applies and $F(u)$ is actually a $C^{1,\alpha}$ $(n-2)$-dimensional manifold. In fact:

Theorem 4. *Assume* $3/2 \leq \mu < 2$. *Then there exists a neighborhood of the origin* B_ρ *and a cone of tangential directions* $\Gamma'(e_{n-1}, \theta)$ *with axis* e *and opening* $\theta \geq \pi/3$, *such that, for every* $\tau \in \Gamma'(e_{n-1}, \theta)$, *we have*

$$D_\tau u \geq 0 \qquad in \;\; B_\rho \; .$$

In particular, in B_ρ, $F(u)$ *is the graph of a Lipschitz function* $x_{n-1} = f(x_1, \cdots, x_{n-2})$.

7. Some current development

In this section we describe some very recent results and some current development on the optimal regularity of solution and free boundary in the obstacle problem for the fractional Laplacian $(-\Delta)^s$, $0 < s \leq 1$

$$\tag{5} \begin{cases} u \geq \varphi & \text{in } \mathbb{R}^n \\ (-\Delta)^s u \geq 0\;, & \\ & \text{in } \mathbb{R}^n \\ (u-\psi)(-\Delta)^s u = 0 & \\ u \to 0 & \text{as } |x| \to \infty\;. \end{cases}$$

The expected optimal regularity of the solution is $C^{1,s}$. The first important study of this problem has been made by Silvestre ([11]). In his thesis Silvestre proves the $C^{1,\alpha}$ regularity of the solution for any $\alpha < s$. Moreover, if the coincidence set is convex, then he proves optimal regularity. In addition, he proves that u is a semiconvex supersolution of $(-\Delta)^s$.

To achieve optimal regularity in general, Caffarelli and Silvestre ([7]) resort to the same idea expressed in section 3, trying to rediscover the operator $(-\Delta)^s$ as a sort of Dirichlet-Neumann map for a suitable operator.

In fact, this is possible. Let $a = 1 - 2s$ and write points in $\mathbb{R}^{n+1}$ as $(x, y) \in \mathbb{R}^n \times \mathbb{R}$. It turns out that this if we consider the problem

$$\tag{6} \begin{cases} L_a u = \operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u(x,0) = f(x) & \text{in } \mathbb{R}^n \\ u \to 0 & \text{as } |x| \to \infty \end{cases}$$

then,

$$\tag{7} (-\Delta)^s f = \lim_{y \to 0+} -y^a u_y \; .$$

Note that this reduces to the normal derivative case if $s = 1/2$. Morally, formula (7) gives a way to *localize* the non local pseudodifferential operator $(-\Delta)^s$. More precisely, the extension problem (6) allows the use of local p.d.e. methods, greatly simplifying the proof of various important properties of solutions to $(-\Delta)^s f = 0$.

Typical examples in this directions are the Harnack's inequality and the Boundary Harnack's principle ([2]).

From (7), the obstacle problem (5) can be recasted in the following equivalent way:

$$\begin{cases} u(x,0) \geq \varphi(x) & \text{in } \mathbb{R}^n \\ \operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \lim\limits_{y \to 0+} y^a u_y(x,0) = 0 & \text{if } u(x,0) > \varphi(x) \\ \lim\limits_{y \to 0+} y^a u_y(x,0) \leq 0 & \text{in } \mathbb{R}^n \end{cases}$$

which corresponds to a a thin obstacle problem for the operator L_a.

Optimal regularity of solution and free boundary for this problem is object of investigations of Caffarelli, Salsa and Silvestre in a current work, with the idea of using the same strategy in [5] for the Laplace operator.

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