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Cone criterion for non-divergence equations modeled on Hörmander vector fields

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Abstract[1]. We are concerned with an exterior cone criterion of regularity for the Dirichlet problems related to a class of sub-elliptic operators. Our criterion also provides an uniform boundary estimate of the moduli of continuity of the solutions.

1. Introduction

In this note we present some results obtained in collaboration with E. Lanconelli. Further developments will appear in the forthcoming paper [20]. We are concerned with an exterior cone criterion of regularity of boundary points for the Dirichlet problems related to a class of PDO including operators in the form

$$\mathcal{H} = \sum_{i,j=1}^{m} a_{i,j}(x,t) X_i X_j + \sum_{j=1}^{m} b_j(x,t) X_j + q(x,t) - \partial_t \ , \ (x,t) \in \mathbb{R}^{N+1} \ , \tag{1.1}$$

$$\mathcal{L} = \sum_{i,j=1}^{m} a_{i,j}(x) X_i X_j + \sum_{j=1}^{m} b_j(x) X_j + q(x) \ ,x \in \mathbb{R}^N \ .$$

Here $X_1, \cdots, X_m$ are Hörmander vector fields in $\mathbb{R}^N$ and $a_{i,j}$, b_j, q are Hölder continuous functions (with respect to the Carnot-Carathéodory distance d induced by $X_1, \cdots, X_m$) such that $\Lambda^{-1}|\eta|^2 \leq \sum a_{i,j}\, \eta_i \eta_j \leq \Lambda |\eta|^2$, for some $\Lambda > 0$ and for every $\eta \in \mathbb{R}^m$.

Theorem 1.1. *Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ satisfying the exterior d-cone condition* (3.4) *at any point of its "parabolic boundary" $\partial_p\Omega$.* [2] *For every continuous function φ in $\partial_p\Omega$ and every d-Hölder continuous function f in a neighborhood of $\overline{\Omega}$, there exists a unique solution u to the problem*

$$\begin{cases} \mathcal{H}u = f & in\, \Omega \ , \\ u = \varphi & in\, \partial_p\Omega \ . \end{cases}$$

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[2]We assume for simplicity that Ω is a cylinder. Otherwise the term "parabolic boundary" loses its meaning. Anyway the theorem still holds true for general open sets Ω, defining $\partial_p\Omega$, for instance, as the closure of the set $\{(x,t) \in \partial\Omega \ | \ t < \sup_{(\xi,\tau)\in\Omega} \tau\}$.

Moreover u is locally d-Hölder continuous together with its derivatives along $X_1, \cdots$, X_m *up to second order, and along* ∂_t (*up to first order*).

An analogous result holds for the stationary operators $\mathcal{L}$, under more restrictive hypotheses. We refer to Theorem 4.1, Theorem 4.7 and Theorem 4.9 for the precise statements. Our criterion also provides an uniform estimate of the moduli of continuity at the boundary of the solutions to the Dirichlet problems. This estimate is uniform with respect to the operators (varying in suitable classes) and it turns out to be a crucial step in the proof of an invariant Harnack inequality for the operators in (1.1) (which is given in the forthcoming papers [3, 9]). This was actually the starting motivation for this work.

Our operators arise in many theoretical and applied settings sharing a sub-Riemannian underlying geometry: e.g. in diffusion theory, mathematical models for finance and for human vision, control theory, geometric theory of several complex variables (see e.g. [10, 11, 14, 15, 18, 19, 21, 22, 25, 28] and references therein). In particular the *nonvariational* operators in (1.1) appear as linearizations of the Levi curvature equations, which are fully nonlinear second order PDE which can be written in the following form (see [21]):

$$(1.2) \qquad \sum_{i,j=1}^{2n} a_{i,j}(Du, D^2u) X_i X_j u = K(x, u, Du) \qquad \text{in } \mathbb{R}^{2n+1} .$$

(In (1.2), the vector fields X_j also depend on Du). This (non-elliptic) equation is the complex-analogue of the classical Monge-Ampère one, as its solutions are functions u whose graphs are CR manifolds in $\mathbb{C}^{n+1}$ with prescribed Levi curvature K (see e.g. [21] and references therein). Existence of classical solutions to (1.2) is still a widely open problem. This paper is part of a project aimed to provide the linear framework for the Levi-Monge-Ampère equation (1.2) (see also [1, 2, 3, 9] and references therein).

Several results concerning the divergence form counterpart of (1.1) are present in the literature. On the contrary (at least to the authors knowledge) very few papers are devoted to the non-divergence form operators (1.1) (beside the ones just cited, we can only quote [6, 7, 8, 29]). As far as the model operators $\sum X_j^2$, $\sum X_j^2 - \partial_t$, some classical references are [5, 13, 16, 17, 23, 26, 27].

We now give a plan of the paper and write down some of the results. For more precise statements and notation explanations, we directly refer to the respective sections. In Section 2 we provide a few preliminary potential-theoretic results for some degenerate-elliptic hypoelliptic operators in the form

$$(1.3) \quad \mathcal{H} = \sum_{i,j=1}^{N} a_{i,j}(x,t) \partial^2_{x_i,x_j} + \sum_{j=1}^{N} b_j(x,t) \partial_{x_j} + q(x,t) - \partial_t \ , \ (x,t) \in \mathbb{R}^N \times]T_1, T_2[\, ,$$

under the assumption that there exists a global well-behaved fundamental solution Γ for $\mathcal{H}$. The main goal of this section is to construct (in the case $q \le 0$) equilibrium potentials $\Gamma * \mu_K$ for $\mathcal{H}$ satisfying, besides other properties, the following one:

$$(1.4) \qquad |\widetilde{K}| \le c \, \mu_K(K) \, ,$$

for every compact set $K = \widetilde{K} \times \{\lambda\}$, $\widetilde{K} \subset\subset \mathbb{R}^N$ (see Theorem 2.11, see also Proposition 2.8).

Section 3 is devoted to our exterior cone criterion, starting from (1.4). We consider a family $\mathcal{F}$ of operators $\mathcal{H}$ as in Section 2, whose fundamental solutions are assumed to satisfy suitable estimates, uniform in $\mathcal{H} \in \mathcal{F}$, given in terms of some underlying distance d on $\mathbb{R}^N$. The main result of this section reads as follows.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^{N+1}$ be a bounded open set and let*

$$(1.5) \qquad A = \{(x_0, t_0) \in \partial\Omega \mid \exists\, M, T, \theta > 0 \; : \; \forall\, \lambda \in]0, T[\text{ we have} \\ \left|\{x \in B_d(x_0, \sqrt{M\lambda}) \; : \; (x, t_0 - \lambda) \notin \overline{\Omega}\}\right| \;\geq\; \theta \left|B_d(x_0, \sqrt{M\lambda})\right|\Big\} \; .$$

In other words A is the set of boundary points satisfying the exterior d-cone condition. Then, for every continuous datum φ on $\partial\Omega$ and for every operator $\mathcal{H}$ in the family $\mathcal{F}$, there exists a classical solution $u^{\mathcal{H}}$ to

$$\begin{cases} \mathcal{H}u = 0 & \text{in } \Omega \, , \\ u = \varphi & \text{in } A \, . \end{cases}$$

Moreover, we have the following uniform estimates of the moduli of continuity of the solutions $u^{\mathcal{H}}$ at any boundary point $(x_0, t_0) \in A$: for every $\varepsilon > 0$ there exists $\rho > 0$ such that

$$(1.6) \qquad \sup_{\mathcal{H} \in \mathcal{F}} |u^{\mathcal{H}}(x,t) - \varphi(x_0,t_0)| \leq \varepsilon \;\text{ if }\; (x,t) \in \Omega \;\;,\;\; d(x,x_0)^4 + (t-t_0)^2 < \rho \, ,$$

where ρ depends on φ only through a bound for its norm in $\partial\Omega$ and through its modulus of continuity at (x_0, t_0).

Finally, in Section 4 we furnish some examples of applications of the previous results to the operators $\mathcal{H}$ and $\mathcal{L}$ in (1.1), for which fundamental solutions have been constructed in [2, 9]. The family $\mathcal{F}$ of the operators $\mathcal{H}$ in (1.1) which have smooth coefficients and zero order term $q \leq 0$, turns out to satisfy the assumptions of Section 3. Hence Theorem 1.2 holds for this family. Starting from here, we then show that we can remove the assumption on the smoothness of the coefficients and the assumption $q \leq 0$.

Theorem 1.3. *The statement of Theorem 1.2 holds true if $\mathcal{F}$ is the family of the non-divergence form operators $\mathcal{H}$ in* (1.1)*. The uniform estimates* (1.6) *depends on $\mathcal{H}$ only through the vector fields $X_1 \cdots, X_m$, the constant Λ and the Hölder constants of the coefficients $a_{i,j}, b_j, q$.*

As a consequence, using the results in [2, 9], we are then able to prove Theorem 1.1.

The above theorems also allow to prove some partial results of the same kind for the stationary operator $\mathcal{L}$ in (1.1). In this setting the exterior d-cone condition at a point $x_0 \in \partial\Omega$ takes the form

$$(1.7) \qquad \exists\, r_0, \theta > 0 \; : \; |B_d(x_0, r) \setminus \overline{\Omega}| \;\geq\; \theta\, |B_d(x_0, r)| \qquad \forall\, r \in]0, r_0[\, .$$

In particular we obtain the stationary counterpart of Theorem 1.3, under the additional hypothesis $q \leq 0$. We also obtain a stationary counterpart of Theorem 1.1 in the setting of Carnot groups using the results in [2]. In this case we also give a further example of application, showing that $\mathcal{L}$ endows any bounded domain of $\mathbb{R}^N$ with a structure of β-harmonic space (in the sense of [12]). We directly refer to Theorem 4.7 and Theorem 4.9 for the precise statements of these results.

2. Equilibrium potentials

Throughout this section we shall denote by $\mathcal{H}$ a partial differential operator in the form

$$\mathcal{H} = \sum_{i,j=1}^{N} a_{i,j}(x,t)\partial^2_{x_i,x_j} + \sum_{j=1}^{N} b_j(x,t)\partial_{x_j} + q(x,t) - \partial_t \ , \tag{2.1}$$

with smooth real-valued coefficients $a_{i,j} = a_{j,i}$, b_j, q, defined in some strip $E = \mathbb{R}^N \times]T_1, T_2[\ \ (-\infty \le T_1 < T_2 \le \infty)$. We shall make the following assumptions on $\mathcal{H}$:

(H2.1) *$\mathcal{H}$ is hypoelliptic, not totally degenerate (i.e. $\forall\, z \in E \ \exists\, i,j \ : \ a_{i,j}(z) \neq 0$) and degenerate-elliptic (i.e. $\sum_{i,j=1}^{N} a_{i,j}(z)\xi_i\xi_j \ge 0 \ \forall\, z \in E \ \forall\, \xi \in \mathbb{R}^N$).*

(H2.2) *There exists a Borel function $\Gamma : E \times E \to [0, \infty[$ with the following properties:*

(2.2) *For every $\zeta \in E$ we have $\Gamma(\cdot, \zeta) \in L^1_{\mathrm{loc}}(E)$. Moreover, for every compact set $K \subseteq E$, the function $\zeta \mapsto \int_K \Gamma(\cdot, \zeta)$ is locally bounded in E.*

(2.3) *For every $\zeta \in E$ we have $\mathcal{H}(\Gamma(\cdot,\zeta)) = -\delta_\zeta$ in $\mathcal{D}'(E)$ (i.e. Γ is a fundamental solution for $\mathcal{H}$). In particular, by (H2.1), $\Gamma(\cdot,\zeta) \in C^\infty(E \setminus \{\zeta\})$.*

(2.4) $\Gamma(x,t,\xi,\tau) = 0$ *iff* $t \le \tau$.

(2.5) *Γ is bounded on $K_1 \times K_2$, if K_1, K_2 are disjoint compact subsets of E.*

(2.6) *$\Gamma(z,\zeta) \to 0$, as $|z| \to \infty$, uniformly in $(z,\zeta) \in S \times K$, for every compact set $K \subseteq E$ and for every strip $S = \mathbb{R}^N \times [t_1, t_2]$, with $T_1 < t_1 < t_2 < T_2$.*

(2.7) *For every $T > 0$ there exists a positive constant $\beta(T)$ such that*

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\, d\xi \le \beta(T) \ \ , \ \ \beta(T)^{-1} \le \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\, dx \le \beta(T) \ ,$$

if $0 < t - \tau < T$.

Remark 2.1. The above assumptions are satisfied, e.g., when $\mathcal{H}$ is the operator in (4.1), if the coefficients of $\mathcal{H}$ are assumed to be smooth, see Proposition 4.2.

Remark 2.2. Assumption (2.6) can be slightly weakened in the following way: for every compact set $K \subseteq E$ and for every strip $S = \mathbb{R}^N \times [t_1, t_2]$, with $T_1 < t_1 < t_2 < T_2$, we have

$$\Gamma(z,\zeta_0) \to 0 \ \text{, as } \ |z| \to \infty \ \ , \ \ \text{uniformly in } z \in S \ \ , \ \ \text{for every fixed } \zeta_0 \in E \ , \tag{2.8}$$

(2.9) there exists a compact set $F \subseteq \mathbb{R}^{N+1}$ such that Γ is bounded in $(S \setminus F) \times K$.

Remark 2.3. Assumption (2.7) can be slightly weakened in the following way: for every compact set $K \subseteq E$ there exists a positive constant $\beta(K)$ such that

$$\exists\, \alpha \ : \ \tau < \alpha < T_2 \ \ , \ \ \beta(K)^{-1} \le \int_{\mathbb{R}^N} \Gamma(x,\alpha,\xi,\tau)\, dx \le \beta(K) \ \ \forall\, (\xi,\tau) \in K \ , \tag{2.10}$$

(2.11) $\exists\, \varepsilon > 0 \ : \ \int_F \Gamma(x,t,\xi,\lambda)\, d\xi \le \beta(K)$ for every compact set $F \subseteq \mathbb{R}^N$ and $\lambda \in \mathbb{R}$ such that $F \times \{\lambda\} \subseteq K$ and for every $(x,t) \in K$ such that $|t - \lambda| < \varepsilon$.

For any given open set $\Omega \subseteq E$, we denote by $\mathrm{P}(\Omega)$ the set of the $\mathcal{H}$-harmonic functions on Ω, i.e. the set of the smooth solutions u to the equation $\mathcal{H}u = 0$ in Ω. Clearly P is a harmonic sheaf on E. Moreover[3], we denote by $\mathrm{P}^*(\Omega)$ ($\overline{\mathrm{P}}(\Omega)$) the set of the $\mathcal{H}$-hyperharmonic ($\mathcal{H}$-superharmonic) functions on Ω. The following result is essentially contained in [24].

Proposition 2.4. *For every bounded open set Ω such that $\overline{\Omega} \subseteq E$ we have*

(i) (Ω, P) *is a β-harmonic space in which the axiom of Doob holds.*
(ii) *Given a function $u : \Omega \to]-\infty, \infty]$, we have*[4]

$$(u \in L^1_{\mathrm{loc}}(\Omega)\ ,\ \mathcal{H}u \le 0 \mathit{in}\, \mathcal{D}'(\Omega)) \quad \Leftrightarrow \quad (\exists\, v \in \overline{P}(\Omega)\ :\ u = v\ a.e.\ in\, \Omega)$$

We explicitly remark that the hypothesis [24, (iv) page 86] can be replaced by the assumption that $\mathcal{H}$ is hypoelliptic. Moreover, the hypothesis [24, (v) page 86] is satisfied by taking $\theta(x,t) = e^{(m+1)t}$, $\theta^*(x,t) = e^{-(m^*+1)t}$, where $m = \max_{\overline{\Omega}} |q|$, $m^* = \max_{\overline{\Omega}} |q^*|$ (being q^* the zero order term of the adjoint operator $\mathcal{H}^*$), so that $\theta, \theta^* > 0$, $\mathcal{H}\theta, \mathcal{H}^*\theta^* < 0$ in Ω.

Remark 2.5. If u is a C^2 function on some open set $\Omega \subseteq E$, such that $\mathcal{H}u \le 0$ in Ω, then $u \in \overline{\mathrm{P}}(\Omega)$. This immediately follows from the definition of $\overline{\mathrm{P}}(\Omega)$ and from the classical Picone's weak maximum principle for degenerate-elliptic equations, using the barrier function θ defined above.

From abstract potential theory we know that the following minimum principle for $\mathcal{H}$-hyperharmonic functions holds on every bounded open set Ω such that $\overline{\Omega} \subseteq E$:

$$u \in \mathrm{P}^*(\Omega) \quad \liminf_{\partial\Omega} u \ge 0 \quad \Rightarrow \quad u \ge 0 \text{in } \Omega \tag{2.12}$$

Starting from (2.12), we can easily derive the following "hyperparabolic" minimum principles.

Proposition 2.6. *Let Ω be a bounded open set such that $\overline{\Omega} \subseteq \mathbb{R}^N \times]T_1, T_2]$ and let $t_0 \in]T_1, T_2]$. If $u \in \mathrm{P}^*(\Omega \cap \{t < t_0\})$ and $\liminf_{(\partial\Omega)\cap\{t<t_0\}} u \ge 0$, then $u \ge 0$ in $\Omega \cap \{t < t_0\}$.*

Proposition 2.7. *Let $S = \mathbb{R}^N \times]t_1, t_2[\setminus F$, where $T_1 < t_1 < t_2 < T_2$ and F is a closed subset of $\mathbb{R}^{N+1}$. Suppose that the zero order term q of $\mathcal{H}$ is bounded above in S. If $u \in \mathrm{P}^*(S)$, $\liminf_{(\partial S)\cap\{t<t_2\}} u \ge 0$ and $\liminf_{|z|\to\infty} u(z) \ge 0$, then $u \ge 0$ in S.*

In the sequel we shall denote by $\mathcal{M}$ the set of the (positive) Radon measures on E, compactly supported in E. Moreover, for any given compact set $K \subseteq E$, we shall denote by $\mathcal{M}(K)$ the set of the measures $\mu \in \mathcal{M}$ supported in K. In the following proposition we collect some properties of the Γ-potentials of measures $\mu \in \mathcal{M}$.

Proposition 2.8. *Let $\mu \in \mathcal{M}$ and define*

$$\Gamma * \mu(z) = \int_E \Gamma(z,\zeta)\, d\mu(\zeta)\ , \quad z \in E\ .$$

[3] We refer the reader to [12] for any unexplained notation from abstract potential theory

[4] This means that $\int u\mathcal{H}^*\varphi \le 0$ for every nonnegative test function $\varphi \in C_0^\infty(\Omega)$, where $\mathcal{H}^*$ is the formal adjoint operator of $\mathcal{H}$.

Then we have

$$\Gamma * \mu \in L^1_{\mathrm{loc}}(E) \quad , \quad \mathcal{H}(\Gamma * \mu) = -\mu \text{ in } \mathcal{D}'(E) \,, \tag{2.13}$$

$$\Gamma * \mu \in \mathrm{P}(E \setminus \operatorname{supp} \mu) \,, \tag{2.14}$$

$$\Gamma * \mu \in \overline{P}(E) \,, \tag{2.15}$$

$$\Gamma * \mu(z) \to 0 \,, \text{ as } |z| \to \infty \,, \text{ uniformly in } z \in \mathbb{R}^N \times [t_1, t_2] \,, \text{ for every } T_1 < t_1 < t_2 < T_2 \,. \tag{2.16}$$

Let $\Omega \subseteq E$ be an open set and let $u \in \overline{\mathrm{P}}(\Omega)$. From Proposition 2.4-(ii), we infer the existence of a (unique) Radon measure $\mu[u]$ on Ω such that $\mathcal{H}u = -\mu[u]$ in $\mathcal{D}'(\Omega)$ (the Riesz measure of u). Moreover, using also the hypoellipticity of $\mathcal{H}$ and (2.13), it is standard to derive the following Riesz-type representation theorem.

Theorem 2.9. *Let $\Omega \subseteq E$ be an open set and let $u \in \overline{\mathrm{P}}(\Omega)$. For every bounded Borel set A such that $\overline{A} \subseteq \Omega$, there exists $h \in \mathrm{P}(\mathring{A})$ such that $u = h + \Gamma * (\mu[u]_{|A})$ a.e. in $\mathring{A}$.*

We are now ready to approach the core of this section, namely the construction of equilibrium potentials. Let us suppose that $q \leq 0$ in E (recall that q is the zero order term of $\mathcal{H}$), so that $1 \in \overline{\mathrm{P}}(E)$. For any given compact set $K \subseteq E$, we set

$$W_K = \inf_{\mathrm{P}hi_K} v \quad , \quad U_K = \min\{W_K, \liminf W_K\} \,, \tag{2.17}$$

where $\mathrm{P}hi_K = \{v \in \overline{\mathrm{P}}(E) \;|\; v \geq 0 \text{in } E \,,\; v \geq 1 \text{in } K\}$. We obviously have

$$0 \leq U_K \leq W_K \leq 1 \text{ in } E \quad , \quad W_K = 1 \text{ in } K \quad , \quad U_K = W_K = 1 \text{ in } \mathring{K} \,. \tag{2.18}$$

Moreover it is standard to prove that

$$U_K|_{E \setminus K} = W_K|_{E \setminus K} \in \mathrm{P}(E \setminus K) \quad , \quad U_K \in \overline{\mathrm{P}}(E) \,. \tag{2.19}$$

From the properties of Γ, we easily obtain the following

Proposition 2.10. *In the above hypotheses, we have*

$$U_K = 0 \text{ in } K^- := \{(x,t) \in E \;|\; t \leq \tau \;\forall\, (\xi, \tau) \in K\} \,, \tag{2.20}$$

$$U_k(z) \to 0 \,, \text{ as } |z| \to \infty \,, \text{ uniformly in } z \in S \,, \text{ for every strip } S = \mathbb{R}^N \times [t_1, t_2] \,, \text{ with } T_1 < t_1 < t_2 < T_2 \,. \tag{2.21}$$

Let us denote for brevity by $\mu_K = \mu[U_K]$ the Riesz measure of U_K. Then $\mu_K \in \mathcal{M}(K)$, by (2.19). The following is the main result of this section.

Theorem 2.11. *Suppose that $q \leq 0$ in E. For any given compact set $K \subseteq E$, we have:*

$$U_K = \Gamma * \mu_K \quad \text{a.e. in } E \,, \tag{2.22}$$

$$U_K = \Gamma * \mu_K \quad \text{in } E \setminus \partial K \,, \tag{2.23}$$

$$\Gamma * \mu_K \leq 1 \text{ in } E \quad , \quad \Gamma * \mu_K = 1 \text{ in } \mathring{K} \,. \tag{2.24}$$

Moreover, if K is of the form $K = \widetilde{K} \times \{\lambda\}$, for some $\lambda \in \mathbb{R}$ and some compact set $\widetilde{K} \subseteq \mathbb{R}^N$, then the N-dimensional Lebesgue measure $|\widetilde{K}|$ of $\widetilde{K}$ satisfies the following estimate

$$|\widetilde{K}| \leq \beta(F)^3 \mu_K(K) , \tag{2.25}$$

if $F \subseteq E$ is a fixed compact set such that $K \subseteq \overset{\circ}{F}$ ($\beta(F)$ is defined in Remark 2.3).

We explicitly remark that μ_K does not vanish inside K (if $q \neq 0$).

3. A uniform regularity criterion for boundary points

Throughout this section we shall denote by $\mathcal{F}$ a family of partial differential operators as in Section 2, in the form

$$\mathcal{H} = \sum_{i,j=1}^{N} a_{i,j}(x,t)\partial^2_{x_i,x_j} + \sum_{j=1}^{N} b_j(x,t)\partial_{x_j} + q(x,t) - \partial_t ,$$

with smooth real-valued coefficients $a_{i,j} = a_{j,i}$, b_j and $q \leq 0$, defined in some strip $E = \mathbb{R}^N \times]T_1, T_2[$ ($-\infty \leq T_1 < T_2 \leq \infty$ fixed) and satisfying assumptions (H2.1)-(H2.2) of Section 2 (see also Remarks 2.2 and 2.3). Moreover we shall make the following hypotheses on $\mathcal{F}$:

(H3.1) *The constant $\beta(K)$ in Remark 2.3 can be made independent on $\mathcal{H}$ (i.e. uniform in $\mathcal{H} \in \mathcal{F}$).*

(H3.2) *There exists a distance d on $\mathbb{R}^N$ inducing the Euclidean topology, such that, setting $d_p(x,t,\xi,\tau) = (d(x,\xi)^4 + (t-\tau)^2)^{1/4}$ for every $(x,t), (\xi,\tau) \in \mathbb{R}^{N+1}$, the following estimates hold*[5]*:*

(3.1) *For every compact set $K \subseteq E$ and for every $\sigma > 0$ there exists a positive constant $C_1(\sigma, K)$ such that $\Gamma^{\mathcal{H}}(z,\zeta) \leq C_1(\sigma,K)$ for every $\mathcal{H} \in \mathcal{F}$ and for every $z, \zeta \in K$ such that $d_p(z,\zeta) \geq \sigma$.*

(3.2) *For every compact set $K \subseteq E$ and for every $M > 0$ there exists a positive constant $C_2(M,K)$ such that*[6] *$\Gamma^{\mathcal{H}}(x,t,\xi,\tau) \geq C_2(M,K)^{-1}|B_d(x, \sqrt{M(t-\tau)})|^{-1}$ for every $\mathcal{H} \in \mathcal{F}$ and for every $(x,t), (\xi,\tau) \in K$, satisfying $d(x,\xi)^2 \leq M(t-\tau)$, $0 < t-\tau < 1$.*

(3.3) *For every $z_0 = (x_0,t_0) \in E$ and $M, \eta > 0$ there exists a positive constant $\delta = C_3(\eta, M, z_0)^{-1}$ such that $\left|(\Gamma^{\mathcal{H}}(z,\zeta)/\Gamma^{\mathcal{H}}(z_0,\zeta)) - 1\right| \leq \eta$ for every $\mathcal{H} \in \mathcal{F}$ and for every $z = (x,t), \zeta = (\xi,\tau) \in E$ satisfying $d_p(z,z_0) \leq \delta\, d_p(z_0,\zeta) \leq \delta^2$, $d(x,\xi)^2 \leq M(t-\tau)$, $d(x_0,\xi)^2 \leq M(t_0-\tau) > 0$.*

Remark 3.1. The above hypotheses are satisfied, e.g., when $\mathcal{F}$ is the family $\mathcal{F}_s^-$ of the PDO in the form (4.1) with smooth coefficients and zero order term $q \leq 0$, see Proposition 4.2.

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[5]Given $\mathcal{H} \in \mathcal{F}$, we shall use all the notation introduced in Section 2, adding the superscript $^{\mathcal{H}}$. For instance $\Gamma^{\mathcal{H}}$ will denote the fundamental solution for $\mathcal{H}$ as in (H2.2). The superscript will be sometimes omitted when there is no risk of confusion.

[6]We denote by $B_d(x,r) = \{y \in \mathbb{R}^N \mid d(x,y) < r\}$ the balls in the metric d and by $|\cdot|$ the N-dimensional Lebesgue measure.

Let now Ω be a bounded open set such that $\overline{\Omega} \subseteq E$ and let $z_0 \in \partial\Omega$. Ω and $z_0 = (x_0, t_0)$ will be fixed throughout the section and we shall assume that Ω satisfies the following *exterior d-cone condition* at z_0:

(3.4) there exist $M, T, \theta > 0$ such that, for every $\lambda \in]0, T[$, we have
$$|\{x \in B_d(x_0, \sqrt{M\lambda}) \ : \ (x, t_0 - \lambda) \notin \overline{\Omega}\}| \ \geq \ \theta \, |B_d(x_0, \sqrt{M\lambda})| \, .$$

Hereafter in this section, we shall denote by $\mathbf{c}$ any positive constant that depends only on E, Ω, z_0, d, on the constants M, T, θ in (3.4) and on the constants β, C_1, C_2, C_3 in (H3.1)-(H3.2) (more precisely on $\beta(K_0)$, $C_2(M, K_0)$ and on the functions $\sigma \mapsto C_1(\sigma, K_0)$ and $\eta \mapsto C_3(\eta, 8M, z_0)$, being $K_0 \subseteq E$ a fixed compact neighborhood of $\overline{\Omega}$). Moreover, we shall use the notation $\mathbf{c}(f_1, \cdots, f_n)$ for constants also depending on $f_1, \cdots, f_n$.

The following theorem is the main result of this section.

Theorem 3.2. *Under the above hypotheses, let $\varphi \in C(\partial\Omega, \mathbb{R})$. For any $\mathcal{H} \in \mathcal{F}$, let us denote by $u^{\mathcal{H}} = {}^{\mathcal{H}}H^{\Omega}_{\varphi}$ the Perron-Wiener generalized solution to the Dirichlet problem*
$$\begin{cases} \mathcal{H}u = 0 & in \, \Omega \, , \\ \ u = \varphi & in \, \partial\Omega \, . \end{cases}$$
Then, for every $\varepsilon > 0$ there exists a positive constant $\rho = \mathbf{c}(\varepsilon, \eta)^{-1}$ such that
$$\sup_{\mathcal{H} \in \mathcal{F}} |u^{\mathcal{H}}(z) - \varphi(z_0)| \leq \varepsilon (1 + 2 \max_{\partial\Omega} |\varphi|) \quad , \quad \forall \, z \in \Omega \, : \, d_p(z, z_0) \leq \rho \, .$$
Here η denotes a given bound for the modulus of continuity of φ at z_0 (i.e. a function $\eta : [0, \delta] \to \mathbb{R}$ such that $\eta(s) \to 0$ as $s \to 0$ and $|\varphi(z) - \varphi(z_0)| \leq \eta(d_p(z, z_0))$ for every $z \in \partial\Omega$ such that $d_p(z, z_0) \leq \delta$).

We recall that the Perron-Wiener function ${}^{\mathcal{H}}H^{\Omega}_{\varphi} \in \mathrm{P}^{\mathcal{H}}(\Omega)$ (i.e. it is a smooth solution to $\mathcal{H}u = 0$ in Ω), since $(\overset{\circ}{K}_0, \ \mathrm{P}^{\mathcal{H}})$ is a β-harmonic space by Proposition 2.4-(i).

We shall now briefly sketch the steps of the proof of the above theorem. Let us fix some notation. For every $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ and $R > 0$, we set $P_R(\zeta) = \{(x, t) \in \mathbb{R}^{N+1} \ | \ d(x, \xi)^2 \leq R(t - \tau)\}$, $P^-_R(\zeta) = \{(x, t) \in \mathbb{R}^{N+1} \ | \ d(x, \xi)^2 \leq R(\tau - t)\}$. For every $\lambda \in]0, T]$ and $\mathcal{H} \in \mathcal{F}$, we set
$$G_\lambda = (P^-_M(z_0) \cap \mathbb{R}^N \times [t_0 - \lambda, t_0]) \setminus \Omega \quad , \quad u^{\mathcal{H}}_\lambda = V^{\mathcal{H}}_{G_\lambda} = \Gamma^{\mathcal{H}} * \mu^{\mathcal{H}}_{G_\lambda}$$
(recall Theorem 2.11). It is not restrictive to suppose $T < 1/2$ small enough so that $\{(x, t) \ : \ d(x, x_0)^2 \leq 2MT \ , \ |t - t_0| \leq T\} \subseteq \overset{\circ}{K}_0$ (T, M have been introduced in (3.4)). Next lemma can be easily proved using the estimates (3.1), (3.2) and (2.24).

Lemma 3.3. *For every $\sigma > 0$ there exists a positive constant $\lambda_\sigma = \mathbf{c}(\sigma)^{-1}$ ($< T$) such that $\sup_{\mathcal{H} \in \mathcal{F}} u^{\mathcal{H}}_\lambda(z) \leq 1/2$ for every $\lambda \in]0, \lambda_\sigma[$ and for every $z \in \Omega$ such that $d_p(z, z_0) \geq \sigma$.*

In force of the d-cone condition (3.4), for every $\lambda \in]0, T[$ we can find a compact set $\widetilde{F}_\lambda \subseteq \widetilde{A}_\lambda := \{x \in B_d(x_0, \sqrt{M\lambda}) \ : \ (x, t_0 - \lambda) \notin \overline{\Omega}\}$ such that

$$|\widetilde{F}_\lambda| \geq \frac{1}{2} \, |\widetilde{A}_\lambda| \geq \frac{\theta}{2} \, |B_d(x_0, \sqrt{M\lambda})| \, . \tag{3.5}$$

In the following we shall denote $F_\lambda = \widetilde{F}_\lambda \times \{t_0 - \lambda\}$.

Making use of (3.3) and (2.24) one can prove the following

Lemma 3.4. *For every $\eta > 0$ there exists a natural number $q_\eta = \mathbf{c}(\eta)$ such that the following statement holds for every $\lambda \in]0, T[$, $p, q \in \mathbb{N}$ such that $q \geq q_\eta$. There exists a positive constant $\rho = \mathbf{c}(\eta, \lambda, p, q)^{-1}$ such that, setting*

$$w^{\mathcal{H}} = \sum_{k=1}^{p} v_k^{\mathcal{H}} \quad , \quad v_k^{\mathcal{H}} = \Gamma^{\mathcal{H}} * \mu^{\mathcal{H}}_{F_{\lambda^{kq}}} \, ,$$

we have

$$w^{\mathcal{H}}(z) \leq \quad 1 + (1+\eta) w^{\mathcal{H}}(z_0) \quad \forall\, z \in \overset{\circ}{G}_T \, ,$$

$$w^{\mathcal{H}}(z) \geq \quad (1-\eta) w^{\mathcal{H}}(z_0) \quad \forall\, z \in E \, : \, d_p(z, z_0) \leq \rho \, ,$$

for every $\mathcal{H} \in \mathcal{F}$.

Exploiting Theorem 2.11 and the d-cone condition, along with (3.2) and (3.5), this allows to prove

Lemma 3.5. *For every $\eta > 0$ and $\lambda \in]0, T[$ there exists a positive constant $\rho = \mathbf{c}(\lambda, \eta)^{-1}$ such that $u_\lambda^{\mathcal{H}}(z) \geq 1 - \eta$ for every $\mathcal{H} \in \mathcal{F}$ and for every $z \in E$ such that $d_p(z, z_0) \leq \rho$.*

For every $\mathcal{H} \in \mathcal{F}$ we set $w^{\mathcal{H}} = \sum_{k=1}^{\infty} 2^{-k} u^{\mathcal{H}}_{2^{-k}T^k}$. From (2.24) it follows that the above series converges and we have $0 \leq w^{\mathcal{H}} \leq 1$. Moreover, by Proposition 2.4-(i) and (2.14), $w^{\mathcal{H}} \in \mathrm{P}^{\mathcal{H}}(\Omega)$.

Lemma 3.6. *Setting $v^{\mathcal{H}} = 1 - w^{\mathcal{H}}$, we have $0 < v^{\mathcal{H}} \leq 1$ in Ω, for every $\mathcal{H} \in \mathcal{F}$. Moreover for every $\varepsilon, \sigma > 0$ there exists $\rho = \mathbf{c}(\varepsilon, \sigma)^{-1}$ such that*

$$0 < \sup_{z \in \Omega,\, d_p(z,z_0) \leq \rho} v^{\mathcal{H}}(z) \leq \varepsilon \inf_{z \in \Omega\, ,\, d_p(z,z_0) \geq \sigma} v^{\mathcal{H}}(z) \quad \forall\, \mathcal{H} \in \mathcal{F} \, .$$

This follows from Lemma 3.3 and Lemma 3.5.

We are now in position to give the proof of Theorem 3.2. There exists $\sigma = \mathbf{c}(\varepsilon, \eta)^{-1}$ such that $|\varphi(z) - \varphi(z_0)| \leq \varepsilon$ for every $z \in \partial\Omega$ such that $d_p(z, z_0) \leq \sigma$. Let v be as in Lemma 3.6 and set

$$\lambda = 2 I_\sigma^{-1} \max_{\partial\Omega} |\varphi| \quad , \quad I_\sigma = \inf_{z \in \Omega,\, d_p(z,z_0) \geq \sigma} v(z) \, ,$$

$$\overline{h} = \varphi(z_0) + \varepsilon + \lambda v \quad , \quad \underline{h} = \varphi(z_0) - \varepsilon - \lambda v \, .$$

Observing that $\lambda \geq |\varphi(z_0)|$ (since $v \leq 1$) and recalling that $v = 1 - w$, $w \in \mathrm{P}(\Omega)$, $1 \in \overline{\mathrm{P}}(\Omega)$ (the zero order terms q of the operators $\mathcal{H} \in \mathcal{F}$ are ≤ 0 by assumption), we see that $\overline{h}, -\underline{h} \in \overline{\mathrm{P}}(\Omega)$. Moreover

$$\liminf_{\Omega \ni z \to \zeta} \overline{h}(z) \geq \varphi(z) \geq \limsup_{\Omega \ni z \to \zeta} \underline{h}(z) \quad \forall\, \zeta \in \partial\Omega \, .$$

Indeed, if $d_p(\zeta, z_0) \leq \sigma$, this follows immediately from the choice of σ recalling that $v \geq 0$ in Ω. Otherwise, we have $v \geq I_\sigma$ in a neighborhood of ζ and the claim follows from the definition of λ.

Observing also that $\overline{h}, \underline{h}$ are bounded in Ω, we then obtain $\underline{h} \leq H_\varphi^\Omega \leq \overline{h}$ in Ω, by definition of the Perron-Wiener function H_φ^Ω. We finally exploit Lemma 3.6 and find $\rho = \mathbf{c}(\varepsilon, \sigma)^{-1}$ such that

$$|H_\varphi^\Omega(z) - \varphi(z_0)| \leq \varepsilon + \lambda v(z) \leq \varepsilon(1 + \lambda I_\sigma) = \varepsilon(1 + 2 \max_{\partial\Omega} |\varphi|) \, ,$$

for every $z \in \Omega$ such that $d_p(z, z_0) \leq \rho$.

4. Applications to non-divergence Hörmander operators

Let $X_1, \cdots, X_m$ be a system of Hörmander vector fields in $\mathbb{R}^N$ (i.e. of smooth vector fields in $\mathbb{R}^N$ such that $\operatorname{rank}\operatorname{Lie}\{X_1, \cdots, X_m\}(x) = N$ for every $x \in \mathbb{R}^N$) and let us denote by d the related Carnot-Carathéodory control distance. Let us also fix the parameters $\alpha \in]0, 1[$, $k > 0$, $\Lambda > 1$.

Throughout this section we shall denote by $\mathcal{F} = \mathcal{F}(\alpha, k, \Lambda)$ the family of the partial differential operators in the form

$$\mathcal{H} = \sum_{i,j=1}^{m} a_{i,j}(x,t) X_i X_j + \sum_{j=1}^{m} b_j(x,t) X_j + q(x,t) - \partial_t \ , \tag{4.1}$$

with d-Hölder continuous coefficients $a_{i,j} = a_{j,i},\ b_j,\ q$ satisfying

$$\Lambda^{-1}|\eta|^2 \le \sum_{i,j=1}^{m} a_{i,j}(z)\eta_i\eta_j \le \Lambda|\eta|^2 \quad \forall\, \eta \in \mathbb{R}^m \ , \ \forall\, z \in \mathbb{R}^{N+1} \ ,$$

$$\|a_{i,j}\|_{\Gamma^\alpha(\mathbb{R}^{N+1})} \ , \ \ \|b_j\|_{\Gamma^\alpha(\mathbb{R}^{N+1})} \ , \ \ \|q\|_{\Gamma^\alpha(\mathbb{R}^{N+1})} \le k \ .$$

We refer to (4.2) below for the definition of the d-Hölder norm $\|\cdot\|_{\Gamma^\alpha}$.

We shall assume that outside some compact set of $\mathbb{R}^N$ our vector fields are simply $X_j = \partial_{x_j}$ for $j = 1, \cdots, N$, $X_j = 0$ for $j = N+1, \cdots, m$. We explicitly remark that, given any system of Hörmander vector fields $Z_1, \cdots, Z_q$ defined in a neighborhood of some bounded domain D of $\mathbb{R}^N$, it is always possible to "extend" it to a new system $X_1, \cdots, X_m$ ($m = N + q$) of Hörmander vector fields in $\mathbb{R}^N$ satisfying the above assumption and such that $X_{N+j} = Z_j$ in D, for $j = 1, \cdots, q$ (we refer to [9] for a proof of this fact). Because of the local nature of our results below, this assumption is therefore not restrictive.

We can prove the following

Theorem 4.1. *Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ satisfying the exterior d-cone condition* (3.4) *at any point of $\partial_p\Omega$ (the constants M, T, θ in* (3.4) *are allowed to depend on the point). Let $\varphi \in C(\partial_p\Omega)$ and let $f \in \Gamma^\beta(\Omega_0)$ for some $0 < \beta \le \alpha$ and for some neighborhood Ω_0 of $\overline{\Omega}$. Then, for every PDO $\mathcal{H} \in \mathcal{F}$, there exists a (unique) solution $u \in \Gamma^{2+\beta}_{\rm loc}(\Omega) \cap C(\Omega \cup \partial_p\Omega)$ to the problem*

$$\begin{cases} \mathcal{H}u = f & \text{in } \Omega \ , \\ u = \varphi & \text{in } \partial_p\Omega \ . \end{cases}$$

Here $\Gamma^\beta(\Omega)$ denotes the space of functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{\Gamma^\beta(\Omega)} := \sup_\Omega |u| + \sup_{(x,t)\neq(x',t')\in\Omega} \frac{|u(x,t) - u(x',t')|}{d(x,x')^\beta + |t-t'|^{\beta/2}} < \infty \ . \tag{4.2}$$

Moreover, $\Gamma^{2+\beta}(\Omega)$ denotes the space of functions $u : \Omega \to \mathbb{R}$ which belong to $\Gamma^\beta(\Omega)$ together with any Lie-derivative along the vector fields $X_1, \cdots, X_m$ up to second order, and along ∂_t up to first order. Analogously one defines $\Gamma^{2+\beta}_{\rm loc}(\Omega)$. We say that u has Lie-derivative along X_j at $z_0 \in \Omega$, if $u \circ \gamma$ is differentiable at 0, where γ is the integral curve of X_j such that $\gamma(0) = z_0$.

Let us introduce some more notation. We shall denote by $\mathcal{F}_s$ the family of the PDO $\mathcal{H} \in \mathcal{F}$ with smooth coefficients $a_{i,j}, b_j, q$. Moreover $\mathcal{F}_s^-$ will denote the family of the PDO $\mathcal{H} \in \mathcal{F}_s$ with zero order term $q \le 0$.

Proposition 4.2. *The family $\mathcal{F}_s^-$ satisfies all the assumptions of Section 2 and Section 3 (with $E = \mathbb{R}^{N+1}$ and with constants β, C_1, C_2, C_3 only depending on α, k, Λ and $X_1, \cdots, X_m$). In particular the uniform regularity criterion for boundary points Theorem 3.2 holds for this family.*

Indeed any $\mathcal{H} \in \mathcal{F}_s$ can be written in the form

$$\mathcal{H} = \sum_{j=1}^{m} Y_j^2 + Y_0 + q \, ,$$

where $Y_i = \sum_{j=1}^m r_{ij}(x,t) X_j, \ i = 1, \cdots, m$, being $(r_{ij})_{i,j}$ a smooth symmetric square root of the matrix $(a_{i,j})_{i,j}$, and

$$Y_0 = -\partial_t + \sum_{k=1}^{m} (b_j(x,t) - \sum_{j=1}^{m} Y_j r_{j,k}(x,t)) X_k \, .$$

Moreover $Y_0, \cdots, Y_m$ turn out to be a system of Hörmander vector fields in $\mathbb{R}^{N+1}$. Hence (H2.1) follows from the celebrated Hörmander hypoellipticity Theorem [13] (we explicitly remark that $\mathcal{H}$ is not totally degenerate since $Y_1, \cdots, Y_m$ are Hörmander in $\mathbb{R}^N$, for every fixed t, see also [5]). On the other hand, (H2.2) is a consequence of the results in [2, 9]. Indeed in those papers, well-behaved global fundamental solutions $\Gamma^{\mathcal{H}}$ for $\mathcal{H} \in \mathcal{F}(\alpha, k, \Lambda)$ are constructed, having uniform Gaussian bounds

$$C(T)^{-1} |B_d(x, \sqrt{t-\tau})|^{-1} \exp\left(-c\, d(x,\xi)^2/(t-\tau)\right) \leq \Gamma^{\mathcal{H}}(x,t,\xi,\tau)$$

$$\leq C(T) |B_d(x, \sqrt{t-\tau})|^{-1} \exp\left(-c^{-1} d(x,\xi)^2/(t-\tau)\right) \, , \quad \text{if } 0 < t-\tau < T$$

(the same estimates hold for $\Gamma^{\mathcal{H}}(\xi,t,x,\tau)$). Here c is a positive constant only depending on α, k, Λ and $X_1, \cdots, X_m$ (while $c(T)$ may also depend on T). These uniform estimates in particular ensure that also (H3.1), (3.1) and (3.2) hold. As far as the proof of (2.7), we explicitly remark that the same estimates as above holds for the fundamental solution Γ_0 of the model operator $\sum_{j=1}^m X_j^2 - \partial_t$, which furthermore satisfies $\int_{\mathbb{R}^N} \Gamma_0(x,t,\xi,\tau)\, d\xi = 1$. Finally (3.3) follows from analogous uniform bounds for the derivatives of Γ along the vector fields $X_1, \cdots, X_m$ and ∂_t, also proved in the above cited papers.

A detailed proof of (3.3) is given in [9]. All the constants in (H3.1)-(H3.2) turn out to depend only on α, k, Λ and $X_1, \cdots, X_m$.

The following corollary states that we can remove the hypothesis $q \leq 0$ in the above d-cone criterion (by the change of variables $u^{\mathcal{H}} = e^{kt} v^{\mathcal{H}}$: since $|q| \leq k$, setting $\widehat{\mathcal{H}} = \mathcal{H} - k$ we have $\widehat{\mathcal{H}} \in \mathcal{F}_s^-(\alpha, 2k, \Lambda)$ for every $\mathcal{H} \in \mathcal{F}_s(\alpha, k, \Lambda)$).

Corollary 4.3. *Let Ω be as in Theorem 4.1 and let $\varphi \in C(\partial_p \Omega)$. Then, for every PDO $\mathcal{H} \in \mathcal{F}_s$, there exists a (unique) solution $u^{\mathcal{H}} \in C^\infty(\Omega) \cap C(\Omega \cup \partial_p \Omega)$ to the problem*

$$\begin{cases} \mathcal{H} u = 0 & \text{in } \Omega \, , \\ u = \varphi & \text{in } \partial_p \Omega \, . \end{cases}$$

Moreover, for every $z_0 \in \partial_p \Omega$ and for every $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\sup_{\mathcal{H} \in \mathcal{F}_s} |u^{\mathcal{H}}(z) - \varphi(z_0)| \leq \varepsilon \quad \forall\, z \in \Omega \, : \, d_p(z, z_0) \leq \rho \, . \tag{4.3}$$

The constant ρ only depends on ε, on the modulus of continuity of φ at z_0, on the maximum of $|\varphi|$ in $\partial_p\Omega$, on Ω, z_0, on the d-cone constants of Ω at z_0 (see (3.4)*) and on $\alpha, k, \Lambda, X_1, \cdots, X_m$.*

Remark 4.4. In the above corollary, $u^{\mathcal{H}} = {}^{\mathcal{H}}H^{\Omega}_{\widetilde{\varphi}}$, the Perron-Wiener generalized solution, being $\widetilde{\varphi}$ any continuous extension of φ to $\partial\Omega$.

Remark 4.5. We shall see in a while that, using the uniformity in $\mathcal{H} \in \mathcal{F}_s$ given by (4.3), we can remove also the assumption on the smoothness of the coefficients $a_{i,j}, b_j, q$: more precisely Corollary 4.3 holds even if $\mathcal{F}_s$ is replaced by $\mathcal{F}$ and $C^\infty(\Omega)$ is replaced by $\Gamma^{2+\alpha}_{\text{loc}}(\Omega)$. We also remark that, if the d-cone condition (3.4) is assumed to hold only at a point $z_0 \in \partial\Omega$, we are still able to find a solution $u \in \Gamma^{2+\alpha}_{\text{loc}}(\Omega)$ to the equation $\mathcal{H}u = 0$ in Ω satisfying (4.3) at z_0.

Remark 4.6. Corollary 4.3 (and in particular the uniform estimate of the moduli of continuity in (4.3)) is a crucial step in the proof of an invariant Harnack inequality for $\mathcal{H} \in \mathcal{F}$ (in the case $q = 0$). We refer to [3, 9] (see in particular [3, Lemma 5.1]) for the complete proof.

We can now prove Theorem 4.1. We first treat the case $f = 0$. Let

$$\mathcal{H}^\varepsilon = \sum_{i,j=1}^m a^\varepsilon_{i,j}(x,t)X_iX_j + \sum_{j=1}^m b^\varepsilon_j(x,t)X_j + q^\varepsilon(x,t) - \partial_t ,$$

be operators obtained regularizing the coefficients of $\mathcal{H}$. Taking suitable mollifiers, one can prove that $\mathcal{H}^\varepsilon \in \mathcal{F}_s(\alpha, c\,k, \Lambda)$ for every $\mathcal{H} \in \mathcal{F}(\alpha, k, \Lambda)$, beside having $a^\varepsilon_{i,j}, b^\varepsilon_j, q^\varepsilon$ uniformly convergent on compact sets to $a_{i,j}, b_j, q$, as $\varepsilon \to 0^+$ (see [9]). We can then apply Corollary 4.3 and find solutions u_ε to $\mathcal{H}^\varepsilon u_\varepsilon = 0$ in Ω, $u_\varepsilon = \varphi$ in $\partial_p\Omega$, satisfying

$$\sup_{0<\varepsilon<1} |u_\varepsilon(z) - \varphi(z_0)| \to 0 \quad , \quad \text{as } z \to z_0 \tag{4.4}$$

for every $z_0 \in \partial_p\Omega$. We can now use the a priori estimates proved in [8] and get

$$\|u_\varepsilon\|_{\Gamma^{2+\alpha}(\overline{O})} \le c(O, \Omega, \alpha, k, \Lambda, X_1, \cdots, X_m) \sup_\Omega |u_\varepsilon|$$

for every bounded domain $O \subset \overline{O} \subset \Omega$. Moreover, comparing u_ε with the function $e^{k(t-t^*)} \max_{\partial_p\Omega} |\varphi|$ where $t^* = \inf_{(x,t)\in\Omega} t$ (see e.g. Proposition 2.6), we see that $\sup_\Omega |u_\varepsilon| \le c(\Omega, \varphi, k)$. As a consequence, for some $\varepsilon_k \to 0$, u_{ε_k} converges in $\Gamma^{2+\beta}_{\text{loc}}(\Omega)$, for every $0 < \beta < \alpha$, to some function $u \in \Gamma^{2+\alpha}_{\text{loc}}(\Omega)$ satisfying $\mathcal{H}u = 0$ in Ω. Moreover (4.4) implies that u takes the boundary value φ in $\partial_p\Omega$. Furthermore the above arguments show that (4.3) holds with $\mathcal{F}_s$ replaced by $\mathcal{F}$. Finally, observing that the uniqueness of $u^{\mathcal{H}}$ follows from the maximum principle for $\mathcal{H} \in \mathcal{F}$ (in the class $\Gamma^{2+\alpha}_{\text{loc}}$) proved in [9], also Remark 4.5 is proved.

We now complete the proof of Theorem 4.1, considering the case of a general $f \in \Gamma^\beta(\Omega_0)$. Let $\psi \in C^\infty_0(\Omega_0)$ be a cut-off function such that $\psi = 1$ in $\overline{\Omega}$. Then $\widetilde{f} = f\psi \in \Gamma^\beta(\mathbb{R}^{N+1})$. We set $v(x,t) = -\int_{\mathbb{R}^N\times[T_1,t]} \Gamma(x,t;\xi,\tau)\,\widetilde{f}(\xi,\tau)\,d\xi\,d\tau$, where T_1 is a fixed time "below Ω_0" and Γ is the fundamental solution for $\mathcal{H}$ constructed in [9], so that $v \in \Gamma^{2+\beta}_{\text{loc}}(\mathbb{R}^N \times]T_1, \infty[)$, $\mathcal{H}v = \widetilde{f}$ in $\mathbb{R}^N \times]T_1, \infty[$. Since we already know that Theorem 4.1 holds in the case $f = 0$, we can find a solution $w \in \Gamma^{2+\alpha}_{\text{loc}}(\Omega) \cap C(\Omega \cup \partial_p\Omega)$ to the problem $\mathcal{H}w = 0$ in Ω, $w = \varphi - v$ in $\partial_p\Omega$. It is now sufficient to set $u = v + w$ (uniqueness follows from the above recalled maximum principle for $\mathcal{H} \in \mathcal{F}$, in the class $\Gamma^{2+\beta}_{\text{loc}}$).

We now turn to study the stationary case. Our results below are certainly not optimal. They have to be intended as further examples of application of the preceding material. Let us denote by $\mathcal{G} = \mathcal{G}(\alpha, k, \Lambda)$ the family of the partial differential operators in the form

$$\mathcal{L} = \sum_{i,j=1}^{m} a_{i,j}(x) X_i X_j + \sum_{j=1}^{m} b_j(x) X_j + q(x) , \tag{4.5}$$

with d-Hölder continuous coefficients $a_{i,j} = a_{j,i},\ b_j,\ q$ satisfying

$$\Lambda^{-1}|\eta|^2 \le \sum_{i,j=1}^{m} a_{i,j}(x)\eta_i\eta_j \le \Lambda|\eta|^2 \quad \forall\, \eta \in \mathbb{R}^m ,\ \forall\, x \in \mathbb{R}^N , \tag{4.6}$$

$$\|a_{i,j}\|_{\Gamma^\alpha(\mathbb{R}^N)} ,\ \|b_j\|_{\Gamma^\alpha(\mathbb{R}^N)} ,\ \|q\|_{\Gamma^\alpha(\mathbb{R}^N)} \le k .$$

(For the definition of the d-Hölder spaces, we refer for brevity to the analogous definition in the evolutive case, see (4.2)). Moreover we shall denote by $\mathcal{G}_s$ the family of PDO $\mathcal{L} \in \mathcal{G}$ with smooth coefficients $a_{i,j}, b_j, q$, and by $\mathcal{G}^-$ (respectively $\mathcal{G}_s^-$) the family of PDO $\mathcal{L} \in \mathcal{G}$ (respectively $\mathcal{G}_s$) with zero order term $q \le 0$.

We shall say that an open set $\Omega \subseteq \mathbb{R}^N$ satisfies the *exterior d-cone condition* at a point $x_0 \in \partial\Omega$, if

$$\exists\, r_0, \theta > 0\ :\ |B_d(x_0, r) \setminus \overline{\Omega}| \ \ge\ \theta\, |B_d(x_0, r)| \quad \forall\, r \in]0, r_0[. \tag{4.7}$$

We can prove the following

Theorem 4.7. *Let Ω be a bounded open subset of $\mathbb{R}^N$ satisfying the exterior d-cone condition* (4.7) *at any point of its boundary (the constants r_0, θ in* (4.7) *are allowed to depend on the point). Let $\varphi \in C(\partial\Omega)$. Then, for every PDO $\mathcal{L} \in \mathcal{G}^-$, there exists a solution $u^{\mathcal{L}} \in \Gamma^{2+\alpha}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ to the Dirichlet problem*

$$\mathcal{L}u = 0 \quad \text{in } \Omega ,$$

$$u = \varphi \quad \text{in } \partial\Omega ,$$

satisfying

$$\max_{\overline{\Omega}} |u^{\mathcal{L}}| = \max_{\partial\Omega} |\varphi| . \tag{4.8}$$

Moreover, for every $x_0 \in \partial\Omega$ we have

$$\sup_{\mathcal{L}\in\mathcal{G}^-} |u^{\mathcal{L}}(x) - \varphi(x_0)| \to 0 \quad , \quad \text{as } x \to x_0 . \tag{4.9}$$

Using the above theorem and the Harnack inequality, we can prove the following

Remark 4.8. Given $\mathcal{L} \in \mathcal{G}^-$, we set $\mathrm{H}^{\mathcal{L}}(\Omega) = \{u \in \Gamma^2(\Omega) \mid \mathcal{L}u = 0 \text{ in } \Omega\}$. Then $\mathrm{H}^{\mathcal{L}}$ is a harmonic sheaf on $\mathbb{R}^N$, satisfying the regularity axiom. If $q = 0$ (the zero order term of $\mathcal{L}$), then $\mathrm{H}^{\mathcal{L}}$ satisfies also the Brelot convergence axiom.

In order to give a further example of application of our d-cone criterion proved in Section 3, we now specialize to the following simpler setting: we shall assume that the vector fields $X_1, \cdots, X_m$ ($m > 2$) are the generators of a Carnot group, so that we can use the results in [2], ensuring that the operator

$$\mathcal{L} = \sum_{i,j=1}^{m} a_{i,j}(x) X_i X_j$$

(with Hölder continuous coefficients $a_{i,j}$ as in (4.6)) has a well-behaved (local) fundamental solution γ.

Theorem 4.9.

(i) *Let Ω be a bounded open subset of $\mathbb{R}^N$. Then $(\Omega, \mathrm{H}^{\mathcal{L}})$ is a β-harmonic space in which the Brelot convergence axiom holds.*

(ii) *If moreover Ω satisfies the exterior d-cone condition (4.7) at any point of its boundary, then for every $\varphi \in C(\partial\Omega)$, $f \in C^\infty(\overline{\Omega})$, there exists a unique solution $u \in \Gamma^{2+\alpha}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ to the problem*

$$\mathcal{L}u = f \;\; in\, \Omega \quad , \quad u = \varphi \;\; in\, \partial\Omega \, .$$

Recalling Remark 4.8, in order to prove the first statement of the theorem, we are only left to see that the following separation axiom is fulfilled: for every $x \neq y \in \Omega$ there exists a nonnegative continuous $\mathcal{L}$-superharmonic function v in Ω such that $v(x) \neq v(y)$. The functions v can be constructed as truncated of the fundamental solution γ, exploiting the fact that γ behaves like d^{2-Q} near the pole (see [2, 3]). Here Q is the homogeneous dimension of the Carnot group. We explicitly remark that, in proving that γ is $\mathcal{L}$-hyperharmonic in Ω, we use the weak maximum principle for $\mathcal{L}$ in the weak class of regularity Γ^2 proved in [4] (since we are working in a Carnot group, this maximum principle holds on any bounded open set, because we can always construct global barriers).

Let now $\psi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that $\psi = 1$ in $\overline{\Omega}$ and such that $\widetilde{f} := f\psi \in C_0^\infty(\mathbb{R}^N)$. We set $v(x) = -\int_{\mathbb{R}^N} \gamma(x,\xi)\widetilde{f}(\xi)\, d\xi$, where γ is the fundamental solution of $\mathcal{L}$ in Ω constructed in [2], so that $v \in \Gamma^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ is a solution to $\mathcal{L}v = \widetilde{f} = f$ in Ω. We now apply Theorem 4.7 and find a solution $w \in \Gamma^{2+\alpha}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ to the problem $\mathcal{L}w = 0$ in Ω, $w = \varphi - v$ in $\partial\Omega$. It is now sufficient to set $u = v + w$ (uniqueness follows from the above mentioned weak maximum principle for $\mathcal{L}$ in the class Γ^2).

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