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Yang-Mills fields on 3-dimensional nondegenerate CR manifolds

by Elisabetta Barletta and Sorin Dragomir

Abstract[1]. Given a Hermitian vector bundle $(E, h) \to M$ over a 3-dimensional nondegenerate CR manifold M and a connection $D \in \mathcal{C}(E, h)$ we show[2] that if $\Pi^* D \in \mathcal{C}(\Pi^* E, \Pi^* h)$ is a Yang-Mills field on the Kählerian manifold $M \times \mathbb{R}^+$ (obtained from M by symplectization) and $T \rfloor R^D = 0$ then D is a Yang-Mills field on M whose curvature has a zero trace ($\Lambda_\theta R^D = 0$). Moreover $\Pi^* D$ is (anti) selfdual if and only if D is flat ($R^D = 0$).

1. Introduction

Yang-Mills fields on CR manifolds were first studied by H. Urakawa, [13]-[15]. Given a compact strictly pseudoconvex CR manifold M, on which a contact form θ has been fixed, and a Hermitian vector bundle $(E, h) \to M$, a *Yang-Mills field* is a critical point of the Yang-Mills functional

$$\mathcal{YM} : \mathcal{C}(E, h) \to [0, +\infty) \ , \ \mathcal{YM}(D) = \frac{1}{2} \int_M \|R^D\|^2 \, d\,\mathrm{vol}(g_\theta) \ , \tag{1}$$

where $R^D \in \Omega^2(\mathrm{Ad}(E))$ is the curvature of the connection $D \in \mathcal{C}(E, h)$ and g_θ is the Webster metric, a Riemannian metric associated to θ (a natural contraction of the Levi form of M). The Euler-Lagrange equations of the variational principle $\delta\mathcal{YM}(D) = 0$ are

$$\delta^D R^D = 0 \tag{2}$$

(the *Yang-Mills equations* on M) where δ^D is the formal adjoint of the differential operator $d^D : \Omega^1(\mathrm{Ad}(E)) \to \Omega^2(\mathrm{Ad}(E))$ (an analog of the ordinary exterior differentiation operator, well defined for $\mathrm{Ad}(E)$-valued forms). An inhomogeneous version of the equations (2) has been considered in [4]. Much of the theory in [13] relies on the formal analogy between strictly pseudoconvex CR manifolds (eventually with a vanishing pseudohermitian torsion i.e. Sasakian manifolds) and Kählerian manifolds (where a rather developed theory already exists, cf. e.g. S. Donaldson, [3]). It was unknown for instance whether, given a domain $\Omega \subset \mathbb{C}^n$, Yang-Mills fields on $\partial\Omega$ are boundary values of Yang-Mills fields on Ω (endowed with the Bergman metric). Precisely let $\Omega = \{\rho < 0\}$ be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^2$ i.e. $\rho \in C^\infty(U)$ where $U \subseteq \mathbb{C}^2$ is an open subset and

$$\Omega = \{z \in U : \rho(z) < 0\} \ , \ \partial\Omega = \{z \in U : \rho(z) = 0\} \ ,$$
$$(\rho_{z_1}(z), \rho_{z_2}(z)) \neq 0 \ , \ z \in \partial\Omega \ ,$$

[1]Authors' address: E. Barletta, Università degli Studi della Basilicata, Dipartimento di Matematica, Campus Macchia Romana, 85100 Potenza, Italy; e-mail: barletta@unibas.it .
S. Dragomir, Università degli Studi della Basilicata, Dipartimento di Matematica, Campus Macchia Romana, 85100 Potenza, Italy; e-mail: dragomir@unibas.it .
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$$(\partial\overline{\partial}\rho)(Z, \overline{Z}) \neq 0 \ , \ Z \in T_{1,0}(\partial\Omega) \ , \ Z \neq 0 \, .$$

Let g be the Bergman metric of Ω. Let $p : F \to U$ be a holomorphic vector bundle and h a Hermitian metric on F. Let us consider the Dirichlet problem for the Yang-Mills equations on (Ω, g)

$$\delta^D R^D = 0 \ \text{ in } \ \Omega \, , \tag{3}$$

$$D = D_0 \ \text{ on } \ \partial\Omega \, , \tag{4}$$

where $D_0 \in \mathcal{C}(E, h)$ is a given C^∞ metric connection in the CR-holomorphic vector bundle $E = p^{-1}(\partial\Omega) \to \partial\Omega$. A recent finding is

Theorem 1 (E. Barletta et al., [2])**.** *If a solution D to the Dirichlet problem* (3)-(4) *exists then its boundary values D_0 satisfies*

$$\delta_b^{D_0} R^{D_0} = 0 \ \text{ on } \ \partial\Omega \, . \tag{5}$$

Moreover if $T \rfloor R^{D_0} = 0$ then D_0 is a Yang-Mills field on $\partial\Omega$.

Here the differential operator $\delta_b^{D_0}$ (the boundary analog to δ^D) is given by (48) and T is the characteristic direction of $\theta = (i/2)(\overline{\partial} - \partial)\rho$. The final test of the theory (such as built in [13]-[15]) is related to the use of the Fefferman metric in CR geometry (see Section 2 for the relevant definitions). That is one should investigate whether integrating along the fibre in $\widehat{\mathcal{YM}}(\pi^* D)$ (with $D \in \mathcal{C}(E, h)$) leads to the functional (1). Here

$$\widehat{\mathcal{YM}}(\mathbb{D}) = \frac{1}{2} \int_{C(M)} \langle R^{\mathbb{D}}, R^{\mathbb{D}} \rangle \, d\,\mathrm{vol}(F_\theta) \ , \ \mathbb{D} \in \mathcal{C}(\pi^* E, \pi^* h) \, .$$

Also $S^1 \to C(M) \xrightarrow{\pi} M$ is the canonical circle bundle over M and F_θ is the Fefferman metric of (M, θ) (a Lorentz metric on $C(M)$). This approach has been quite successful elsewhere (e.g. D. Jerison & J.M. Lee, [7]-[8], or E. Barletta et al., [1]). Indeed both the CR Yamabe problem (cf. D. Jerison & J.M. Lee, *op. cit.*) and subelliptic harmonic maps (cf. J. Jost & C-J. Xu, [10]) were recognized to arise by a quite natural projection process (via $\pi : C(M) \to M$) from the ordinary Yamabe problem, respectively from ordinary harmonic maps, with respect to the Fefferman metric F_θ on $C(M)$. As it turns out (cf. again [2]) integration along the fibre of $C(M)$ in the ordinary Yang-Mills functional (with respect to the Fefferman metric) leads to the *pseudo Yang-Mills functional*

$$\mathcal{PYM} : \mathcal{C}(E, h) \to [0, +\infty) \ , \ \mathcal{PYM}(D) = \frac{1}{2} \int_M \|\pi_H R^D\|^2 \, \theta \wedge d\theta \, ,$$

rather than to $\mathcal{YM} : \mathcal{C}(E, h) \to [0, +\infty)$. Precisely one finds that

$$\widehat{\mathcal{YM}}(\pi^* D) = 2\pi \, \mathcal{PYM}(D) \ , \ D \in \mathcal{C}(E, h) \, .$$

Here $\pi_H : \Omega^\bullet(\mathrm{Ad}(E)) \to \Omega^\bullet(\mathrm{Ad}(E))/\mathcal{I}_\theta^\bullet$ is the natural projection and $\mathcal{I}_\theta^\bullet \subset \Omega^\bullet(\mathrm{Ad}(E))$ is the ideal generated by θ. A critical point $D \in \mathcal{C}(E, h)$ on $\mathcal{PYM}$ is a *pseudo Yang-Mills field.* In the presence of an admissible coframe $\Omega^\bullet(\mathrm{Ad}(E))/\mathcal{I}_\theta^\bullet$ may be identified to

$$\Omega_H^\bullet(\mathrm{Ad}(E)) = \{\omega \in \Omega^\bullet(\mathrm{Ad}(E)) : T \rfloor \omega = 0\} \, .$$

To emphasize on the relationship to the Yang-Mills theory we devote the remainder of the Introduction to a brief derivation of the Euler-Lagrange equations of the variational principle $\delta \, \mathcal{PYM}(D) = 0$ (the notations and conventions are made clear in Sections 2 and 3). To this end let $\varphi \in \Omega^1(\mathrm{Ad}(E))$ and let us set

$$D^{t,\varphi} = D + t\varphi \in \mathcal{C}(E, h) \ , \ t \in \mathbb{R} \, .$$

If $u \in \Omega^0(E)$ and $\omega_{t,\varphi} = D^{t,\varphi} u$ then

$$(R^{D + t\varphi} u)(X, Y) = (d^{D + t\varphi} \omega_{t,\varphi})(X, Y) =$$

$$= \frac{1}{2} \{D_X^{t,\varphi}(\omega_{t,\varphi}(Y)) - D_Y^{t,\varphi}(\omega_{t,\varphi}(X)) - \omega_{t,\varphi}([X, Y])\} =$$

$$= \frac{1}{2}\{D_X^{t;\varphi}(D_Y u + t\,\varphi_Y u) - D_Y^{t;\varphi}(D_X u + t\,\varphi_X u) - D_{[X,Y]}u - t\,\varphi_{[X,Y]}u\}$$

where $\varphi_X = \varphi(X)$ for any $X \in T(M)$. We also set as customary

$$[\varphi \wedge \psi]_{X,Y} = [\varphi_X, \psi_Y] - [\varphi_Y, \psi_X] \ \ , \ \ \varphi, \psi \in \Omega^1(\mathrm{Ad}(E)) \ .$$

Then

$$(R^{D+t\varphi}u)(X,Y) = \frac{1}{2}\{D_X D_Y u - D_Y D_X u - D_{[X,Y]}u\} +$$
$$+\frac{t}{2}\{D_X(\varphi_Y u) - \varphi_Y D_X u - D_Y(\varphi_X u) + \varphi_X D_Y u - \varphi_{[X,Y]}u\} +$$
$$+\frac{t^2}{2}\{\varphi_X\varphi_Y - \varphi_Y\varphi_X\}u$$

where from

$$R^{D+t\varphi} = \frac{1}{2}\,R^D + t\,d^D\varphi + \frac{t^2}{4}\,[\varphi \wedge \varphi] \ .$$

Consequently

$$\|\pi_H R^{d+t\varphi}\|^2 = \frac{1}{4}\,\|\pi_H R^D\|^2 + t\langle \pi_H R^D \, , \, \pi_H d^D \varphi\rangle + O(t^2)$$

and then for any critical point $D \in \mathcal{C}(E,h)$ of $\mathcal{PYM}$

$$0 = \frac{d}{dt}\,\{\mathcal{PYM}(D + t\varphi)\}_{t=0} =$$
$$= \frac{1}{2}\int_M \frac{d}{dt}\left\{\|\pi_H R^{D+t\varphi}\|^2\right\}_{t=0} \theta \wedge d\theta =$$
$$= \frac{1}{2}\int_M \langle \pi_H R^D \, , \, d^D\varphi\rangle \, \theta \wedge d\theta = \frac{1}{2}\int_M \langle \delta^D \pi_H R^D \, , \, \varphi\rangle \, \theta \wedge d\theta$$

for any $\varphi \in \Omega^1(\mathrm{Ad}(E))$. Therefore

(6) $$\delta^D \pi_H R^D = 0 \ .$$

Of course (6) makes sense when M is noncompact, as well, and the C^∞ solutions $D \in \mathcal{C}(E,h)$ to (6) are the pseudo Yang-Mills fields on M. Closing a circle of ideas any pseudo Yang-Mills field $D \in \mathcal{C}(E,h)$ such that $T \rfloor R^D = 0$ is a Yang-Mills field on M.

In this paper we look for new solutions to the Yang-Mills equations (2) on a strictly pseudoconvex CR manifold, though only confined to the 3-dimensional case. We exploit the so called symplectization process, associating to each 3-dimensional nondegenerate CR manifold M the 4-dimensional Kählerian manifold $N = M \times \mathbb{R}^+$, and relate the solutions to the Yang-Mills equations on N to the solutions of (2) (cf. our Theorem 2).

2. CR manifolds and pseudohermitian geometry

Let M be a real 3-dimensional C^∞ differentiable manifold. A *CR structure* on M is a complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ of complex rank 1 such that

(7) $$T_{1,0}(M) \cap T_{0,1}(M) = (0) \ ,$$

where $T_{0,1}(M) = \{\overline{Z} : Z \in T_{1,0}(M)\}$ and $\overline{Z}$ is the complex conjugate of Z. A pair $(M, T_{1,0}(M))$ is a *CR manifold*. The *Levi distribution* is the subbundle $H(M) \subset T(M)$ of real rank 2 given by

(8) $$H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\} \ .$$

It carries the complex structure

$$\phi : H(M) \to H(M) \ \ , \ \ \phi(Z + \overline{Z}) = i(Z - \overline{Z}) \ \ , \ \ Z \in T_{1,0}(M) \ ,$$

$(i = \sqrt{-1})$. Let us assume that M is oriented so that the conormal bundle

$$H(M)^\perp_x = \{\omega \in T^*_x(M) : \mathrm{Ker}(\omega) \supseteq H(M)_x\} \ \ , \ \ x \in M \ ,$$

is trivial i.e. $H(M)^\perp \approx M \times \mathbb{R}$. A *pseudohermitian structure* is a globally defined nowhere vanishing cross section $\theta \in \Gamma^\infty(H(M)^\perp)$. The *Levi form* is

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}) \ , \ Z, W \in T_{1,0}(M) \ . \tag{9}$$

A CR manifold is *nondegenerate* if L_θ is nondegenerate for some θ. As $H(M)^\perp \to M$ is a real line bundle any other pseudohermitian structure $\hat{\theta}$ is related to θ by

$$\hat{\theta} = f\,\theta \tag{10}$$

for some C^∞ function $f : M \to \mathbb{R}\setminus\{0\}$ hence $L_{\hat{\theta}} = f\, L_\theta$. Therefore nondegeneracy is a *CR invariant* property i.e. it is invariant under a transformation (10) of the pseudohermitian structure. Given a nondegenerate CR manifold M any pseudohermitian structure θ on M is a *contact form* i.e. $\theta \wedge d\theta$ is a volume form on M. For all local calculations let $T_1 : U \to T(M) \otimes \mathbb{C}$ be a complex vector field defined on the open subset $U \subseteq M$ such that

$$T_1(x) \in T_{1,0}(M)_x \ , \ T_1(x) \neq 0 \ , \ x \in U \ .$$

Let us consider the C^∞ function $g_{1\bar{1}} : U \to \mathbb{C}$ given by

$$g_{1\bar{1}} = L_\theta(T_1, T_{\bar{1}}) \ , \ T_{\bar{1}} = \overline{T}_1 \ .$$

When M is nondegenerate $g_{1\bar{1}}(x) \neq 0$ at any $x \in U$. We may assume without loss of generality that $g_{1\bar{1}}(x) < 0$. The global statement is of course that a contact form θ may be chosen such that L_θ is negative definite.

Let M be a 3-dimensional nondegenerate CR manifold and θ a contact form on M. The *characteristic direction* of $d\theta$ is the unique nowhere zero globally defined vector field T on M, transverse to $H(M)$, determined by

$$\theta(T) = 1 \ , \ T \rfloor d\theta = 0 \ , \tag{11}$$

where $T\rfloor$ denotes the interior product with T. Then $\{T_1, T_{\bar{1}}, T\}$ is a local frame of $T(M) \otimes \mathbb{C}$ on U. The *Webster metric* is the semi-Riemannian metric g_θ given by

$$g_\theta(X, Y) = -(d\theta)(X, \phi Y) \ , \ g_\theta(X, T) = 0 \ , \ g_\theta(T, T) = 1 \ , \tag{12}$$

for any $X, Y \in H(M)$. Through the remainder of this paper M will denote a 3-dimensional oriented CR manifold on which a contact form θ has been fixed such that the Levi form L_θ is negative definite. Then the Webster metric g_θ is a Riemannian metric on M. By a result in [2] there is a constant $c \neq 0$ depending only on the orientation of M such that

$$\theta \wedge d\theta = c\, d\,\mathrm{vol}(g_\theta) \ ,$$

where $d\,\mathrm{vol}(g_\theta)$ is the canonical volume form of the Riemannian manifold (M, g_θ). The *Tanaka-Webster connection* of (M, θ) is the unique linear connection ∇ on M satisfying i) $H(M)$ is parallel with respect to ∇, ii) $\nabla\phi = 0$, $\nabla g_\theta = 0$, and iii) the torsion T_∇ of ∇ is *pure* i.e.

$$T_\nabla(Z, \overline{W}) = 2iL_\theta(Z, \overline{W}) \ , \ Z, W \in T_{1,0}(M) \ , \tag{13}$$

$$\tau \circ \phi + \phi \circ \tau = 0 \ , \tag{14}$$

where

$$\tau(X) = T_\nabla(T, X) \ , \ X \in T(M) \ , \tag{15}$$

is the *pseudohermitian torsion* of ∇. Cf. e.g. N. Tanaka, [12]

$$\tau\, T_{1,0}(M) \subseteq T_{0,1}(M) \ . \tag{16}$$

The local manifestation of the pseudohermitian torsion is the C^∞ function $A_1^{\bar{1}} : U \to \mathbb{C}$ given by $\tau(T_1) = A_1^{\bar{1}} T_{\bar{1}}$.

A complex p-form $\eta \in \Gamma^\infty(\Lambda^p T^*(M) \otimes \mathbb{C})$ is a $(p, 0)$-*form* on M if $T_{0,1}(M) \rfloor \eta = 0$. The top degree $(p, 0)$-forms are the $(2, 0)$-forms. Let θ^1 be the complex 1-form on U given by

$$\theta^1(T_1) = 1 \ , \ \theta^1(T_{\bar{1}}) = 0 \ , \ \theta^1(T) = 0 \ , \tag{17}$$

and let us set $\theta^{\bar{1}} = \overline{\theta^1}$ so that $\{\theta^1,\ \theta^{\bar{1}},\ \theta\}$ is a local frame of $T^*(M) \otimes \mathbb{C}$ on U. Then θ^1 given by (17) is referred to as the *admissible* complex 1-form associated to T_1 on (M, θ). A $(2,0)$-form η on M may be locally represented as

$$\eta = f\ \theta \wedge \theta^1$$

for some C^∞ function $f : U \to \mathbb{C}$. Let $\mathbb{C} \to K(M) \to M$ be the *canonical bundle* of M i.e.

$$K(M)_x = \{\omega \in \Lambda^2 T_x^*(M) \otimes \mathbb{C} : T_{0,1}(M)_x \rfloor \omega = 0\} \ \ , \ \ x \in M \ .$$

Then $(2,0)$-forms are C^∞ sections in $K(M)$. There is a natural free action of $\mathbb{R}^+$ (the multiplicative positive reals) on $K(M) \setminus \{\text{zero section}\}$. Let $C(M)$ be the quotient space and $\pi : C(M) \to M$ the projection. Then $C(M)$ is the total space of a principal S^1-bundle $S^1 \to C(M) \to M$ (the *canonical circle bundle* over M) whose locally trivial structure is described by the C^∞ diffeomorphism

$$\pi^{-1}(U) \to U \times S^1 \quad , \quad [\omega] \to (x\, ,\, \frac{\lambda}{|\lambda|})\ , \tag{18}$$
$$\omega = \lambda\, (\theta \wedge \theta^1 \wedge \theta^{\bar{1}})_x \quad , \quad \lambda \in \mathbb{C} \setminus \{0\} \ \ , \ \ x \in U \ ,$$

where $[\omega]$ is the class mod $\mathbb{R}^+$ of $\omega \in K(M)_x$, $\omega \neq 0$. Let us set

$$G_\theta(X, Y) = (d\theta)(X, \phi Y) \ \ , \ \ X, Y \in H(M)\ ,$$
$$G_\theta(T, V) = 0 \ \ , \ \ V \in T(M)\ .$$

Then G_θ is a symmetric degenerate $(0,2)$-tensor field on M. Let $\gamma : \pi^{-1}(U) \to \mathbb{R}$ be a local fibre coordinate on $C(M)$ i.e. with the notations in (18)

$$\gamma([\omega]) = \arg(\frac{\lambda}{|\lambda|}) \ \ , \ \ [\omega] \in \pi^{-1}(U)\ ,$$

where $\arg : S^1 \to [0, 2\pi)$ is the principal argument. Let us consider the real 1-form on $\pi^{-1}(U)$ given by

$$\frac{1}{3}\left\{ d\gamma + \pi^* \left(i\omega_1^1 - \frac{i}{2}\, g^{1\bar{1}}\, dg_{1\bar{1}} - \frac{R}{8}\,\theta \right) \right\} \tag{19}$$

where we adopted the following notations. First $\omega_1^1 \in \Gamma^\infty(U, T^*(M) \otimes \mathbb{C})$ is the connection 1-form of the Tanaka-Webster connection ∇ i.e.

$$\nabla T_1 = \omega_1^1 \otimes T_1\ .$$

Next $g^{1\bar{1}} = 1/g_{1\bar{1}}$. Finally R is the *pseudohermitian scalar curvature* i.e. the C^∞ function $R : M \to \mathbb{R}$ locally given by

$$R = g^{1\bar{1}}\, R_{1\bar{1}}\ . \tag{20}$$

Here

$$\mathrm{Ric}(X, Y) = \mathrm{trace}\{Z \in T(M) \mapsto R^\nabla(Z, X)Y\} \ \ , \ \ X, Y \in T(M)\ ,$$
$$R_{1\bar{1}} = \mathrm{Ric}(T_1\, ,\, T_{\bar{1}})\ .$$

Also R^∇ is the curvature tensor field of ∇. The 1-form (19) is the local manifestation on $\pi^{-1}(U)$ of a globally defined real 1-form σ on $C(M)$ which turns out to be a connection 1-form in the principal bundle $S^1 \to C(M) \to M$.

Let $\pi^* G_\theta$ be the pullback of G_θ to $C(M)$. It is a symmetric $(0,2)$-tensor field on $C(M)$. Also $\pi^* G_\theta$ is degenerate in that both the tangent to the S^1-action and just any lift of T are "perpendicular" on anything. By a construction due to J.M. Lee, [11], one may use σ to modify $\pi^* G_\theta$ into a semi-Riemannian metric F_θ on $C(M)$ by setting

$$F_\theta = -\pi^* G_\theta + 2(\pi^*\theta) \odot \sigma\ . \tag{21}$$

Here $\odot$ is the symmetric tensor product e.g. if α, β are two 1-forms then $\alpha \odot \beta = \frac{1}{2}\{\alpha \otimes \beta + \beta \otimes \alpha\}$. Then F_θ is a Lorentz metric on $C(M)$ (the *Fefferman metric* of (M, θ)) changing conformally

$$F_{\hat{\theta}} = e^{u \circ \pi} F_\theta$$

under a transformation (10) of contact form (cf. J.M. Lee, *op. cit.*) with $f = e^u$, $u \in C^\infty(M)$.

3. Yang-Mills fields

Let $E \to M$ be a complex vector bundle over a 3-dimensional nondegenerate CR manifold M, as in the previous section. A *$\overline{\partial}$-operator* on E is a first order differential operator

$$\overline{\partial}_E : \Gamma^\infty(E) \to \Gamma^\infty(T_{0,1}(M)^* \otimes E)$$

such that

$$\overline{\partial}_E(fu) = (\overline{\partial}_b f) \otimes u + f\, \overline{\partial}_E u \tag{22}$$

for any $f \in C^\infty(M, \mathbb{C})$ and any $u \in \Gamma^\infty(E)$. Here

$$\overline{\partial}_b : C^\infty(M, \mathbb{C}) \to \Gamma^\infty(T_{0,1}(M)^*)$$

is the *tangential Cauchy-Riemann operator* i.e.

$$(\overline{\partial}_b f)\overline{Z} = \overline{Z}(f) \ \ , \ \ f \in C^\infty(M, \mathbb{C}) \ \ , \ \ Z \in T_{1,0}(M)\ .$$

A function $f \in C^\infty(M, \mathbb{C})$ is a *CR function* if $\overline{\partial}_b f = 0$. A pair $(E, \overline{\partial}_E)$ consisting of a complex vector bundle over a CR manifold and a $\overline{\partial}$-operator is a *CR-holomorphic* vector bundle. A section $u \in \Gamma^\infty(E)$ is *CR-holomorphic* if $\overline{\partial}_E u = 0$. A remark is in order. If M is a real $(2n+1)$-dimensional manifold ($n \geq 1$) and $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ has complex rank n the requirement (7) defines the notion of an *almost CR structure* on M (respectively the requirement (22) furnishes the notion commonly referred to as a *pre-$\overline{\partial}$-operator*) and appropriate integrability conditions must be imposed. Nevertheless when $n = 1$ these integrability conditions are identically satisfied i.e.

$$[\Gamma^\infty(T_{1,0}(M)), \Gamma^\infty(T_{1,0}(M))] \subseteq \Gamma^\infty(T_{1,0}(M))$$

and

$$[\overline{Z}, \overline{W}] \cdot u = \overline{Z} \cdot \overline{W} \cdot u - \overline{W} \cdot \overline{Z} \cdot u$$

for any $Z, W \in T_{1,0}(M)$ and $u \in \Gamma^\infty(E)$. Here $\overline{Z} \cdot u$ is short for $(\overline{\partial}_E u)\overline{Z}$. If $E \to M$ is a complex vector bundle we set

$$\Omega^p(E) = \Gamma^\infty(\Lambda^p T^*(M) \otimes E) \ \ , \ \ 0 \leq p \leq 3\ ,$$

so that $\Omega^0(E) = \Gamma^\infty(E)$. Let h be a Hermitian metric in E. A connection in E

$$D : \Omega^0(E) \to \Omega^1(E)$$

is *metric* if $Dh = 0$ i.e.

$$X(h(u, v)) = h(D_X u, v) + h(u, D_X v) \ \ , \ \ u, v \in \Omega^0(E) \ \ , \ \ X \in T(M)\ .$$

Let $\mathcal{C}(E, h)$ be the affine space of all metric connections in (E, h). Let

$$\mathrm{Ad}(E)_x = \{S \in \mathrm{End}_{\mathbb{C}}(E_x) : \langle Su, v\rangle + \langle u, Sv\rangle = 0\} \ \ , \ \ x \in M\ ,$$

where $h_x = \langle\ ,\ \rangle$. Let us assume M to be compact. The *Yang-Mills functional* $\mathcal{YM} : \mathcal{C}(E, h) \to [0, +\infty)$ is

$$\mathcal{YM}(D) = \frac{1}{2} \int_M \|R^D\|^2 \, d\,\mathrm{vol}(g_\theta) \tag{23}$$

where $R^D \in \Omega^2(\mathrm{Ad}(E))$ is the curvature 2-form of $D \in \mathcal{C}(E, h)$. The Euler-Lagrange equations associated to the variational principle $\delta\mathcal{YM}(D) = 0$ are

$$\delta^D R^D = 0 \tag{24}$$

where $\delta^D : \Omega^2(\mathrm{Ad}(E)) \to \Omega^1(\mathrm{Ad}(E))$ is the formal adjoint of $d^D : \Omega^1(\mathrm{Ad}(E)) \to \Omega^2(\mathrm{Ad}(E))$ with respect to the inner product

$$(\alpha, \beta) = \int_M g_\theta^*(\alpha, \overline{\beta})\, d\,\mathrm{vol}(g_\theta) .$$

We recall that given a connection D in $E \to M$ the differential operator

$$d^D : \Omega^p(E) \to \Omega^{p+1}(E) \ \ , \ \ 0 \le p \le 2 ,$$

is given by

$$(d^D\omega)(X_1, \cdots, X_{p+1}) =$$
$$= \frac{1}{p+1}\{\sum_{i=1}^{p+1}(-1)^{i+1}D_{X_i}(\omega(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1})) +$$
$$+ \sum_{1 \le i < j \le p+1}(-1)^{i+j}\omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{p+1})\}$$

for any $\omega \in \Omega^p(E)$ and any $X_i \in T(M)$. A hat indicates as usual the suppression of a term. A solution $D \in \mathcal{C}(E, h)$ to (24) (the *Yang-Mills equations*) on a not necessarily compact CR manifold M is a *Yang-Mills field* on M. The first attempt to build solutions to the Yang-Mills equations on a strictly pseudoconvex CR manifold by purely differential geometric methods is due to H. Urakawa, [13]. Let $(E, \overline{\partial}_E)$ be a CR-holomorphic vector bundle and h a Hermitian metric in E. A connection $D \in \mathcal{C}(E, h)$ is *Hermitian* if $D^{0,1} = \overline{\partial}_E$. Here $D^{0,1}u$ is the restriction of Du to $T_{0,1}(M)$. By a result of N. Tanaka, [12], there is a unique Hermitian connection in E (the *Tanaka connection* of $(E, \overline{\partial}_E, h)$) such that

$$\Lambda_\theta R^D = 0 . \tag{25}$$

If $F \to M$ is a vector bundle and $\varphi \in \Gamma^\infty(T^*(M) \otimes T^*(M) \otimes F)$ a F-valued bilinear form then the *trace* of φ is given by

$$i\,(\Lambda_\theta \varphi)_x = \varphi(E_1, E_{\bar{1}})_x \ \ , \ \ x \in U ,$$

where E_1 is a nowhere zero complex vector field of type $(1,0)$ on U such that $L_\theta(E_1, E_{\bar{1}}) = 1$. If $\hat{\theta}$ is another contact form on M given by (10) then $\Lambda_{\hat{\theta}}\varphi = (1/f)\Lambda_\theta\varphi$ hence (25) is a CR invariant requirement. Hence the Tanaka connection is a CR invariant. H. Urakawa looked (cf. *op. cit.*) for Hermitian solutions to (24) such that

$$T \rfloor R^D = 0 . \tag{26}$$

His finding is that the only such solution is the Tanaka connection of $(E, \overline{\partial}_E, h)$. Nevertheless solutions D which are merely metric (and perhaps obey to (26)) might exist. A variant of the Yang-Mills theory (based on the concept of conjugate connections) seeking for critical points to (23) which aren't necessary metrical was proposed in [5], with applications to Einstein-Weyl geometry and affine geometry. The problem of producing non metric Yang-Mills fields on a CR manifold is open.

4. New solutions to the Yang-Mills equations by symplectization

Let M be a 3-dimensional nondegenerate CR manifold and θ a contact form on M such that L_θ is negative definite. Let us set $N = M \times \mathbb{R}^+$ and let $\Pi : N \to M$ be the projection. Let Ω be the symplectic 2-form on N given by $\Omega = d(t\,\Pi^*\theta)$. If X is a vector field on M we set

$$X^\uparrow_{(x,t)} = (d_x\alpha_t)X_x \ \ , \ \ (x,t) \in N ,$$

where $\alpha_t : M \to N$ is given by $\alpha_t(x) = (x,t)$. Let $\xi = T^\uparrow$. One has the decomposition

$$T(N) = H(M)^\uparrow \oplus \mathbb{R}\xi \oplus \mathbb{R}\frac{\partial}{\partial t} .$$

A complex structure J on N is given by

$$J\,X^{\uparrow} = (\phi X)^{\uparrow} \ , \ \ X \in H(M) \ , \ \ J\xi = t\,\frac{\partial}{\partial t} \ , \ \ J\frac{\partial}{\partial t} = -\frac{1}{t}\,\xi \ . \tag{27}$$

Note that Ω is J-invariant hence

$$g(X,Y) = \Omega(JX,Y) \ , \ \ X,Y \in T(N) \ , \tag{28}$$

is a Kähler metric on (N,J) with an exact Kähler 2-form. The process of building a Kähler manifold starting from the data $(M, T_{1,0}(M), \theta)$ is commonly referred to as *symplectization* and is due essentially to S.M. Webster, [17]. Given a Hermitian vector bundle $(E,h) \to M$ we shall examine the relationship among Yang-Mills fields $D \in \mathcal{C}(E,\,h)$ and Yang-Mills fields $\mathbb{D} \in \mathcal{C}(\Pi^*E,\,\Pi^*h)$.

Let $(E, \overline{\partial}_E) \to M$ be a CR-holomorphic vector bundle. Let $u \in \Omega^0(E)$ be a C^∞ section. Its *natural lift* is the section $\hat{u} \in \Omega^0(\Pi^*E)$ given by $\hat{u} = u \circ \Pi$. There is a natural morphism

$$\Pi^* : \Omega^p(E) \to \Omega^p(\Pi^*E) \ , \ \ 0 \leq p \leq 3 \ ,$$

given by $\Pi^*(\omega \otimes u) = (\Pi^*\omega)\otimes\hat{u}$ (followed by extension by additivity). Let $\sigma = (\sigma_1, \cdots, \sigma_r)$ be a local frame of $E \to M$ defined on the open subset $U \subseteq M$ and let $s_j = \hat{\sigma}_j$. Then $s = (s_1, \cdots, s_r)$ is a local frame of $\Pi^*E \to N$ defined on $\Pi^{-1}(U)$. There is a natural differential operator

$$\overline{\partial} : \Omega^0(\Pi^*E) \to \Gamma^\infty(T^{0,1}(N)^* \otimes \Pi^*E)$$

locally given by

$$\overline{\partial} v = (\overline{\partial} f^j) \otimes s_j + f^j \ \Pi^*\overline{\partial}_E \sigma_j \ ,$$

for any $v \in \Omega^0(\Pi^*E)$ locally represented as $v = f^j s_j$ for some C^∞ functions $f^j : U \to \mathbb{C}$. However in general $s_j \notin \mathrm{Ker}(\overline{\partial})$. Let $p : E \to M$ be the projection. We have

Proposition 1. *Assume that $E \to M$ admits a local trivialization atlas $\{\Phi : p^{-1}(U) \to U \times \mathbb{C}^r\}$ such that each section $\sigma_j(x) = \Phi^{-1}(x, e_j)$ is CR-holomorphic (i.e. $\overline{\partial}_E \sigma_j = 0$). Then the pullback bundle $\Pi^*E \to N$ is holomorphic (and s is a holomorphic frame).*

Indeed the transition functions $a^i_j : U \to \mathbb{C}$ of an atlas of E as in Proposition 1 are CR functions. Also the transition functions of the induced atlas on Π^*E are $g^i_j = a^i_j \circ \Pi$. Finally $\overline{\partial}(f \circ \Pi) = \Pi^*\overline{\partial}_b f$ for any $f \in C^\infty(M, \mathbb{C})$ hence Π^*E is a holomorphic vector bundle and $\overline{\partial} s_j = 0$.

A Hermitian metric h in E induces a Hermitian metric Π^*h in Π^*E locally given by $(\Pi^*h)(s_j, s_k) = h(\sigma_j, \sigma_k) \circ \Pi$. Also if $D \in \mathcal{C}(E,h)$ is locally represented as $D\sigma_j = A^i_j \otimes \sigma_i$ then

$$(\Pi^*D)s_j = (\Pi^*A^i_j) \otimes s_i$$

determines a globally defined connection $\Pi^*D \in \mathcal{C}(\Pi^*E,\,\Pi^*h)$. We shall establish the following

Theorem 2. *Let $(E,h) \to M$ be a Hermitian vector bundle over a 3-dimensional nondegenerate CR manifold M and $D \in \mathcal{C}(E,h)$ a metric connection. If Π^*D is a Yang-Mills field in $\Pi^*E \to N$ and $T \rfloor R^D = 0$ then D is a Yang-Mills field in $E \to M$ such that $\Lambda_\theta R^D = 0$.*

As well known any selfdual or antiselfdual (depending on the sign of $c_2(\mathbb{E})[N]$) connection in a $\mathrm{SU}(r)$ vector bundle $\mathbb{E} \to N$ over a compact oriented 4-dimensional manifold N is an absolute minimum for the Yang-Mills functional (cf. e.g. Theorem 3.2 in [9], p. 123). A serious drawback in CR geometry is the lack of a suitable notion of selfduality (and the previous statement has no CR analog as yet). Although $C(M)$ is 4-dimensional (and Yang-Mills fields in $\pi^*E \to C(M)$ are indeed related to the solutions to (24) cf. [2] and the Introduction) a moment's thought shows that the Hodge operator $* : \Lambda^2 T^*C(M) \to \Lambda^2 T^*C(M)$ (associated to the Fefferman metric F_θ) is an almost complex (rather than an almost product) structure i.e. $*^2 = -1$, killing a hope to introduce an appropriate notion of selfduality. Going back to the symplectization process one may attempt to call $D \in \mathcal{C}(E,h)$ *selfdual* (respectively *anti selfdual*) if $\Pi^* D$ is selfdual (respectively anti selfdual) i.e.

$$* R^{\Pi^* D} = \pm R^{\Pi^* D} . \tag{29}$$

Here $* : \Lambda^2 T^*(N) \to \Lambda^2 T^*(N)$ is the Hodge operator on $N = M \times \mathbb{R}^+$ with respect to the Riemannian metric g given by (28). However we may show that

Theorem 3. *Let $(E,h) \to M$ be a Hermitian vector bundle over a 3-dimensional CR manifold M. If $D \in \mathcal{C}(E,h)$ satisfies (29) then D is flat ($R^D = 0$).*

Theorems 2 and 3 will be proved in the next two sections. We first describe the relationship among the Riemannian metrics g and g_θ and express the Levi-Civita connection of (N,g) in terms of the Tanaka-Webster connection of (M,θ).

Proposition 2. *Let M be a 3-dimensional nondegenerate CR manifold and $N = M \times \mathbb{R}^+$. The Riemannian metric g given by (28) satisfies*

$$g(\xi,\xi) = g(\partial/\partial t\, ,\, \partial/\partial t) = 1/(2t) \ \ , \ \ g(\xi\, ,\, \partial/\partial t) = 0 \ , \tag{30}$$

$$g(X^\uparrow, \xi) = 0 \ \ , \ \ g(X^\uparrow\, ,\, \partial/\partial t) = 0 \ , \tag{31}$$

$$g(X^\uparrow\, ,\, Y^\uparrow) = t g_\theta(X,Y) \circ \Pi \ \ , \tag{32}$$

for any $X, Y \in H(M)$.

Corollary 1. *The Levi-Civita connection ∇^g of (N,g) is expressed by*

$$\begin{aligned} \nabla^g_{X^\uparrow} Y^\uparrow = {} & (\nabla_X Y)^\uparrow - t g_\theta(X,Y)\, \partial/\partial t + \\ & + \{2t^2 A(X,Y) - (d\theta)(X,Y)\}\xi \ , \end{aligned} \tag{33}$$

$$\nabla^g_{X^\uparrow} \xi = (\tau X)^\uparrow - \frac{1}{2t^2}\, (\phi X)^\uparrow \ \ , \ \ \nabla^g_{X^\uparrow} \partial/\partial t = \frac{1}{2t}\, X^\uparrow \ , \tag{34}$$

$$\nabla^g_\xi \xi = \nabla^g_\xi \partial/\partial t = \nabla^g_{\partial/\partial t} \partial/\partial t = 0 \ , \tag{35}$$

where $A(X,Y) = G_\theta(X, \tau Y)$, for any $X, Y \in H(M)$.

Proof of Proposition 2. Note that

$$\Omega = dt \wedge \Pi^*\theta + t\, \Pi^* d\theta \ . \tag{36}$$

Then (by (27) and $(d\Pi)\partial/\partial t = 0$)

$$g(\xi,\xi) = \Omega(J\xi,\xi) = t\Omega(\partial/\partial t\, ,\, \xi) = 1/(2t) \ ,$$

$$g(\partial/\partial t\, ,\, \partial/\partial t) = \Omega(J\partial/\partial t\, ,\, \partial/\partial t) = -\frac{1}{t}\, \Omega(\xi\, ,\, \partial/\partial t) = 1/(2t) \ ,$$

$$g(\xi\, ,\, \partial/\partial t) = \Omega(J\xi\, ,\, \partial/\partial t) = 0 \ ,$$

proving (30). Moreover $Y^\uparrow(t) = 0$ for any $Y \in T(M)$ hence

$$g(X^\uparrow, \xi) = \Omega(JX^\uparrow, \xi) = \Omega((\phi X)^\uparrow, \xi) = t\, (d\theta)(X,T) \circ \Pi = 0 \ .$$

Similarly

$$g(X^\uparrow,\, \partial/\partial t) = -\frac{1}{2}\, \theta(\phi X) \circ \Pi = 0$$

as ϕ is $H(M)$-valued. Finally

$$g(X^\uparrow, Y^\uparrow) = \Omega((\phi X)^\uparrow, Y^\uparrow) = t\, (d\theta)(\phi X, Y) \circ \Pi = -t\, G_\theta(X,Y) \circ \Pi$$

proving (32).

Proof of Corollary 1. The Levi-Civita connection ∇^g is given by

$$2\,g(\nabla^g_U V\,,\,W) = U(g(V,W)) + V(g(U,W)) - W(g(U,V)) + \tag{37}$$
$$+g([U,V],W) + g([W,U],V) + g([W,V],U)$$

for any $U,V,W \in T(N)$. Note that

$$[X^\uparrow, Y^\uparrow] = [X,Y]^\uparrow$$

for any $X,Y \in T(M)$. Let $X,Y,Z \in H(M)$. Let us replace (U,V,W) in (37) by $(X^\uparrow, Y^\uparrow, Z^\uparrow)$. Then (by (30)-(31))

$$g(\nabla_{X^\uparrow} Y^\uparrow\,,\,Z^\uparrow) = t\,g_\theta(\nabla^\theta_X Y\,,\,Z)\circ\Pi \tag{38}$$

where ∇^θ is the Levi-Civita connection of (M, g_θ). Let $\pi_H : T(M) \to H(M)$ be the projection associated to the direct sum decomposition

$$T(M) = H(M) \oplus \mathbb{R}T\,.$$

Then $Z^\uparrow = (\pi_H Z)^\uparrow + (\theta(Z)\circ\Pi)\xi$ for any $Z \in T(M)$. Next the identity (37) for $(U,V,W) = (X^\uparrow, Y^\uparrow, \xi)$ yields

$$2\,g(\nabla^g_{X^\uparrow} Y^\uparrow, \xi) = -t\,T(g_\theta(X,Y))\circ\Pi + \frac{1}{2t}\,\theta([X,Y])\circ\Pi +$$
$$+t\{g_\theta(Y,[T,X]) + g_\theta(X,[T,Y])\}\circ\Pi\,.$$

Moreover, taking into account the identities

$$2(d\theta)(X,Y) = -\theta([X,Y])\,,$$
$$2g_\theta(\nabla^\theta_X Y, T) = -T(g_\theta(X,Y)) + \theta([X,Y]) +$$
$$+g_\theta(Y,[T,X]) + g_\theta(X,[T,Y])\,,$$

we obtain

$$g(\nabla^g_{X^\uparrow} Y^\uparrow, \xi) = t g_\theta(\nabla^\theta_X Y, Y)\circ\Pi - \frac{2t^2+1}{2t}\,(d\theta)(X,Y)\circ\Pi \tag{39}$$

for any $X,Y \in H(M)$. Next, as $[X^\uparrow, \partial/\partial t] = 0$ for any $X \in T(M)$ the identity (37) for $(U,V,W) = (X^\uparrow, Y^\uparrow, \partial/\partial t)$ leads to

$$2g(\nabla^g_{X^\uparrow} Y^\uparrow\,,\,\frac{\partial}{\partial t}) = -g_\theta(X,Y)\circ\Pi\,. \tag{40}$$

The Levi-Civita connection ∇^θ of (M, g_θ) and the Tanaka-Webster connection ∇ of (M,θ) are related by (cf. e.g. [6], Chapter 1)

$$\nabla^\theta = \nabla - (A + d\theta)\otimes T + \tau\otimes\theta + 2\,\theta\odot\phi\,. \tag{41}$$

Then $\nabla^\theta_X Y = \nabla_X Y - \{A(X,Y) + (d\theta)(X,Y)\}T$ for any $X,Y \in H(M)$ hence (38)) may be written

$$g(\nabla^g_{X^\uparrow} X^\uparrow\,,\,Z^\uparrow) = t\,g_\theta(\nabla_X Y\,,\,Z)\circ\Pi\,. \tag{42}$$

Summing up, the identities (39)-(40)) and (42) lead to (33). Q.e.d. Similarly, to prove (34) we set $(U,V,W) = (X^\uparrow, \xi, Z^\uparrow)$ in (37) and take into account the identity

$$2g_\theta(\nabla^\theta_X T, Z) = T(g_\theta(X,Z)) + g_\theta([X,T],Z) + g_\theta([Z,T],X) + \theta([Z,X])$$

so that to yield

$$g(\nabla^g_{X^\uparrow}\xi, Z^\uparrow) = t g_\theta(\nabla^\theta_X T, Z) + \frac{2t^2+1}{2t}\,(d\theta)(X,Z)\,.$$

Next we use

$$\nabla^\theta_X T = \tau(X) + \phi X\quad,\quad X \in T(M)\,, \tag{43}$$

and $g_\theta(\phi X, Z) = -(d\theta)(X,Z)$ to conclude that

$$g(\nabla^g_{X^\uparrow}\xi, Z^\uparrow) = -tA(X,Z) + \frac{1}{2t}\,(d\theta)(X,Z) \tag{44}$$

for any $X, Z \in H(M)$. Moreover (as $\nabla^g g = 0$)

$$2g(\nabla^g_{X^\uparrow}\xi, \xi) = X^\uparrow(g(\xi,\xi)) = X^\uparrow(\frac{1}{2t}) = 0$$

and the identity (37) for $(U, V, W) = (X^\uparrow, \xi, \partial/\partial t)$ lead to

$$g(\nabla^g_{X^\uparrow}\xi, \xi) = 0 \ \ , \ \ g(\nabla^g_{X^\uparrow}\xi, \frac{\partial}{\partial t}) = 0 \ . \tag{45}$$

Summing up, (44)-(45) imply the first of the identities (34) in Corollary 1. To prove the remaining identity in (34) we use again $\nabla^g g = 0$ together with (33) and the first part of (34). Then the second part of (34) is a consequence of

$$2g(\nabla^g_{X^\uparrow}\frac{\partial}{\partial t} \ \ , \ \ Y^\uparrow) = g_\theta(X,Y)\circ\Pi \ ,$$

$$g(\nabla^g_{X^\uparrow}\frac{\partial}{\partial t} \ \ , \ \ \xi) = 0 \ \ , \ \ g(\nabla^g_{X^\uparrow}\frac{\partial}{\partial t} \ \ , \ \ \frac{\partial}{\partial t}) = 0 \ .$$

The proof of (35) is similar.

5. The Yang-Mills equations for $\Pi^* E \to N$

The Yang-Mills equations on N are

$$\delta^{\mathbb{D}} R^{\mathbb{D}} = 0 \ \ , \ \ \mathbb{D} \in \mathcal{C}(\Pi^* E, \Pi^* h) \ , \tag{46}$$

where

$$(\delta^{\mathbb{D}}\varphi)_V v = -\sum_{j=1}^{4}(\mathbb{D}_{E_j}\varphi)(E_j, V)v =$$

$$= -\sum_{j=1}^{4}\{\mathbb{D}_{E_j}\varphi(E_j,V)v - \varphi(E_j,V)\mathbb{D}_{E_j}v -$$

$$-\varphi(\nabla^g_{E_j}E_j, V)v - \varphi(E_j, \nabla^g_{E_j}V)v\}$$

for any $\varphi \in \Omega^2(\mathrm{Ad}(\Pi^* E))$ and any $V \in T(N)$, $v \in \Omega^0(\Pi^* E)$. Here $\{E_j : 1 \le j \le 4\}$ is a local orthonormal frame of $(T(N), g)$. We shall show that

Lemma 1. *Let $D \in \mathcal{C}(E,h)$. If $\Pi^* D$ is a Yang-Mills field on N then*

$$\delta^D_b R^D = 0 \tag{47}$$

where the differential operator

$$\delta^D_b : \Omega^2(\mathrm{Ad}(E)) \to \Omega^1(\mathrm{Ad}(E))$$

is given by

$$\begin{aligned}(\delta^D_b\varphi)_X u \ &= -\sum_{a=1}^{2}(D_{X_a}\varphi)(X_a, X)u = -\sum_{a=1}^{2}\{D_{X_a}\varphi(X_a,X)u - \\ &-\varphi(X_a,X)D_{X_a}u - \varphi(\nabla_{X_a}X_a, X)u - \varphi(X_a, \nabla_{X_a}X)u\}\end{aligned} \tag{48}$$

for any $\varphi \in \Omega^2(\mathrm{Ad}(E))$ and any $X \in T(M)$, $u \in \Omega^0(E)$, where $\{X_1, X_2\}$ is a local ortonormal frame of $(H(M), G_\theta)$.

Here ∇ is the Tanaka-Webster connection of (M,θ). We claim that Lemma 1 implies Theorem 2. A comparison between

$$(\delta^D\varphi)_X u = -\sum_{A=1}^{3}(D_{X_A}\varphi)(X_A,X)u = -\sum_{A=1}^{3}\{D_{X_A}\varphi(X_A,X)u-$$

$$-\varphi(\nabla^\theta_{X_A}X_A,X)u-\varphi(X_A,\nabla^\theta_{X_A}X)u-\varphi(X_A,X)D_{X_A}u$$

(where $\{X_A : 1\leq A\leq 3\}$ is a local orthonormal frame of $(T(M),g_\theta)$) and (48) shows that

Proposition 3. *For any* $\varphi\in\Omega^2_H(\mathrm{Ad}(E))$ *one has*

$$(\delta^D\varphi)_X u = (\delta^D_b\varphi)_X u \ \ , \ \ X\in T(M)\ .$$

Consequently if $D\in\mathcal{C}(E,h)$ *satisfies* (47) *and* $T\rfloor R^D=0$ *then* D *is a Yang-Mills field on* M.

Proof of Proposition 3. Let T_1 be a locally defined complex vector field on M such that $T_1\in T_{1,0}(M)$ and $L_\theta(T_1,T_{\bar 1})=-1$. Let us set

$$X_1=\frac{1}{\sqrt{2}}(T_1+T_{\bar 1})\ \ ,\ \ X_2=\phi X_1\ ,$$

so that $\{X_1,X_2,T\}$ is a local orthonormal frame of $(T(M),g_\theta)$. For any $\varphi\in\Omega^2_H(\mathrm{Ad}(E))$ (by $T\rfloor\varphi=0$ and (43))

$$(D_T\varphi)(T,X)u = D_T\varphi(T,X)u-\varphi(T,X)D_Tu\varphi-$$

$$-(\nabla^\theta_T T,X)u-\varphi(T,\nabla^\theta_T X)u=0$$

(indeed (43) yields $\nabla^\theta_T T=0$). Also (by (41))

$$\nabla^\theta_{X_a}X=\nabla_{X_a}X-\{A(X_a,X)+(d\theta)(X_a,X)\}T+\tau(X_a)\theta(X)\ ,$$

$$\sum_{a=1}^{2}\nabla^\theta_{X_a}X_a=\sum_{a=1}^{2}\nabla_{X_a}X_a-\mathrm{trace}(\tau)T$$

and $\mathrm{trace}(\tau)=0$ (cf. e.g. Chapter 1 in [6]). Thus

$$(\delta^D\varphi)_X u = -\sum_{a=1}^{2}(D_{X_a}\varphi)(X_a,X)u=$$

$$=-\sum_a\{D_{X_a}\varphi(X_a,X)u-\varphi(X_a,X)D_{X_a}u-$$

$$-\varphi(\nabla^\theta_{X_a}X_a,X)u-\varphi(X_a,\nabla^\theta_{X_a}X)u\}=(\delta^D_b\varphi)_X u+$$

$$+\sum_a\{[A(X_a,X)+(d\theta)(X_a,X)]\varphi(X_a,T)u-\theta(X)\varphi(X_a,\tau(X_a))u\}$$

hence

$$(\delta^D\varphi)_X u=(\delta^D_b\varphi)_X u-\theta(X)\ \mathrm{trace}_{g_\theta}\ \varphi(\cdot\,,\,\tau\,\cdot)u\ ,$$

for any $X\in T(M)$. Up to this point the argument is quite general (and holds on a strictly pseudoconvex CR manifold or arbitrary CR dimension). As a peculiarity of the 3-dimensional case $\mathrm{trace}_{g_\theta}\ \varphi(\cdot\,,\,\tau\,\cdot)=0$. Indeed

$$\mathrm{trace}_{g_\theta}\ \varphi(\cdot\,,\,\tau\,\cdot)=\sum_{a=1}^{2}\varphi(X_a,\tau(X_a))=$$

$$=\frac{1}{2}\{A^1_{\bar 1}\,\varphi(T_1,T_1)+A^{\bar 1}_1\,\varphi(T_{\bar 1},T_{\bar 1})\}=0$$

as φ is skew. Proposition 3 is proved.

Proof of Lemma 1. Let $\{X_1, X_2\}$ be the orthonormal frame adopted in the proof of Proposition 3 and let us set

$$E_1 = \frac{1}{\sqrt{t}}\, X_1^{\uparrow} \ , \ \ E_2 = JE_1 \ , \ \ E_0 = \frac{1}{\sqrt{2t}}\, \xi \ , \ \ E_3 = \frac{1}{\sqrt{2t}}\, \frac{\partial}{\partial t} \ . \tag{49}$$

Let $\mathbb{D} = \Pi^* D$ and $X \in T(M)$. Then

$$(\delta^{\mathbb{D}} R^{\mathbb{D}})_{X^{\uparrow}}\, \hat{u} = -\sum_{j=0}^{3} (\mathbb{D}_{E_j} R^{\mathbb{D}})(E_j, X^{\uparrow})\hat{u}$$

or (by (49))

$$\begin{aligned}(\delta^{\mathbb{D}} R^{\mathbb{D}})_{X^{\uparrow}}\, \hat{u} = -\frac{1}{2t}\, \{(\mathbb{D}_{\xi} R^{\mathbb{D}})(\xi, X^{\uparrow}) + (\mathbb{D}_{\partial/\partial t} R^{\mathbb{D}})(\frac{\partial}{\partial t}\, , X^{\uparrow}) + \\ +2(\mathbb{D}_{T_1^{\uparrow}} R^{\mathbb{D}})(T_{\bar{1}}^{\uparrow}, X^{\uparrow}) + 2(\mathbb{D}_{T_{\bar{1}}^{\uparrow}} R^{\mathbb{D}})(T_1^{\uparrow}, X^{\uparrow})\}\hat{u}\ .\end{aligned} \tag{50}$$

We compute the terms in the right hand side of (50). As

$$R^{\mathbb{D}} \hat{u} = \Pi^* R^D u \ , \ \ \mathbb{D}\hat{u} = \Pi^* D u\ , \tag{51}$$

one has for any $X, Y \in T(M)$

$$\mathbb{D}_{X^{\uparrow}}\, \hat{u} = (D_X u)^{\uparrow} \ , \ \ R^{\mathbb{D}}(X^{\uparrow}, Y^{\uparrow})\hat{u} = (R^D(X, Y)u)^{\uparrow}\ .$$

We distinguish two cases as I) $X \in H(M)$ or II) $X = T$. In the first case

$$\begin{aligned}&(\mathbb{D}_{T_1^{\uparrow}} R^{\mathbb{D}})(T_{\bar{1}}^{\uparrow}, X^{\uparrow})\hat{u} = \\ &= \mathbb{D}_{T_1^{\uparrow}} (R^{\mathbb{D}}(T_{\bar{1}}^{\uparrow}, X^{\uparrow})\hat{u}) - R^{\mathbb{D}}(T_{\bar{1}}^{\uparrow}, X^{\uparrow})\mathbb{D}_{T_1^{\uparrow}}\, \hat{u} - \\ &-R^{\mathbb{D}}(\nabla^g_{T_1^{\uparrow}} T_{\bar{1}}^{\uparrow}, X^{\uparrow}) - R^{\mathbb{D}}(T_{\bar{1}}^{\uparrow}, \nabla^g_{T_1^{\uparrow}} X^{\uparrow})\hat{u} =\end{aligned}$$

by (33) in Corollary 1

$$\begin{aligned}&= \mathbb{D}_{T_1^{\uparrow}} (R^D(T_{\bar{1}}, X)u)^{\uparrow} - (R^D(T_{\bar{1}}, X)D_{T_1} u)^{\uparrow} - \\ &-R^{\mathbb{D}}((\nabla_{T_1} T_{\bar{1}})^{\uparrow} + tG_{\theta}(T_1, T_{\bar{1}})\frac{\partial}{\partial t} + \{2t^2 A(T_1, T_{\bar{1}}) - (d\theta)(T_1, T_{\bar{1}})\}\xi\, , \, X^{\uparrow})\hat{u} - \\ &-R^{\mathbb{D}}(T_{\bar{1}}\, , \, (\nabla_{T_1} X)^{\uparrow} + tG_{\theta}(T_1, X)\frac{\partial}{\partial t} + \{2t^2 A(T_1, X) - (d\theta)(T_1, X)\}\xi)\hat{u}\ .\end{aligned}$$

Note that (by (51))

$$\partial/\partial t \,\rfloor\, R^{\mathbb{D}} = 0 \ , \ \ R^{\mathbb{D}}(X^{\uparrow}, \xi)\hat{u} = (R^D(X, T)u)^{\uparrow}\ .$$

Also (by (16))

$$\begin{aligned}A(T_1, T_{\bar{1}}) = 0 \ , \ \ (d\theta)(T_1, T_{\bar{1}}) = -i\ , \\ A(T_1, X)T_{\bar{1}} = -(\tau X)^{0,1} \ , \ \ (d\theta)(T_1, X)T_{\bar{1}} = -iX^{0,1}\ .\end{aligned}$$

Here if $X \in H(M)$ is represented as $X = Z + \overline{Z}$ with $Z \in T_{1,0}(M)$ then we adopt the notations $X^{1,0} = Z$ and $X^{0,1} = \overline{Z}$. We may conclude that

$$\begin{aligned}(\mathbb{D}_{T_1^{\uparrow}} R^{\mathbb{D}})(T_{\bar{1}}^{\uparrow}, X^{\uparrow})\hat{u} = \Big((D_{T_1} R^D)(T_{\bar{1}}, X)u\Big)^{\uparrow} + \\ +2t^2 \Big(R^D((\tau X)^{0,1}, T)u\Big)^{\uparrow} - i\Big(R^D(X^{0,1}, T)u\Big)^{\uparrow}\end{aligned}$$

and then

$$\begin{aligned}&(\mathbb{D}_{T_1^{\uparrow}} R^{\mathbb{D}})(T_{\bar{1}}^{\uparrow}, X^{\uparrow})\hat{u} + (\mathbb{D}_{T_{\bar{1}}^{\uparrow}} R^{\mathbb{D}})(T_1^{\uparrow}, X^{\uparrow})\hat{u} = \\ &= -\frac{1}{2}\, \Big((\delta_b^D R^D)_X u\Big)^{\uparrow} + 2t^2 \Big(R^D(\tau X, T)u\Big)^{\uparrow} + \Big(R^D(\phi X, T)u\Big)^{\uparrow}\ .\end{aligned} \tag{52}$$

It remains that we compute the first two terms in the left hand side of (50). Using $\partial/\partial t \rfloor R^{\mathbb{D}} = 0$ and (35) in Corollary 1

$$(\mathbb{D}_{\partial/\partial t} R^{\mathbb{D}})(\frac{\partial}{\partial t}\,,\, X^{\uparrow})\hat{u} = -R^{\mathbb{D}}(\frac{\partial}{\partial t}\,,\, \nabla^g_{\partial/\partial t} X^{\uparrow})\hat{u} =$$

by (34) in Corollary 1

$$= -\frac{1}{2t}\, R^{\mathbb{D}}(\frac{\partial}{\partial t}\,,\, X^{\uparrow})\hat{u} = 0\,.$$

Similarly (by (34)-(35))

$$(\mathbb{D}_\xi R^{\mathbb{D}})(\xi, X^{\uparrow})\hat{u} = \mathbb{D}_\xi R^{\mathbb{D}}(\xi, X^{\uparrow})\hat{u} - R^{\mathbb{D}}(\xi, \nabla^g_\xi X^{\uparrow})\hat{u} - R^{\mathbb{D}}(\xi, X^{\uparrow})\mathbb{D}_\xi \hat{u} =$$

$$= \Big(D_T R^D(T,X)u - R^D(T,X) D_T u\Big)^{\uparrow} -$$

$$-R^{\mathbb{D}}(\xi\,,\, [\xi, X^{\uparrow}] + (\tau X)^{\uparrow} - \frac{1}{2t^2}\,(\phi X)^{\uparrow})\hat{u} =$$

$$= \Big(D_T R^D(T,X)u - R^D(T,X) D_T u -$$

$$-R^D(T, [T,X] + \tau X)u\Big)^{\uparrow} + \frac{1}{2t^2}\,\Big(R^D(T,\phi X)u\Big)^{\uparrow}\,.$$

On the other hand (by $\nabla T = 0$) $[T,X] + \tau(X) = \nabla_T X$ so that

$$(\mathbb{D}_\xi R^{\mathbb{D}})(\xi, X^{\uparrow})\hat{u} =$$

(53)
$$= \Big((D_T R^D)(T,X)u\Big)^{\uparrow} + \frac{1}{2t^2}\,\Big(R^D(T,\phi X)u\Big)^{\uparrow}\,.$$

Summing up (by (52)-(53) and the hypothesis $T \rfloor R^D = 0$)

(54)
$$(\delta^{\mathbb{D}} R^{\mathbb{D}})_{X^{\uparrow}}\hat{u} = \frac{1}{2t}\,\Big((\delta_b^D R^D)_X u\Big)^{\uparrow}\ ,\ \ X \in H(M)\,.$$

For any $\varphi \in \Omega^2_H(\mathrm{Ad}(E))$ one has $T \rfloor \delta_b^D \varphi = 0$. In particular (by $T \rfloor R^D = 0$) if $\delta^{\mathbb{D}} R^{\mathbb{D}} = 0$ then $\delta_b^D R^D = 0$. Lemma 1 is proved.

Let D be as in Theorem 2. Then (by Lemma 1 and Proposition 3) D is a Yang-Mills field on M. To complete the proof of Theorem 2 note that

$$(\mathbb{D}_{\partial/\partial t} R^{\mathbb{D}})(\frac{\partial}{\partial t}\,,\, \xi)\hat{u} = 0\ \ ,\ \ (\mathbb{D}_{T_1^{\uparrow}} R^{\mathbb{D}})(T_{\bar{1}}^{\uparrow}, \xi)\hat{u} = -\frac{i}{2t^2}\,\Big(R^D(T_1, T_{\bar{1}})u\Big)^{\uparrow}\ ,$$

imply

(55)
$$(\delta^{\mathbb{D}} R^{\mathbb{D}})_\xi \hat{u} = -\frac{1}{t^3}\,\Big((\Lambda_\theta R^D)u\Big)^{\uparrow}\,.$$

6. Selfdual connections on $N = M \times \mathbb{R}^+$

Let $\mathbb{D}$ be a connection in $\Pi^* E \to N$. Then $R^{\mathbb{D}} = \mathbb{F}^i_j \otimes E^j_i$ for some 2-forms $\mathbb{F}^i_j$ on $\Pi^{-1}(U)$ where $E^i_j \in \Gamma^\infty(U, \mathrm{End}(\Pi^* E))$ are given by $E^i_j = [\delta^i_\ell \delta^k_j]_{1 \le k,\ell \le r}$ (with respect to the local frame $s = (s_1, \cdots, s_r)$). Consequently $\mathbb{D}$ is (anti) selfdual if and only if

(56)
$$*\mathbb{F}^i_j = \pm\, \mathbb{F}^i_j\,.$$

Any orientation of M induces naturally an orientation of N. Let $\Psi = \theta \wedge d\theta$. Then $\omega = (\Pi^*\Psi) \wedge dt$ is a volume form on N. Then $\omega = f\ d\,\mathrm{vol}(g)$ for some $f \in C^\infty(N)$. The Hodge operator $* : \Lambda^2 T^*(N) \to \Lambda^2 T^*(N)$ is given by

$$(*\varphi)(V,W)\ d\,\mathrm{vol}(g) = \varphi \wedge \lambda \wedge \mu\ \ ,\ \ V,W \in T(N)\,,$$

where $\lambda^\sharp = V$ and $\mu^\sharp = W$. Here $\sharp$ indicates the raising of indices by g. Thus

$$(*\mathbb{F}^i_j)(V,W)\ \omega = f\ \mathbb{F}^i_j \wedge \lambda \wedge \mu\,.$$

We recall (cf. e.g. Chapter 1 in [6]) that

$$\Psi = 2i g_{1\bar{1}}\, \theta \wedge \theta^1 \wedge \theta^{\bar{1}} \qquad (g_{1\bar{1}} = -1)\,.$$

Then $\omega(\xi, T_1^\uparrow, T_{\bar 1}^\uparrow, \frac{\partial}{\partial t}) = (1/4)\ \Psi(T, T_1, T_{\bar 1}) = -i/12$ yields

$$(*\mathbb{F}^i_j)(V, W) = 12 i f (\mathbb{F}^i_j \wedge \lambda \wedge \mu)(\xi, T_1^\uparrow, T_{\bar 1}^\uparrow, \frac{\partial}{\partial t}) \ . \tag{57}$$

We establish the following

Proposition 4. *Let $\mathbb{D}$ be a connection in $\Pi^* E \to N$. Then*

$$(*\mathbb{F}^i_j)(X^\uparrow, Y^\uparrow) = t^2 f g_\theta(\phi X, Y)\mathbb{F}^i_j(\partial/\partial t\, ,\, \xi)\ , \tag{58}$$

$$(*\mathbb{F}^i_j)(X^\uparrow, \xi) = (f/2)\ \mathbb{F}^i_j((\phi X)^\uparrow\, ,\, \partial/\partial t)\ , \tag{59}$$

$$(*\mathbb{F}^i_j)(X^\uparrow\, ,\, \partial/\partial t) = (f/2)\mathbb{F}^i_j(\xi, (\phi X)^\uparrow)\ , \tag{60}$$

$$(*\mathbb{F}^i_j)(\xi\, ,\, \partial/\partial t) = (if/(4t^2))\ \mathbb{F}^i_j(T_1^\uparrow\, ,\, T_{\bar 1}^\uparrow)\ , \tag{61}$$

for any $X, Y \in H(M)$.

Proof. For instance (58) follows from (57) for $V = X^\uparrow$ and $V = Y^\uparrow$. Indeed if this is the case then $\lambda(\partial/\partial t) = \mu(\partial/\partial t) = 0$ and $\lambda(\xi) = \mu(\xi) = 0$ (by Corollary 1) hence

$$(*\mathbb{F}^i_j)(X^\uparrow, Y^\uparrow) = i f \mathbb{F}^i_j(\partial/\partial t\, ,\, \xi)\{\lambda(T_{\bar 1}^\uparrow)\mu(T_1^\uparrow) - \lambda(T_1^\uparrow)\mu(T_{\bar 1}^\uparrow)\}\ .$$

On the other hand

$$i\{\lambda(T_{\bar 1}^\uparrow)\mu(T_1^\uparrow) - \lambda(T_1^\uparrow)\mu(T_{\bar 1}^\uparrow)\} =$$
$$= i t^2\{g_\theta(X, T_{\bar 1}) g_\theta(Y, T_1) - g_\theta(X, T_1) g_\theta(Y, T_{\bar 1})\} =$$
$$= t^2 i g_\theta(Y, X^{1,0} - X^{0,1}) = t^2 g_\theta(Y, \phi X)\ .$$

The proof of (59)-(61) is similar. As a consequence of (58)- (61)

Corollary 2. *$\mathbb{D}$ is $\pm$ selfdual if and only if*

$$t^2 f\ g_\theta(\phi X, Y)\mathbb{F}^i_j(\partial/\partial t\, ,\, \xi) = \pm \mathbb{F}^i_j(X^\uparrow, Y^\uparrow)\ , \tag{62}$$

$$f\ \mathbb{F}^i_j((\phi X)^\uparrow,\, \partial/\partial t) = \pm 2\mathbb{F}^i_j(X^\uparrow, \xi)\ , \tag{63}$$

$$f\ \mathbb{F}^i_j(X, (\phi X)^\uparrow) = \pm 2\mathbb{F}^i_j(X^\uparrow,\, \partial/\partial t)\ , \tag{64}$$

$$i f\ \mathbb{F}^i_j(T_1^\uparrow, T_{\bar 1}^\uparrow) = \pm 4 t^2 \mathbb{F}^i_j(\xi,\, \partial/\partial t)\ , \tag{65}$$

for any $X, Y \in H(M)$.

Clearly Corollary 2 implies Theorem 3. Indeed if $\mathbb{D} = \Pi^* D$ and $R^D \sigma_j = F^i_j \otimes \sigma_i$ then $\mathbb{F}^i_j = \Pi^* F^i_j$ hence $\partial/\partial t \rfloor R^{\mathbb{D}} = 0$ and then (62)-(65) yield $\mathbb{F}^i_j = 0$.

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