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A new proof of the Takeuchi theorem

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Abstract[1]. In this paper[2], we use the matrix-valued Riccati equation to provide a simple proof of the Takeuchi Theorem, which is a curved version of the Oka lemma. We also use the Fermi-coordinate system and the associated length-function near a geodesic segment.

1. Introduction

In this paper, we present a short proof of the following theorem due to Takeuchi.

Theorem A. (Takeuchi [18], [17]) *Let Ω be a pseudoconvex domain with C^2-smooth boundary in a Kähler manifold M^{2n} and $r = d(x, b\Omega)$. Suppose that the Kähler manifold M^{2n} has holomorphic bisectional curvature ≥ 1. Then the second fundamental form of $b\Omega_{(-t)}$ satisfies:*

$$i\,\partial\overline{\partial}(-r)(\zeta, \bar{\zeta}) \geq r\|\zeta\|^2$$

for all $\zeta \in T_x^{1,0}(b\Omega_{(-t)})$, where $b\Omega_{(-t)} = \{x \in \Omega \mid d(x, b\Omega) = t\}$ for $t \geq 0$. Moreover, we have the curved version of Oka's inequality: $i\partial\overline{\partial}(-\log r)(\zeta, \bar{\zeta}) \geq (1/6)\|\zeta\|^2$ for any $\zeta \in T_x^{1,0}(\Omega)$ and $x \in \Omega$.

Let us recall some preliminary results. For any C^2 smooth function f and a complex vector τ of (1, 0)-type, the Levi form and complex Hessian are related as follows:

$$\mathcal{L}f(\tau, \bar{\tau}) = 4\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}\tau_j\bar{\tau}_k = 2\sqrt{-1}(\partial\overline{\partial}f)(\tau, \bar{\tau}) ,$$

where $\tau = \sum_{j=1}^{n} \tau_j(\partial\ /\partial z_j) \in T^{(1,0)}(M^{2n})$. Notice that the complex Hessian $\sqrt{-1}\,(\partial\overline{\partial}f)$ is independent of the choice of the metrics on M^n.

When M^{2n} admits a Kähler metric $g = \langle\ ,\ \rangle$, both the Levi form $\mathcal{L}f$ and $\sqrt{-1}\,(\partial\overline{\partial}f)$ are related to the real Hessian of f which we now recall.

Since the Kähler metric g is a Hermitian metric, it preserves the complex structure J, i.e., $|JX|^2 = \langle JX, JX\rangle = \langle X, X\rangle = |X|^2$ for any real vector $X \in [T(M^{2n})]_{\mathbb{R}}$. There is a

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natural isometry between $T(M^{2n})]_{\mathbb{R}}$ and $T^{(1,0)}(M^{2n})$ over the real numbers. The map

$$u \longmapsto \tilde{u} = \frac{1}{\sqrt{2}}(u - \sqrt{-1}\,Ju)$$

is a linear isomorphism from $[T(M^{2n})]_{\mathbb{R}}$ to $T^{(1,0)}(M^{2n})$.

Recall that, if the metric is Kähler, for $\tilde{u} = (1/\sqrt{2})(u - \sqrt{-1}\,Ju)$, we have

$$\sqrt{-1}\,\partial\overline{\partial} f(\tilde{u}, \bar{\tilde{u}}) = \text{Hess}(f)(u,u) + \text{Hess}(f)(Ju, Ju)$$

see [9], where $\text{Hess}(f)(X,Y) = XYf - (\nabla_X Y)f = \langle \nabla_X(\nabla f), Y\rangle$ and ∇ is the covariant derivative (the induced connection) determined by the Kähler metric g.

2. A new proof of Takeuchi Theorem

Definition 2.0. Let J be the complex structure of a Kähler manifold M^{2n} and let Ω be a compact C^2-smooth domain with boundary $b\Omega$ in M^{2n}. Suppose that $\nabla\rho$ is the *outward* unit normal vector of Ω along its boundary $b\Omega$. If

$$\langle \nabla_X(\nabla\rho), X\rangle + \langle \nabla_{JX}(\nabla\rho), JX\rangle \geq 0 \tag{2.1}$$

holds for all $X \perp \{\nabla\rho, J(\nabla\rho)\}$ with $X \in T_p(b\Omega)$ and all $p \in b\Omega$, then the domain Ω is said to be pseudoconvex.

A convex subdomain of a Kähler manifold is necessarily pseudoconvex. However, a pseudoconvex domain of a Kähler manifold is not necessarily convex. For instance, any compact C^2-smooth domain Ω of $\mathbb{C}$ is pseudoconvex, because there is no non-zero tangential vectors X with $X \perp \{\nabla\rho, J(\nabla\rho)\}$ in $\mathbb{C}$.

Inspired by Cheeger-Gromoll's work [4] for real Riemannian manifolds, we also study *the inward equidistant evolution* for a peudoconvex domain Ω:

$$\Omega_{(-t)} = \{x \in \Omega \mid d(x, b\Omega) \geq t\}$$

2.1. **Estimates for the complex Hessian of distance functions.** When f has the property $|\nabla f| = |df| = 1$, it is easy to check that integral curves of the gradient flow are geodesics of unit speed. Therefore, $\nabla_{\nabla f}(\nabla f) = 0$ and $\text{Hess}(f)(\nabla f, Y) = \langle \nabla_{\nabla f}(\nabla f), Y\rangle = 0$ for any $Y \in [T(M^{2n})]_{\mathbb{R}}$. In particular, if $f(x) = r(x) = d(x, b\Omega)$ is a distance function, we have

$$\text{Hess}(r)(\nabla r, Y) = 0\ , \tag{2.2}$$

for any $Y \in [T(M^{2n})]_{\mathbb{R}}$.

It is sufficient to estimate $\text{Hess}(r)$ when it is restricted to the tangential subspace $[T(b\Omega_{(-r)})]_{\mathbb{R}}$, where $\Omega_{(-r)} = \{x \in \Omega \mid d(x, b\Omega) \geq r\}$. The real Hessian $\text{Hess}(r)\mid_{[T(b\Omega_{(-r)})]_{\mathbb{R}}}$ is exactly the so-called second fundamental form of $b\Omega_{(-r)}$ in the Kähler manifold M^{2n}. It is well-known that the tangential part of $\text{Hess}(r)$ satisfies the Riccati equation:

$$\nabla_{\nabla r}\text{Hess}(r) + [\text{Hess}(r)]^2 + \mathcal{R} = 0\ ,$$

where $\mathcal{R}$ is a bilinear form related to sectional curvatures of the Kähler metric g (see e.g. Proposition 4.1 in [13]).

The following result was proved by the variational method (e.g., see Takeuchi [18] or Siu [15]). We shall use the Riccati equation to give a new simple proof.

Theorem 2.1. *Let M^{2n} be a Kähler manifold with bisectional curvature ≥ 1. Let $\Omega \subset M^{2n}$ be a pseudoconvex domain with C^2 boundary $b\Omega = \Sigma$ and let $\rho(x) = -d(x, \Sigma)$ for $x \in \Omega$. Then*

$$\mathcal{L}(\rho)(\tau, \bar{\tau}) = 2\sqrt{-1}\,(\partial\overline{\partial}\rho)(\tau, \bar{\tau}) \geq |\rho|\|\tau\|^2, \tag{2.3}$$

for any $\tau \in T^{(1,0)}(b\Omega_{(-|\rho|)})$ and $|\rho| \leq \varepsilon_0$, where ε_0 is sufficiently small.

Proof. Let $Q_0 \in b\Omega$ and Exp_{Q_0} denote the exponential map from $T_{Q_0}(M^{2n})$ to M^{2n}. Let $\sigma : [-t_0, t_0] \to M^{2n}$ be the geodesic given by

$$\sigma(t) = \mathrm{Exp}_{Q_0}(t\nabla\rho) \ , \tag{2.4}$$

for small $t_0 > 0$. We will study how the Levi form $\mathcal{L}(\rho)$ changes along σ. By (0.2), it suffices to analyze $\mathrm{Hess}(\rho)$ along $\sigma(t)$. Recall that $\sigma'(t) = \nabla\rho \mid_{\sigma(t)}$ and

$$\mathrm{Hess}(\rho)(\nabla\rho, \xi) = \langle \tilde{D}_{\nabla\rho}(\nabla\rho), \xi\rangle \equiv 0$$

since σ is geodesic. It remains to discuss $\mathrm{Hess}(\rho)(\xi, \xi)$ for $\xi \perp \nabla\rho$, or equivalently, $\xi \in T(\Sigma_{(-s)})$ where $\Sigma_{(-s)} = \{x \in X \mid \rho(x) = -s\} = b\Omega_{(-s)}$ for some small number $s > 0$. Notice that the second fundamental form of $\Sigma_{(-s)}$ is equal to the $\mathrm{Hess}(\rho)$ restricted to the tangent space $T(\Sigma_{(-s)})$, i.e.,

$$\mathrm{Hess}(\rho)(\xi, \eta) = \langle \nabla_\xi \nabla\rho, \eta\rangle = \Pi_{\Sigma_{(-s)}}(\xi, \eta)$$

for $\xi, \eta \in T(\Sigma_{(-s)})$.

The second fundamental forms of $\Sigma_{(-s)}$ along $\sigma(s)$ satisfy the Riccati equation. We choose an orthonormal frame $e_1(0), \cdots, e_{2n}(0)$ of $T_{Q_0}(M^{2n})$, where $\tilde{e}_k = (1/\sqrt{2})\,[e_{2k} - \sqrt{-1}\,Je_{2k}]$, $k = 1, \cdots, n$. We require that $\tilde{e}_1(0), \cdots, \tilde{e}_{n-1}(0)$ span $T^{(1,0)}_{Q_0}(\Sigma)$ and that $e_{2k-1} = -Je_{2k}$. We also choose

$$e_{2n}(0) = \sigma'(0) = \nabla\rho \mid_{Q_0} \ . \tag{2.5}$$

Let $\{E_k(t)\}$ be a parallel vector field along $\sigma(t)$ with initial condition $E_k(0) = e_k(0)$. Since M is Kähler, we have

$$\begin{aligned} E_{2n}(t) &= \nabla\rho \mid_{\sigma(t)} \quad , \quad E_{2n-1} = -J(\nabla\rho) \ , \\ E_{2j-1}(t) &= -J(E_{2j}(t)) \quad , \quad j = 1, \cdots, n-1 \end{aligned} \tag{2.6}$$

for all $0 \le t \le \varepsilon$.

For each $k = 1, \cdots, 2n-1$, we consider the Jacobi field ξ_k with initial condition

$$\begin{cases} \xi_k(0) & = E_k(0) \ , \\ \xi_k'(0) & = \nabla_{E_k(0)}(\nabla\rho) \ . \end{cases}$$

For any Jacobi field $\xi(s)$, we have

$$\mathrm{Hess}(\rho)(\xi, \xi) = \Pi_{\Sigma_{(-s)}}(\xi, \xi) = \langle \nabla_\xi \nabla\rho, \xi\rangle = \langle \nabla_{\nabla\rho}\xi, \xi\rangle = \langle \xi'(s), \xi(s)\rangle \ . \tag{2.7}$$

Let $A(s) = (a_{jk}(s))$ be the matrix-valued function defined by

$$\xi_k(s) = \sum_{j=1}^{2n-1} a_{jk}(s) E_j(s)$$

and the curvature matrix $R(s) = (R_{ij}(s))$ defined by

$$R(\sigma', E_i)\sigma' = \sum_{j=1}^{2n-1} R_{ji} E_j \ .$$

With the notation above, we have using the Jacobi equation

$$0 = \xi_k'' + R(\sigma', \xi_k)\sigma' = \sum_{j=1}^{2n-1} a_{jk}'' E_j(s) + \sum_{i,j=1}^{2n-1} R_{ji} a_{ik} E_j \ .$$

Thus we have the matrix expression of the Jacobi equation

$$A''(s) + R(s)A(s) = 0 \ . \tag{2.8}$$

Let

$$B(s) = A'(s)A^{-1}(s) = (b_{ij}(s)) \ .$$

Then

$$\Pi_{\Sigma_{(-s)}}(E_i, E_j) = b_{ij}(s) \ .$$

Using (2.7), we get

$$\Pi_{\Sigma_{(-s)}}(\xi, \xi) = \langle A'(s)A^{-1}(s)\xi, \xi\rangle = \langle B(s)\xi, \xi\rangle \ . \tag{2.9}$$

Thus $B(s)$ is the matrix representation of the second fundamental form $\Pi_{\Sigma_{(-s)}}$ with respect to the orthonormal basis $E_1(s), \cdots, E_{2n-1}(s)$. It follows from (2.8) and (2.9) that

$$0 = A''A^{-1} + R = B' + B^2 + R \ , \tag{2.10}$$

or equivalently,

$$\Pi' + \Pi^2 + R = 0 \ . \tag{2.11}$$

We now apply the above Riccati equation (2.11) to prove Theorem 2.1. If $\tau(s) \in T^{(1,0)}(\Sigma_{(-s)})$, then

$$\tau(s) = \xi(s) - \sqrt{-1}\, J(\xi(s)) \ ,$$

where $\xi = \sum_{k=1}^{2n-2} c_k E_k(s)$ for some $\underline{C} = (c_1, \cdots, c_{2n-2}) \in \mathbb{R}^{2n}$.

Let

$$\lambda_\xi(s) = \Pi(\xi(s), \xi(s))$$

and let

$$\mu_\tau(s) = \mathcal{L}(\rho)(\tau(s), \bar{\tau}(s))$$

be the Levi form in the τ direction. From the assumption that Ω is pseudoconvex, we have

$$\mu_\tau(0) \geq 0 \ .$$

Using (2.11), we get

$$\begin{aligned} \lambda'_\xi(s) &= \langle B'(s)\underline{C}\,, \underline{C}\,\rangle = \langle -B^2\underline{C}\,, \underline{C}\,\rangle - \langle R\underline{C}\,, \underline{C}\,\rangle = \\ &= -\|B\underline{C}\,\|^2 - \langle R\underline{C}\,, \underline{C}\,\rangle \leq -\langle R(\sigma', \xi)\sigma', \xi\rangle \ , \end{aligned} \tag{2.12}$$

where we have used that the second fundamental form is symmetric and $B(s)$ is a symmetric matrix. Similarly, we have

$$\lambda'_{J\xi}(s) \leq -\langle R(\sigma', J\xi)\sigma', J\xi\rangle \ . \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\mu'_\tau(s) \leq -(\langle R(\sigma', \xi)\sigma', \xi\rangle + \langle R(\sigma', J\xi)\sigma', J\xi\rangle) \ .$$

The term $(\langle R(\sigma', \xi)\sigma', \xi\rangle + \langle R(\sigma', J\xi)\sigma', J\xi\rangle)$ is equal to the bisectional curvature (see e.g. Zheng [19]) in the direction of $\tilde{e}_n, \tau$. Thus from our assumption, the bisectional curvature is greater or equal to one. Hence, we have

$$\mu'_\tau(s) \leq -1 \ . \tag{2.14}$$

Using

$$\mu_\tau(0) - \mu_\tau(-\varepsilon) = \int_{-\varepsilon}^0 \mu'_\tau(s)\, ds$$

and (2.14), we have that

$$\mu_\tau(-\varepsilon) = \mu_\tau(0) - \int_{-\varepsilon}^0 \mu'_\tau(s)\, ds \geq 0 - (-1)\varepsilon = \varepsilon$$

for any $0 < \varepsilon < t_0$. Thus

$$\mathcal{L}(\rho)\ |_{\sigma(-\varepsilon)}\ (\tau, \bar{\tau}) \geq \varepsilon$$

for any $0 < \varepsilon \leq t_0$ with $\tau \in T^{(1,0)}(\Sigma_{(-\varepsilon)})$. This proves (2.3) and Theorem 2.1. □

We would also like to extend the inequality (2.3) to the subset of full measure in domain Ω, not just near the boundary $b\Omega$. To do this, we need to recall the definition of cut loci in Riemannian geometry.

Definition 2.2 (Cut loci)**.** Let $\Omega \subset M^m$ be a compact domain with C^2-smooth boundary in a Riemannian manifold (M^m, g). Suppose that $\sigma : [0, \ell] \to \Omega$ is a geodesic of unit speed such that $\sigma(0) \in b\Omega$ and $\sigma'(0)$ is orthogonal to $b\Omega$ at $\sigma(0)$.

(1) The above geodesic segment σ is said to be length-minimizing from $b\Omega$ if $d(\sigma(t), b\Omega) = t$ for any $t \in [0, \ell]$.

(2) Suppose that the above geodesic segment σ is length-minimizing from $b\Omega$. The end-point $Q = \sigma(\ell)$ is said to be a cut point of $b\Omega$ in Ω if $d(\sigma(\ell + \varepsilon), b\Omega) < \ell + \varepsilon$ for any $\varepsilon > 0$.

(3) The subset of all cut points Q described in (2) is called the cut-loci of $b\Omega$ in Ω, denoted by $Cut_\Omega(b\Omega)$.

We need to use the following geometric properties of the cut-loci.

Proposition 2.3 ([3] p. 99, [13])**.** *Let $\Omega \subset M^{2n}$ be a compact domain with C^2-smooth boundary in a C^2-smooth Riemannian manifold (M^m, g). Then*

(1) *The cut-loci of $b\Omega$ in Ω is a closed subset of zero measure.*

(2) *There is a nearest point projection: $\mathcal{P}_{b\Omega} : [\bar{\Omega} - Cut_\Omega(b\Omega)] \to b\Omega$; i.e., for each $Q \notin Cut_\Omega(b\Omega)$, there exists the unique nearest point $P_Q = \mathcal{P}_{b\Omega}(Q) \in b\Omega$ such that $d(Q, b\Omega) = d(Q, P_Q)$.*

The proof of Theorem 2.1 also implies the following:

Corollary 2.4. *Let M^{2n} be a Kähler manifold with holomorphic bisectional curvature ≥ 1. Suppose that $\Omega \subset M^{2n}$ is a pseudoconvex domain with C^2 boundary $b\Omega = \Sigma$. Let $\Omega_{(-t)} = \{x \in \Omega \mid d(x, b\Omega) = d(x, \Sigma) \geq |t|\}$ and $\rho(x) = -r(x) = -d(x, \Sigma)$. Then*

$$\mathcal{L}(\rho) \mid_Q (\tau, \bar{\tau}) \geq |\rho| \|\tau\|^2 \tag{2.3'}$$

for any $\tau \in T_Q^{(1,0)}(b\Omega_{(\rho)})$ and $Q \notin Cut_\Omega(b\Omega)$.

Notice that neither Theorem 2.1 nor Corollary 2.4 have estimates of complex Hessian $\mathcal{L}(-r)$ on complex normal directions. In fact, we already have $\text{Hess}(r)(\nabla r, Y) = 0$ for any Y. Furthermore, one can also construct an example of pseudoconvex domain $\Omega \subset \mathbb{C}$, for which the signed distance function $\rho(x) = \rho_{b\Omega}(x)$ has the property $\text{Hess}(\rho)(J\nabla\rho, J\nabla\rho) \mid_Q < 0$ for some $Q \in \Omega$. In such an example, we have $i\partial\overline{\partial}(\rho)(\widetilde{\nabla\rho}, \overline{\widetilde{\nabla\rho}}) \mid_Q < 0$ for some $Q \in \Omega$.

In order to find a plurisubharmonic function f (i.e., $\mathcal{L}f \geq 0$ on Ω), Oka considers $[-\log r]$ instead of the signed distance function ρ. Therefore, in next subsection, we estimate $\mathcal{L}(-\log r)(\tau, \bar{\tau}) = 2i\partial\overline{\partial}[-\log r](\tau, \bar{\tau})$. It will be shown that $\mathcal{L}(-\log r)$ is strictly positive definite in all directions.

2.2. **Estimates for $i\partial\overline{\partial}(-\log r)$ in all directions.** Our goal of this subsection is to show the following

Theorem 2.5. *Let Ω be a pseudoconvex domain with C^2 boundary $b\Omega = \Sigma$ in a Kähler manifold M with the bisectional curvatures ≥ 1 and let $r = d(x, \Sigma)$ be the distance function from $x \in \Omega$ to $b\Omega = \Sigma$. Then*

$$\mathcal{L}(-\log r)(\tilde{\zeta}, \bar{\tilde{\zeta}}) = 2i\partial\overline{\partial}(-\log r)(\tilde{\zeta}, \bar{\tilde{\zeta}}) \geq \frac{1}{3} \|\tilde{\zeta}\|^2 \tag{2.15}$$

for any $\tilde{\zeta} \in T_x^{(1,0)}(\Omega)$ and $x \in \Omega$.

Before we provide the proof of Theorem 2.5, we need to recall an elementary but useful fact, which will be used in the proof.

Fact 2.6 (Calabi [1])**.** Let U be an open set of M, $X \in [T_{Q_0}(M)]_{\mathbb{R}}$, $f : U \to \mathbb{R}$ be a real-valued continuous function and $Q_0 \in U$. If there is another C^2-smooth function $h : U \to \mathbb{R}$ such that (1) $h(\mathrm{Exp}_{Q_0}(sX)) \leq f(\mathrm{Exp}_{Q_0}(sX))$ for $s \in (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$; (2) $f(Q_0) = h(Q_0)$ and (3)

$$\frac{d^2[h(\mathrm{Exp}_{Q_0}(sX))]}{ds^2}(0) \geq C ,$$

then

$$\mathrm{Hess}(f)(X, X) \geq C .$$

We will choose a *nice* lower barrier function $-\ell$ to estimate $\mathrm{Hess}(-r)(X, X)$, where $f(x) = -r(x) = -d(x, b\Omega)$, $h(x) = -\ell(x)$ and $\ell(x) \geq r(x)$.

Proof of Theorem 2.5. We first assume that $x \in U \cap \Omega$, where U is a small neighborhood of $b\Omega$. It is easy to see that for any C^2 function f, we have

$$\mathrm{Hess}(f(\rho))(\xi, \eta) = f'(\rho)\mathrm{Hess}(\rho)(\xi, \eta) + f''(\rho)d\rho(\xi) \otimes d\rho(\eta) .$$

Let $\rho = -r$. Then

$$\mathrm{Hess}(-\log|\rho|)(\xi, \eta) = \frac{1}{|\rho|}\,\mathrm{Hess}(-|\rho|)(\xi, \eta) + \frac{1}{\rho^2}\,d\rho(\xi) \otimes d\rho(\eta) . \tag{2.16}$$

Using the same notation as in the proof of Theorem 2.1, by (2.3) and (2.16) we already have

$$\mathcal{L}(-\log|\rho|)(\tau, \bar{\tau}) \geq \|\tau\|^2 \;\;,\;\; \tau \in T^{(1,0)}(b\Omega_{(-\varepsilon)}) , \tag{2.17}$$

for $0 < \varepsilon < t_0$, where $[d\rho \otimes d\rho](\tau, \bar{\tau}) = 0$.

Let us now handle the remaining case of $\tilde{\zeta} \in [T^{(1,0)}(M) - T^{(1,0)}(b\Omega_{(-\varepsilon)})]$.

Suppose that $\sigma : [0, r_0] \to M$ be the length-minimizing geodesic of unit speed from $b\Omega$ to x, where $r_0 = d(x, b\Omega_0)$. For each $\tilde{\zeta} \in T_x^{(1,0)}(M)$ and a complex number $\lambda \in \mathbb{C}$, we have

$$i\,\partial\overline{\partial}(-\log r)(\lambda\tilde{\zeta}, \overline{\lambda\tilde{\zeta}}) = |\lambda|^2 i\,\partial\overline{\partial}(-\log r)(\tilde{\zeta}, \overline{\tilde{\zeta}}) .$$

Replacing $\tilde{\zeta}$ by $\lambda\tilde{\zeta}$ if needed, we may assume that $\tilde{\zeta} = (1/\sqrt{2})\,(\zeta - \sqrt{-1}\,J\zeta)$ with

$$\zeta = aV_{n-1} + b\sigma'(r_0) , \tag{2.18}$$

where $\{a, b\} \in \mathbb{R}$ are real numbers, we also require

$$V_{n-1} \in [T_x(M)]_{\mathbb{R}} \;\;,\;\; |V_{n-1}| = 1 \;\text{ and }\; V_{n-1} \perp \{\sigma'(r_0), J\sigma'(r_0)\} .$$

For each real vector $V \in [T_{\sigma(r_0)}(M)]_{\mathbb{R}}$, there is a unique parallel vector along σ $\{E_V(t)\}$ with $E_V(r_0) = V$.

When $\zeta = \zeta(r_0) = aV_{n-1} + b\sigma'(r_0)$, we consider the vector field

$$\zeta(t) = aE_{V_{n-1}}(t) + b\,\frac{t}{r_0}\,\sigma'(t) , \tag{2.19}$$

and a family of curves $\{F(s, .)\}$ given by

$$F(s, t) = \mathrm{Exp}_{\sigma(t)}[s\zeta(t)] . \tag{2.20}$$

For sufficiently small $|s| \leq \varepsilon$, each curve $F(s, .) : \mathbb{R} \to M$ must pass $b\Omega$ at time t_s near 0, where $t_s \to 0$ as $s \to 0$. Finally, we let

$$\ell_\zeta(s) = \int_{t_s}^{r_0} \|\frac{\partial F}{\partial t}(s, t)\|\, dt .$$

It is clear that $\ell_\zeta(s) \geq d(\mathrm{Exp}_x(s\zeta), b\Omega)$. By Fact 2.6, we have

$$\mathrm{Hess}(-r)(\zeta,\zeta)|_x \geq \frac{-d^2[\ell_\zeta(s)]}{ds^2}(0) \ . \tag{2.21}$$

It follows from (1.14) of [3], p.20 that

$$\begin{aligned} \frac{d^2[\ell_\zeta(s)]}{ds^2}(0) = \int_0^{r_0} \left\{ \left|\frac{b}{r_0}\right|^2 - \langle R(\sigma',\zeta)\sigma',\zeta\rangle - \left|\frac{b}{r_0}\right|^2 \right\} dt- \\ -\mathrm{Hess}(\rho)(\zeta(0),\zeta(0)) = \\ = -\left\{ \int_0^{r_0} a^2 \langle R(\sigma', E_{V_{n-1}})\sigma', E_{V_{n-1}}\rangle \, dt + \mathrm{Hess}(\rho)(\zeta(0),\zeta(0)) \right\} . \end{aligned} \tag{2.22}$$

By (2.21)-(2.22), we conclude that

$$\begin{aligned} \mathrm{Hess}(-r)(\zeta,\zeta)|_x \geq \\ \geq \int_0^{r_0} a^2 \langle R(\sigma', E_{V_{n-1}})\sigma', E_{V_{n-1}}\rangle \, dt + \mathrm{Hess}(\rho)(\zeta(0),\zeta(0)) \ . \end{aligned} \tag{2.23}$$

By considering the vector field $\{J\zeta(t)\}$ instead, we have $J\zeta(t) \perp \sigma'(t)$ for all t. The formula (1.14) of [3], p. 20 will also give

$$\begin{aligned} \frac{d^2[\ell_{J\zeta}(s)]}{ds^2}(0) &= \int_0^{r_0} \left\{ \left|\frac{b}{r_0}\right|^2 - \langle R(\sigma',J\zeta)\sigma',J\zeta\rangle \right\} dt - \mathrm{Hess}(\rho)(J\zeta(0),J\zeta(0)) = \\ &= -\left\{ \left[-\left(\frac{b^2}{r_0}\right) + \mathrm{Hess}(\rho)(J\zeta(0),J\zeta(0)) \right] + \right. \\ &\quad \left. + \int_0^{r_0} [a^2 \langle R(\sigma', E_{JV_{n-1}})\sigma', E_{JV_{n-1}}\rangle + b^2 \langle R(\sigma', J\sigma')\sigma', J\sigma'\rangle] \, dt \right\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hess}(-r)(J\zeta,J\zeta)|_x \geq \left\{ \left[-\frac{b^2}{r_0} + \mathrm{Hess}(\rho)(J\zeta(0),J\zeta(0)) \right] + \right. \\ \left. + \int_0^{r_0} [a^2 \langle R(\sigma', E_{JV_{n-1}})\sigma', E_{JV_{n-1}}\rangle + b^2 \langle R(\sigma', J\sigma')\sigma', J\sigma'\rangle] \, dt \right\} . \end{aligned} \tag{2.24}$$

Using our assumptions that $b\Omega$ is pseudo-convex and bisectional curvatures ≥ 1, adding (2.23) and (2.24) together, we conclude that that

$$\begin{aligned} i\,\partial\overline{\partial}(-r)(\tilde{\zeta},\overline{\tilde{\zeta}})\,|_x &= \mathrm{Hess}(-r)(\zeta,\zeta)|_x + \mathrm{Hess}(-r)(J\zeta,J\zeta)|_x \geq \\ &\geq -\frac{b^2}{r_0} + (a^2+b^2)r_0 \ . \end{aligned} \tag{2.25}$$

It is easy to see that

$$[d\rho \otimes d\rho](\zeta,\zeta) = b^2 \ . \tag{2.26}$$

By (2.16) and (2.25)-(2.26), we arrive at

$$i\,\partial\overline{\partial}(-\log r)(\tilde{\zeta},\overline{\tilde{\zeta}})\,|_x \geq \left[-\frac{b^2}{r_0^2} + (a^2+b^2) + \frac{b^2}{r_0^2} \right] \geq a^2+b^2 \ . \tag{2.27}$$

This completes the proof of Theorem 2.5 away from the cut-locus. On the cut-locus, the distance function r is not C^2. However, it is well-known (see Proposition 2.3 above) that the cut-locus of $b\Omega$ has measure zero in Ω. Observe that, on the cut-locus, the function

$r(x) = d(x, b\Omega)$ remains to be continuous. By Fact 2.6, one can show that $[-\log r]$ satisfies (2.27) for all $x \in \Omega$. □

2.3. **Proof of Theorem A.** Theorem A is a direct consequence of Theorem 2.1 and Theorem 2.5. We remark that, using methods in our proof of Theorem 2.5, one can derive a new proof of the classical Oka Lemma. Theorem A has many applications (see e.g. [14], [15, 16] and [2]).

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