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A new homogeneous tube domain

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Abstract[1]. Some classical homogeneous tube domains are discussed. These are compared with a new four-dimensional homogeneous tube domain, taken from a recent classification result obtained with Vladimir Ezhov and Alexander Isaev.

This article is based on a talk "Affine and CR Geometry in Examples of Complex Homogeneous Domains" presented at "CR Geometry and Partial Differential Equations", a meeting of Centro Internazionale per la Ricerca Matematica (CIRM), in Levico Terme (Trento) on 16th September 2004. I would like to thank the organisers of the meeting, Marco Peloso, Dmitri Zaitsev, and Giuseppe Zampieri, for their invitation to speak and hospitality.

We shall discuss the tube domain

$$D \equiv \{z = x + iy \in \mathbb{C}^4 \text{ such that } x_1x_2 + {x_1}^2x_3 > {x_4}^2 \text{ and } x_1 > 0\} , \tag{1}$$

putting it into context by comparing it with other, more familiar, tube domains. A major feature of D is that it is homogeneous, i.e. its holomorphic automorphism group acts transitively. This domain arises in recent joint work with Vladimir Ezhov and Alexander Isaev [1], where it occurs as part of a classification. Rather than discuss this classification, here we shall merely consider some key features of D. Detailed proofs of these features may be found in [1]. The corresponding features of various other tube domains are either well-known or may be found in [2].

Let us start with some tube domains in $\mathbb{C}^3$. The unit ball

$$B \equiv \{z \in \mathbb{C}^3 \text{ such that } |z_1|^2 + |z_2|^2 + |z_3|^2 < 1\} \tag{2}$$

is a very well known homogeneous domain. An initial change of coordinates

$$z_1 \mapsto \frac{z_1 - 1}{z_1 + 1} \qquad z_2 \mapsto 2\frac{z_2}{z_1 + 1} \qquad z_3 \mapsto 2\frac{z_3}{z_1 + 1} \tag{3}$$

throws B into the form:

$$B = \{z \in \mathbb{C}^3 \text{ such that } x_1 > |z_2|^2 + |z_3|^2\} ,$$

whereupon the further change

$$z_1 \mapsto 2z_1 - {z_2}^2 - {z_3}^2 \qquad z_2 \mapsto z_2 \qquad z_3 \mapsto z_3$$

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throws B into tube form:

$$B = \{z \in \mathbb{C}^3 \text{ such that } x_1 > {x_2}^2 + {x_3}^2\}\ . \tag{4}$$

The holomorphic automorphism group of B is most easily realised in its original coordinates (2) by viewing $\mathbb{C}^3$ as an affine coordinate patch in $\mathbb{CP}_3$, whence

$$B = \{[Z_1, Z_2, Z_3, Z_\infty] \in \mathbb{CP}_3 \text{ such that } |Z_1|^2 + |Z_2|^2 + |Z_3|^2 < |Z_\infty|^2\}\ .$$

The evident action of $\mathrm{SU}(3,1)$ on $\mathbb{CP}_3$ has B as one of its orbits. The normal subgroup generated by $i \times \mathrm{Id}$ acts trivially and the quotient is the holomorphic automorphism group. In particular, the Lie algebra of holomorphic vector fields on B is $\mathfrak{su}(3,1)$.

The other open orbit for the action of $\mathrm{SU}(3,1)$ on $\mathbb{CP}_3$ is

$$\{[Z_1, Z_2, Z_3, Z_\infty] \in \mathbb{CP}_3 \text{ such that } |Z_1|^2 + |Z_2|^2 + |Z_3|^2 > |Z_\infty|^2\}\ .$$

But if we attempt to view this in $\mathbb{C}^3$ using the same coordinate changes, then we obtain not only

$$\{z \in \mathbb{C}^3 \text{ such that } x_1 < {x_2}^2 + {x_3}^2\}\ , \tag{5}$$

but also some points at infinity. The point is that the rational expressions (3) have singularities outside the unit ball. Nevertheless, the infinitesimal automorphisms yield $\mathfrak{su}(3,1)$, as before, and the only difference is that not all holomorphic vector fields integrate to genuine automorphisms. Those that do comprise a certain parabolic subalgebra and these are responsible for all the genuine holomorphic automorphisms. They are sufficiently plentiful that (5) is a homogeneous tube. As a final variant of this construction,

$$\{z \in \mathbb{C}^3 \text{ such that } x_1 > x_2 x_3\} \tag{6}$$

is a well-known homogeneous tube with holomorphic symmetries $\mathfrak{su}(2,2)$ and whose symmetry group is 4-fold covered by a parabolic subgroup of $\mathrm{SU}(2,2)$.

Directly to verify that the tubes (4), (5), and (6) are homogeneous is straightforward by infinitesimal means, as follows. The three affine vector fields on $\mathbb{R}^3$

$$2x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} \qquad 2x_2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \qquad 2x_3\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$$

are linearly independent except where

$$\begin{vmatrix} 2x_1 & x_2 & x_3 \\ 2x_2 & 1 & 0 \\ 2x_3 & 0 & 1 \end{vmatrix} = 2(x_1 - {x_2}^2 - {x_3}^2)$$

vanishes. Therefore, the corresponding holomorphic affine vector fields

$$2z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} \qquad 2z_2\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \qquad 2z_3\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_3}$$

together with the three basic vector fields in the imaginary directions

$$i\frac{\partial}{\partial z_1} \qquad i\frac{\partial}{\partial z_2} \qquad i\frac{\partial}{\partial z_3}$$

are everywhere linearly independent on the domains (4) and (5). Moreover, these six vector fields are everywhere tangent along the common boundary of (4) and (5).

Integrating these fields gives a transitive holomorphic symmetry group. A similar result holds for (6) starting with the affine fields

$$2x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}\qquad x_3\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\qquad x_2\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3}\ .$$

This reasoning easily extends to show that a tube with affine homogeneous base is always holomorphically homogeneous. For example, the domain

$$\{z\in\mathbb{C}^3 \text{ such that } x_1>x_2x_3+{x_2}^3\}$$

is homogeneous because the affine fields

$$3x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+2x_3\frac{\partial}{\partial x_3}\qquad x_3\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}-3x_2\frac{\partial}{\partial x_3}\qquad x_2\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3}$$

are linearly independent except where $x_1=x_2x_3+{x_2}^3$ and each field is tangent along this boundary. Holomorphically, however, this domain is no different from (6). The change of coordinates

$$z_1\mapsto z_1-\frac{1}{2}{z_2}^3\qquad z_2\mapsto z_2\qquad z_3\mapsto z_3-\frac{3}{2}{z_2}^2$$

effects this isomorphism.

Now let us discuss the tube

$$C\equiv\{z\in\mathbb{C}^3 \text{ such that } {x_1}^2>{x_2}^2+{x_3}^2 \text{ and } x_1>0\}\tag{7}$$

and its variants. As announced by Wilhelm Kaup at the CIRM meeting, this is the tube over the 'ice-cream cone that one buys in the street' and verifying that it is homogeneous is a 'task for babies'. Arguing as above, it is necessary only to write down sufficiently many affine symmetries. Indeed,

$$\begin{gathered}z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}+z_3\frac{\partial}{\partial z_3}\\ z_1\frac{\partial}{\partial z_2}+z_2\frac{\partial}{\partial z_1}\qquad z_2\frac{\partial}{\partial z_3}-z_3\frac{\partial}{\partial z_2}\qquad z_3\frac{\partial}{\partial z_1}+z_1\frac{\partial}{\partial z_3}\\ i\frac{\partial}{\partial z_1}\qquad i\frac{\partial}{\partial z_2}\qquad i\frac{\partial}{\partial z_3}\end{gathered}\tag{8}$$

span the 7-real-dimensional complex affine symmetry algebra and all of them integrate to genuine symmetries. As in the case of the ball (2), there are additional holomorphic automorphisms besides affine transformations. They are most easily viewed by regarding $\mathbb{C}^3$ as an affine coordinate patch on the non-singular quadric $Q_3\subset\mathbb{CP}_4$. Then C is an open orbit of the evident action of $\mathrm{SO}(3,2)$ on Q_3 and its holomorphic automorphism group is a finite-to-one quotient thereof. This example is also the prototype in [3] where it is shown that all local CR automorphisms of the boundary

$$\{z\in\mathbb{C}^3 \text{ such that } {x_1}^2={x_2}^2+{x_3}^2 \text{ and } x_1>0\}$$

also arise through the action of SO(3,2). On C, the infinitesimal action of $\mathfrak{so}(3,2)$ yields three additional holomorphic symmetries:

$$i({z_1}^2+{z_2}^2+{z_3}^2)\frac{\partial}{\partial z_1}+2iz_1z_2\frac{\partial}{\partial z_2}+2iz_1z_3\frac{\partial}{\partial z_3}$$

$$2iz_1z_2\frac{\partial}{\partial z_1}+i({z_1}^2+{z_2}^2-{z_3}^2)\frac{\partial}{\partial z_2}+2iz_2z_3\frac{\partial}{\partial z_3}$$

$$2iz_1z_3\frac{\partial}{\partial z_1}+2iz_2z_3\frac{\partial}{\partial z_2}+i({z_1}^2-{z_2}^2+{z_3}^2)\frac{\partial}{\partial z_3}$$

each of which integrates to a 1-parameter subgroup of symmetries. The first one, for example, is the infinitesimal generator of

$$\begin{aligned} z_1 &\mapsto \frac{z_1+it({z_1}^2-{z_2}^2-{z_3}^2)}{1+2itz_1-t^2({z_1}^2-{z_2}^2-{z_3}^2)}\,, \\ z_2 &\mapsto \frac{z_2}{1+2itz_1-t^2({z_1}^2-{z_2}^2-{z_3}^2)}\,, \\ z_3 &\mapsto \frac{z_3}{1+2itz_1-t^2({z_1}^2-{z_2}^2-{z_3}^2)}\,. \end{aligned}$$

Notice that the denominator of these rational expressions can never vanish:

$$\begin{aligned} 2tx_1-2t^2(x_1y_1-x_2y_2-x_3y_3) &= 0 \\ 1-2ty_1-t^2({x_1}^2-{x_2}^2-{x_3}^2-{y_1}^2+{y_2}^2+{y_3}^2) &= 0 \end{aligned}$$

implies

$$\begin{aligned} 0 &= (x_1y_1-x_2y_2-x_3y_3)^2-2x_1y_1(x_1y_1-x_2y_2-x_3y_3)- \\ &\quad -{x_1}^2({x_1}^2-{x_2}^2-{x_3}^2-{y_1}^2+{y_2}^2+{y_3}^2)= \\ &= -({x_1}^2+{y_2}^2+{y_3}^2)({x_1}^2-{x_2}^2-{x_3}^2)-(x_2y_3-x_3y_2)^2, \end{aligned}$$

which is clearly impossible if ${x_1}^2>{x_2}^2+{x_3}^2$. This observation is false for the tube over the space-like cone

$$\{x\in\mathbb{R}^3 \text{ such that } {x_1}^2<{x_2}^2+{x_3}^2\}\,. \tag{9}$$

It has the same infinitesimal symmetries (isomorphic to $\mathfrak{so}(3,2)$) but only vector fields from the 7-dimensional parabolic subalgebra generated by (8) integrate to genuine holomorphic, indeed polynomial, automorphisms.

To obtain more interesting homogeneous tube domains, it appears we must look in $\mathbb{C}^4$. There are evident generalisations of the domains we have discussed so far but others besides.

The homogeneous domain known as the 'nil-ball'

$$N\equiv\{z\in\mathbb{C}^4 \text{ such that } x_1>z_2\overline{z_3}+z_3\overline{z_2}+|z_4|^2+|z_2|^4\} \tag{10}$$

is due to Penney [4, 5]. As observed in [2], this domain is biholomorphically equivalent to a tube domain:

$$\{z\in\mathbb{C}^4 \text{ such that } x_1>x_2x_3+{x_4}^2+{x_2}^2x_4+\frac{1}{8}{x_2}^4\}\,.$$

Specifically, the holomorphic change of coordinates

$$z_1 \mapsto 4z_1 - 2z_2z_3 - 2{z_4}^2 - {z_2}^2z_4 - \frac{1}{16}\,{z_2}^4 \qquad z_2 \mapsto \frac{1}{2}\,z_2$$

$$z_3 \mapsto 2z_3 + 2z_2z_4 + \frac{1}{4}\,{z_2}^3 \qquad z_4 \mapsto \sqrt{2}\left(z_4 + \frac{1}{4}\,{z_2}^2\right)$$

transforms N into this tube.

In tube form, homogeneity is easily verified. Simply, the real affine vector fields

$$4x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 3x_3\frac{\partial}{\partial x_3} + 2x_4\frac{\partial}{\partial x_4} \qquad 2x_3\frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_2} - 2x_4\frac{\partial}{\partial x_3} - x_2\frac{\partial}{\partial x_4}$$

$$x_2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} \qquad 2x_4\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}$$

are tangent along the boundary and linearly independent except where

$$\begin{vmatrix} 4x_1 & x_2 & 3x_3 & 2x_4 \\ 2x_3 & 2 & -2x_4 & -x_2 \\ x_2 & 0 & 1 & 0 \\ 2x_4 & 0 & -x_2 & 1 \end{vmatrix} = 8(x_1 - x_2x_3 - {x_4}^2 - {x_2}^2x_4 - \frac{1}{8}\,{x_2}^4)$$

vanishes. The full group of complex affine symmetries of N only picks up translations in the imaginary directions and so weighs in at 8-dimensional. The full group of holomorphic automorphisms is 13-dimensional. In both the original coordinates (10) or in tube coordinates, all of these automorphisms are actually polynomial. In fact, in the original coordinates (10) it is easy to check that the 5-dimensional subgroup of linear transformations

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\phi} & 0 & 0 \\ 0 & b & e^{i\phi} & d \\ 0 & -\bar{d}e^{i\phi}e^{i\psi} & 0 & e^{i\psi} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \quad \begin{array}{l} \text{where,} \\ \phi, \psi \in \mathbb{R},\ b, d \in \mathbb{C} \\ \text{and } |d|^2 = -2\mathrm{Re}\,(e^{i\phi}\bar{b})\ , \end{array}$$

preserves N and accounts for the extra symmetries beyond the affine ones in tube coordinates. In [5, p.629] it is shown that the full automorphism group is solvable with a nilpotent subgroup acting simply transitively on the boundary.

The tube domain D defined by (1) is affine homogeneous. The real affine symmetry algebra of the base is 4-dimensional, generated by

$$x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_4} \qquad 2x_2\frac{\partial}{\partial x_2} + 2x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4}$$

$$x_1\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} \qquad 2x_4\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_4}$$

and, like the nil-ball N, with the imaginary translations, the tube domain D enjoys an 8-dimensional complex affine automorphism group. It is shown in [1] that the full holomorphic automorphism group is 10-dimensional. Indeed, the extra 2-dimensions may be seen via a rather curious subgroup:

$$\begin{aligned} z_1 &\mapsto z_1 \\ z_2 &\mapsto z_2 + 2v^2z_1 + i(2v+u){z_1}^2 - 4ivz_1z_4 - 2v^2{z_1}^3 + 2iv - iu \\ z_3 &\mapsto z_3 - i(2u+4v)z_1 + 4ivz_4 + 2v^2{z_1}^2 - 2v^2 + 2iu \\ z_4 &\mapsto z_4 - iv{z_1}^2 + iv\ , \end{aligned}$$

where $u, v \in \mathbb{R}$. As one might suspect from these strange formulae, the algebraic structure of the holomorphic automorphism group of D is obscure.

Finally, let us compare D with these other homogeneous tubes. It is quite different from the ball in $\mathbb{C}^4$, which has a much larger holomorphic symmetry group of dimension 25. The infinitesimal symmetry algebra of this ball is $\mathfrak{su}(4,1)$. One feature they do share is that all infinitesimal symmetries integrate to genuine automorphisms. However, some automorphisms of the ball are necessarily given by rational transformations. All automorphisms of D are polynomial.

The tube D has a little more in common with C, the tube defined by (7) over the ice-cream cone. The domains

$$\{x \in \mathbb{R}^3 \text{ such that } {x_1}^2 > {x_2}^2 + {x_3}^2\} \quad \text{and} \quad \{x \in \mathbb{R}^4 \text{ such that } x_1 x_2 + {x_1}^2 x_3 > {x_4}^2\}$$

are both disconnected, the inequality $x_1 > 0$ serving to pick out a connected component. In addition, both have a singular boundary. The ice-cream cone has a singularity at the origin. The boundary of the base of D is singular along the line

$$\{x_1 = x_2 = x_4 = 0\}\ .$$

Like the ball, the domain C is pseudo-convex. So, perhaps D compares better with the tube over the space-like cone

$$\{x \in \mathbb{R}^4 \text{ such that } {x_1}^2 < {x_2}^2 + {x_3}^2 + {x_4}^2\}\ .$$

However, there are no interesting automorphisms of this tube: the symmetry algebra is isomorphic to $\mathfrak{so}(4,2)$ but only an 11-dimensional subalgebra of this 15-dimensional Lie algebra integrates to genuine automorphisms and they are all affine.

Certainly D has most in common with N. Away from its singular locus ∂D is Levi-indefinite. The nil-ball N has smooth Levi-indefinite boundary. This common feature implies that holomorphic automorphisms extend past the boundary. In both cases, all infinitesimal symmetries integrate to genuine automorphisms. Of course, N is more symmetrical, having an automorphism group of dimension 13 as compared to only 10 for D. The remaining puzzle regarding D is to elucidate the algebraic structure of its automorphism group. Certainly, one can write the general automorphism in coordinates or present the symmetry algebra as generators and relations. This is done in [1]. Ideally, however, one seeks a matrix representation of this group and some closer link to classical homogeneous spaces.

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