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Intrinsic submanifolds, graphs and currents in Heisenberg groups

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Abstract[1]. We describe intrinsically regular submanifolds in Heisenberg groups $\mathbb{H}^n$. Low dimensional and low codimensional submanifolds turn out to be of a very different nature. The first ones are Legendrian surfaces, while low codimensional ones are more general objects, possibly non Euclidean rectifiable. Nevertheless we prove that they are graphs in a natural group way (that is studied here, with a special attention to the notion of intrinsic Lipschitz graph), as well as that an area formula holds for the intrinsic Haudorff measure. Finally, they can be seen as Federer-Fleming currents given a natural complex of differential forms on $\mathbb{H}^n$.

1. Notations

The aim of the present Note[2] is to develop an algebraic and metric theory of intrinsic submanifolds of the Heisenberg groups $\mathbb{H}^n$. Most of the results announced here are fully presented in [10].

Let $\mathbb{H}^n$ be the n-dimensional Heisenberg group. From now on we identify $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$ choosing exponential coordinates. Accordingly, a point $p \in \mathbb{H}^n$ is denoted $p = (p_1, \dots, p_{2n}, p_{2n+1}) = (p', p_{2n+1})$, with $p' \in \mathbb{R}^{2n}$ and $p_{2n+1} \in \mathbb{R}$. If p and $q \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot q = (p' + q', p_{2n+1} + q_{2n+1} + 2\langle Jp', q'\rangle_{\mathbb{R}^{2n}})$$

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is the $2n \times 2n$ symplectic matrix.
We denote as $p^{-1} := (-p', -p_{2n+1})$ the inverse of p and as 0 the identity of $\mathbb{H}^n$.

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For any fixed $q \in \mathbb{H}^n$ and for any $r > 0$ left translations τ_q and non isotropic dilations δ_r are automorphisms of the group defined as

$$\begin{aligned} \tau_q : \quad & \mathbb{H}^n \to \mathbb{H}^n \quad , \qquad p \mapsto \tau_q(p) := q \cdot p \\ \delta_r : \quad & \mathbb{H}^n \to \mathbb{H}^n \quad , \qquad p \mapsto \delta_r p := (rp', r^2 p_{2n+1}) \ . \end{aligned}$$

For a general review on Heisenberg groups and their properties we refer to [22], [12] and to [23].

We denote as $\mathfrak{h}^n$ or, more frequently, as $\mathfrak{h}$ when the dimension n is intended, the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given by

$$\begin{aligned} X_i &= \partial_i + 2(Jp')_i \partial_{2n+1} \ , \\ Y_i &= \partial_{i+n} + 2(Jp')_{i+n} \partial_{2n+1} \ , \\ T &= \partial_{2n+1} \ . \end{aligned}$$

for $i = 1, \dots, n$. The only non-trivial commutation relations among them are

$$[X_j, Y_j] = -4T \quad , \quad \text{for } j = 1, \dots, n \ . \tag{1}$$

Sometimes we will shift notations putting

$$W_i := X_i \quad , \quad W_{i+n} := Y_i \quad , \quad W_{2n+1} := T \quad , \quad \text{for } i = 1, \cdots, n \ .$$

The *horizontal subspace* $\mathfrak{h}_1$ is the vector subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \ . \tag{2}$$

We need a notion of an intrinsic distance on $\mathbb{H}^n$ (the so called Carnot-Carathéodory distance). There are many equivalent definitions of it, one of the possible ones is the following:
An absolutely continuous curve $\gamma : [0, T] \to \mathbb{H}^n$ is a *subunit curve* with respect to $X_1, \dots, X_n, Y_1, \dots, Y_n$ if there are real measurable functions $c_1, \dots, c_{2n}$, defined in $[0, T]$, such that, for a.e. $s \in [0, T]$, $\sum_j c_j^2(s) \le 1$ and

$$\dot\gamma(s) = \sum_{j=1}^{n} c_j(s) X_j(\gamma(s)) + c_{j+n}(s) Y_j(\gamma(s)) \ .$$

Definition 1.1. If $p, q \in \mathbb{H}^n$, their cc-distance $d_c(p, q)$ is

$$d_c(p, q) \stackrel{\text{def}}{=} \inf \ \{T > 0 : \ \gamma \text{ is subunit} \ , \ \gamma(0) = p \ , \ \gamma(T) = q\} \ .$$

The set of subunit curves joining p and q is not empty, by Chow's theorem.

We shall denote respectively by $U_c(p, r)$ and by $B_c(p, r)$ the open and the closed ball associated with d_c.

We remind that for any compact set $K \subset \mathbb{H}^n$ there exist positive constants c_K and C_K such that

$$c_K |p - q|_{\mathbb{R}^{2n+1}} \le d_c(p, q) \le C_K |p - q|_{\mathbb{R}^{2n+1}} \ . \tag{3}$$

Several distances *equivalent* to d_c have been used in the literature. Later on, we shall use the following one, that can also be computed explicitly

$$d_\infty(p, q) = d_\infty(q^{-1} \cdot p, 0) \ ,$$

where, if $p = (p', p_{2n+1}) \in \mathbb{H}^n$, then

$$d_\infty(p, 0) = \max \{ |p'|_{\mathbb{R}^{2n}} \ , |p_{2n+1}|^{1/2} \} \ . \tag{4}$$

As above, $U_\infty(p, r)$ and $B_\infty(p, r)$ are the open and closed balls associated with d_∞.

Both the cc-metric d_c and the metric d_∞ are well behaved with respect to left translations and dilations, that is

$$(5)\qquad \begin{array}{ccc} d_c(z\cdot x, z\cdot y) = d_c(x,y) & , & d_c(\delta_\lambda x, \delta_\lambda y) = \lambda d_c(x,y) \\ d_\infty(z\cdot x, z\cdot y) = d_\infty(x,y) & , & d_\infty(\delta_\lambda x, \delta_\lambda y) = \lambda d_\infty(x,y) \end{array}$$

for $x, y, z \in \mathbb{H}^n$ and $\lambda > 0$.

We recall that, because the topologies induced by d_c, d_∞ and by the Euclidean distance coincide, the topological dimension of $\mathbb{H}^n$ is $2n+1$. On the contrary the Hausdorff dimension of $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$, with respect to the cc-distance d_c or with respect to any other equivalent distance, is the integer

$$(6)\qquad Q = 2n + 2$$

usually called the *homogeneous dimension* of $\mathbb{H}^n$ (see [18]).

For a nonnegative integer k, $\mathcal{L}^k$ denotes the k-dimensional Lebesgue measure. $\mathcal{L}^{2n+1}$ is the bi-invariant Haar measure of $\mathbb{H}^n$. That is, if $E \subset \mathbb{R}^{2n+1}$ is measurable, then $\mathcal{L}^{2n+1}(\tau_p(E)) = \mathcal{L}^{2n+1}(E)$ for all $p \in \mathbb{H}^n$. Moreover, if $\lambda > 0$ then $\mathcal{L}^{2n+1}(\delta_\lambda(E)) = \lambda^{2n+2}\mathcal{L}^{2n+1}(E)$. We explicitly observe that, $\forall p \in \mathbb{H}^n$ and $\forall r > 0$,

$$(7)\qquad \mathcal{L}^{2n+1}(B_c(p,r)) = r^{2n+2}\mathcal{L}^{2n+1}(B_c(p,1)) = r^{2n+2}\mathcal{L}^{2n+1}(B_c(0,1))$$

and as a consequence

$$(8)\qquad \mathcal{L}^{2n+1}(\partial B_c(p,r)) = 0 \text{ and } \mathcal{L}^{2n+1}(B_c(p,r)) = \mathcal{L}^{2n+1}(U_c(p,r))\ .$$

Analogously for the d_∞ distance.

Related with the previously defined distances d_c and d_∞, different Hausdorff measures, obtained following Carathédory's construction as in [6] Section 2.10.2, are used in this paper. For $m \geq 0$, we denote by $\mathcal{H}_E^m$ the m-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$ and by $\mathcal{H}_c^m$ and $\mathcal{H}_\infty^m$ the m-dimensional Hausdorff measures in $\mathbb{H}^n$, obtained, respectively, from the distances d_c and d_∞. Analogously, $\mathcal{S}_E^m$, $\mathcal{S}_c^m$, and $\mathcal{S}_\infty^m$ denote the corresponding spherical Hausdorff measures. We have to be more precise about the constants appearing in the various definitions. Since explicit computations will be carried out only for the measures $\mathcal{S}_\infty^m$, with m a positive integer, we limit ourselves to this case. For each $A \subset \mathbb{H}^n$ and $\delta > 0$, $\mathcal{S}_\infty^m(A) := \lim_{\delta\to 0}\mathcal{S}_{\infty,\delta}^m(A)$, where

$$\mathcal{S}_{\infty,\delta}^m(A) = \inf\left\{\sum_i \zeta(B_\infty(p_i,r_i)) :\ A \subset \bigcup_i B_\infty(p_i,r_i) \text{ and } r_i \leq \delta\right\}$$

and the evaluation function ζ is

$$(9)\qquad \zeta(B_\infty(p,r)) := \begin{cases} \omega_m r^m & \text{if } 1 < m \leq n\ , \\ 2\omega_{m-1} r^m & \text{if } m = n+1\ , \\ 2\omega_{m-2} r^m & \text{if } n+2 \leq m \end{cases}$$

where ω_m is m-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^m$. Translation invariance and homogeneity under dilations of Hausdorff measures follow as usual from (5).

Finally, we recall the definition of *Heisenberg tangent cone* to a set A in a point p.

Definition 1.2. Let $A \subset \mathbb{H}^n$. The *Heisenberg tangent cone* to A in 0 is the set

$$\mathrm{Tan}_{\mathbb{H}}(A,0) \stackrel{\text{def}}{=} \left\{x = \lim_{h\to+\infty} \delta_{r_h} x_h \in \mathbb{H}^n\ ,\ \text{with } r_h \to +\infty \text{ and } x_h \in A\right\}$$

and the cone in a point p is given as

$$\mathrm{Tan}_{\mathbb{H}}(A,p) \stackrel{\text{def}}{=} \tau_p \mathrm{Tan}_{\mathbb{H}}(\tau_{-p}A, 0)\ .$$

2. Regular submanifolds

Let us start with some comments about possible notions of regular submanifolds of a group.

Considering Euclidean regular submanifolds of $\mathbb{H}^n$, identified with the Euclidean space $\mathbb{R}^{2n+1}$, it is never a satisfactory choice: indeed, Euclidean regular submanifolds need not to be group regular; this is absolutely obvious for low dimensional submanifolds: the 1-dimensional, group regular, objects are horizontal curves that are a small subclass of $\mathcal{C}^1$ lines, but, even for low codimensional surfaces, an Euclidean submanifold need not to be group regular due to the presence of the so called *characteristic points* where no intrinsic notion of tangent space to the surface exists (see [4], [14]). On the contrary in Carnot groups exist e.g. 1-codimensional surfaces, sometimes called $\mathbb{H}$-regular or $\mathbb{G}$-regular surfaces, that can be highly irregular as Euclidean objects but that enjoy very nice properties from the group point of view, so that it is very natural to think of them as 1-codimensional regular submanifolds of the group, (see [9], [8], [13]).

What do we mean by 'very nice properties'? The key words here are *intrinsic* and *regular*. First of all, 'intrinsic' will denote properties depending only on the Lie algebra $\mathfrak{h}$. On the other hand, the most natural requirements for a subset $S \subset \mathbb{H}^n$ to be considered as an intrinsic regular submanifold are

(i) S has, at each point, a tangent 'plane' and a normal 'plane';
(ii) tangent and normal 'planes' depend continuously on the point,

where the notions of tangent and normal 'planes' should be intrinsic to $\mathbb{H}^n$, i.e. depending only on the group structure and on the differential structure as given by the horizontal bundle. Since subgroups are the natural counterpart in groups of Euclidean planes through the origin, it seems accordingly natural to ask that

(iii) both the tangent 'plane ' and the normal 'plane' are subgroups (or better cosets of subgroups) of $\mathbb{H}^n$; $\mathbb{H}^n$ is the direct product of them (see later for a precise definition), they are orthogonal to each other in the sense induced by the inner product in $\mathfrak{h}$;
(iv) the tangent 'plane' to S in a point is the limit of group dilations of S centered in that point (see Definition 1.2).

Notice that the explicit requirement of the existence of both a tangent space and a normal space is not pointless. Indeed there are subgroups in $\mathbb{H}^n$, as the T axis for example, without a (normal) complementary subgroup.

We notice also that condition (iv) guarantees that the tangent plane has the natural geometric meaning of 'surface seen at infinite scale', the scale however being meant with respect to intrinsic dilations. This yields that - if S is both an Euclidean smooth manifold and a group regular manifold - the intrinsic tangent plane is usually different from the Euclidean one. On the other hand, as already pointed out, there are sets, 'bad' from the Euclidean point of view, that behave as regular sets with respect to group dilations.

We consider the vector spaces $\mathfrak{h}$ and $\mathfrak{h}_1$,

$$\mathfrak{h} = \text{span}\,\{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}$$

$$\mathfrak{h}_1 = \text{span}\,\{X_1, \dots, X_n, Y_1, \dots, Y_n\}\ ,$$

endowed with the inner product $\langle \cdot, \cdot \rangle$ making the vectors $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ and T orthonormal.

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $X_1, \cdots, Y_n$, T of $\mathfrak{h}$, is the family of covectors $\{dx_1, \cdots, dx_{2n}, \theta\}$ where $\theta := dx_{2n+1} - 2\langle (Jx'), dx' \rangle_{\mathbb{R}^{2n}}$ is the so called *contact form* in $\mathbb{H}^n$. We indicate as $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $dx_1, \cdots, dx_{2n}, \theta$ an orthonormal basis. Sometimes it will be notationally convenient to put $\theta_1 := dx_1, \cdots, \theta_{2n} := dx_{2n}, \theta_{2n+1} := \theta$.

Following Federer (see [6] 1.3), the exterior algebras of $\mathfrak{h}$ and of $\bigwedge^1 \mathfrak{h}$ are the graded algebras indicated, as usual, as $\bigwedge_* \mathfrak{h} = \bigoplus_{k=0}^{2n+1} \bigwedge_k \mathfrak{h}$ and $\bigwedge^* \mathfrak{h} = \bigoplus_{k=0}^{2n+1} \bigwedge^k \mathfrak{h}$ where $\bigwedge_0 \mathfrak{h} = \bigwedge^0 \mathfrak{h} = \mathbb{R}$ and, for $1 \le k \le 2n+1$,

$$\bigwedge_k \mathfrak{h} := \operatorname{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\},$$

$$\bigwedge^k \mathfrak{h} := \operatorname{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n+1\}.$$

The elements of $\bigwedge_k \mathfrak{h}$ and $\bigwedge^k \mathfrak{h}$ are called *k-vectors* and *k-covectors*. As usual, the dual space $\bigwedge^1(\bigwedge_k \mathfrak{h})$ of $\bigwedge_k \mathfrak{h}$ can be naturally identified with $\bigwedge^k \mathfrak{h}$. The action of a k-covector φ on a k-vector v is denoted as $\langle \varphi | v \rangle$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_k \mathfrak{h}$ and to $\bigwedge^k \mathfrak{h}$ making the bases $W_{i_1} \wedge \cdots \wedge W_{i_k}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ orthonormal.

The same construction can be performed starting from the vector subspace $\mathfrak{h}_1 \subset \mathfrak{h}$. This way we obtain the algebras $\bigwedge_* \mathfrak{h}_1 = \bigoplus_{k=1}^{2n} \bigwedge_k \mathfrak{h}_1$ and $\bigwedge^* \mathfrak{h}_1 = \bigoplus_{k=1}^{2n} \bigwedge^k \mathfrak{h}_1$ whose elements are the *horizontal k-vectors* and *horizontal k-covectors*; here

$$\bigwedge_k \mathfrak{h}_1 := \operatorname{span}\{W_{i_1} \wedge \cdots \wedge W_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n\}$$

$$\bigwedge^k \mathfrak{h}_1 := \operatorname{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} : 1 \le i_1 < \cdots < i_k \le 2n\}.$$

and clearly $\bigwedge_k \mathfrak{h}_1 \subset \bigwedge_k \mathfrak{h}$ for $1 \le k \le 2n$.

Definition 2.1. We define linear isomorphisms (see [6] 1.7.8)

$$* : \bigwedge_k \mathfrak{h} \longleftrightarrow \bigwedge_{2n+1-k} \mathfrak{h} \quad \text{and} \quad * : \bigwedge^k \mathfrak{h} \longleftrightarrow \bigwedge^{2n+1-k} \mathfrak{h},$$

for $1 \le k \le 2n$, putting, for $v = \sum_I v_I W_I$ and $\varphi = \sum_I \varphi_I \theta_I$,

$$*v := \sum_I v_I (*W_I) \quad \text{and} \quad *\,\varphi := \sum_I \varphi_I (*\theta_I)$$

where

$$*W_I := (-1)^{\sigma(I)} W_{I^*} \quad \text{and} \quad *\,\theta_I := (-1)^{\sigma(I)} \theta_{I^*}$$

with $I = \{i_1, \cdots, i_k\}$, $1 \le i_1 < \cdots < i_k \le 2n+1$, $W_I = W_{i_1} \wedge \cdots \wedge W_{i_k}$, $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$, $I^* = \{i_1^* < \cdots < i_{2n+1-k}^*\} = \{1, \cdots, 2n+1\} \setminus I$ and $\sigma(I)$ is the number of couples (i_h, i_ℓ^*) with $i_h > i_\ell^*$.

Notice that, if $v = v_1 \wedge \cdots \wedge v_k$ is a simple k-vector, then $*v$ is a simple $(2n+1-k)$-vector. Moreover notice that

$$\text{if } v \in \bigwedge_k \mathfrak{h}_1, \quad \text{then} \quad *\,v = \xi \wedge T, \quad \text{with } \xi \in \bigwedge_{2n-k} \mathfrak{h}_1. \tag{10}$$

If $v \in \bigwedge_k \mathfrak{h}$ we define $v^* \in \bigwedge^k \mathfrak{h}$ by the identity

$$\langle v^* | w \rangle \stackrel{\text{def}}{=} \langle v, w \rangle, \quad \forall\, w \in \bigwedge_k \mathfrak{h}. \tag{11}$$

Analogously we define $\varphi^* \in \bigwedge_k \mathfrak{h}$ for $\varphi \in \bigwedge^k \mathfrak{h}$.

The *symplectic two form* $d\theta \in \bigwedge^2 \mathfrak{h}_1$ is

$$d\theta = 4 \sum_{i=1}^{n} dx_i \wedge dx_{i+n}.$$

We define the vector spaces ${}_H\!\bigwedge_k$ and ${}_H\!\bigwedge^k$ of *integrable k-vectors and k-covectors* as follows

Definition 2.2. For $1 \le k \le n$, we set

$$ {}_H\bigwedge_k \stackrel{\text{def}}{=} \text{span} \left\{ v \in \bigwedge_k \mathfrak{h}_1 \ : \ v \text{ is simple and integrable} \right\} \ , $$

and, for $n+1 \le k \le 2n+1$,

$$ {}_H\bigwedge_k \stackrel{\text{def}}{=} * \left({}_H\bigwedge_{2n+1-k} \right) \ . $$

Integrable covectors are defined by duality: for $1 \le k \le 2n+1$ we set

$$ {}_H\bigwedge^k \stackrel{\text{def}}{=} \left\{ \varphi \in \bigwedge^k \mathfrak{h} \ : \quad \varphi^* \in {}_H\bigwedge_k \right\} \ . $$

We complete the definition putting ${}_H\bigwedge_0 = {}_H\bigwedge^0 = \mathbb{R}$.

Notice that ${}_H\bigwedge_1 = \bigwedge_1 \mathfrak{h}_1 = \mathfrak{h}_1$. On the contrary, for $1 < k \le n$, $0 \ne {}_H\bigwedge_k \subsetneq \bigwedge_k \mathfrak{h}_1$.

If $1 \le k \le n$ and if $w \in {}_H\bigwedge_{2n+1-k}$ is a simple $(2n+1-k)$-vector, then one can choose $w_1, \dots, w_{2n+1-k}$ so that: $w = w_1 \wedge \dots \wedge w_{2n+1-k}$, $w_1 \wedge \dots \wedge w_{2n-k} \in \bigwedge_{2n-k} \mathfrak{h}_1$ and $w_{2n+1-k} = T$.

We can give now the definition of $\mathbb{H}$-regular surfaces in the spirit illustrated above. To this end, we remind first the notion due to P. Pansu of differentiable map between Carnot groups.

Definition 2.3 (Pansu [19])**.** Let $(\mathbb{G}^1, \cdot)$ and $(\mathbb{G}^2, \cdot)$ be Carnot groups with dilation automorphisms δ^1_λ and δ^2_λ. Let $\mathcal{U}$ be an open subset of $\mathbb{G}^1$, and $f : \mathcal{U} \to \mathbb{G}^2$. We say that f is P-differentiable at $p_0 \in \mathcal{U}$ if there is a H-linear map $L : \mathbb{G}^1 \to \mathbb{G}^2$ such that

$$ \lim_{\lambda \to 0} \delta^2_{1/\lambda} \left(f(p_0)^{-1} \cdot f(p_0 \cdot \delta^1_\lambda p) \right) = L(p) $$

uniformly for p in compact subsets of $\mathcal{U}$. In particular, L is unique and we shall write $L := d_H f(p_0)$.

We distinguish here low dimensional and low codimensional surfaces, the first ones being images of open subset of Euclidean spaces while the second ones are level sets of intrinsically regular functions.

Definition 2.4. Let $1 \le k \le n$. A subset $S \subset \mathbb{H}^n$ is a *k-dimensional $\mathbb{H}$-regular surface* (or a $\mathcal{C}^1_{\mathbb{H}}$ surface of dimension k) if for any $p \in S$ there are open sets $\mathcal{U} \subset \mathbb{H}^n$, $\mathcal{V} \subset \mathbb{R}^k$ and a function $\varphi : \mathcal{V} \to \mathcal{U}$ such that $p \in \mathcal{U}$, φ is injective, φ is continuously Pansu differentiable from $\mathbb{R}^k$ to $\mathbb{H}^n$ with $d_H \phi$ injective, and

$$ S \cap \mathcal{U} = \varphi(\mathcal{V}) \ . $$

Definition 2.5. Let $1 \le k \le n$. A subset $S \subset \mathbb{H}^n$ is a *k-codimensional $\mathbb{H}$-regular surface* (or a $\mathcal{C}^1_{\mathbb{H}}$ surface of codimension k or a $\mathcal{C}^1_{\mathbb{H}}$ surface of topological dimension $(2n+1-k)$) if for any $p \in S$ there are an open set $\mathcal{U} \subset \mathbb{H}^n$ and a function $f : \mathcal{U} \to \mathbb{R}^k$ such that $p \in \mathcal{U}$, $f = (f_1, \dots, f_k) \in [\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$, $\nabla_H f_1 \wedge \dots \wedge \nabla_H f_k \ne 0$ in $\mathcal{U}$ (equivalently, $d_H f$ is onto) and

$$ S \cap \mathcal{U} = \{ q \in \mathcal{U} : f(q) = 0 \} \ . $$

Notice that, if $k = 1$, Definition 2.4 gives back the well known notion of horizontal, continuously differentiable, curve. On the other hand, Definition 2.4 cannot be extended to the case $k > n$. Indeed, for $k > n$, as proved in [1] (see also [15]), the set of maps φ satisfying the assumptions of Definition 2.4 is empty. Even more, they show $\mathbb{H}^n$ is *purely k-unrectifiable*, i.e., if $k > n$, for any $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^k, \mathbb{H}^n)$ we have $\mathcal{H}^k_c(f(A)) = 0$ for any $A \subset \mathbb{R}^k$.

In turn Definition 2.5, for $k = 1$, gives the notion of $\mathbb{H}$-regular hypersurface introduced in [7] and [8]. Definition 2.5 - unlike the previous one - could be formally extended to $k > n$, but we restrict ourselves to $1 \le k \le n$ because only in this situation it is possible to prove (see below) that a $\mathcal{C}^1_{\mathbb{H}}$ surface of codimension k is locally a graph in a consistent suitable sense.

The following theorem provides a description of the class of the k-dimensional $\mathbb{H}$-regular surfaces, when $1 \le k \le n$, showing that they are Euclidean submanifolds.

Theorem 2.6. *If S is a k-dimensional $\mathbb{H}$-regular surface, $1 \le k \le n$, then*

(1) *S is an Euclidean k-dimensional submanifold of $\mathbb{R}^{2n+1}$ of class $\mathcal{C}^1$.*
(2) *The Euclidean tangent bundle $\operatorname{Tan} S$ is a subbundle of $\bigwedge_k \mathfrak{h}_1$ and*

$$\operatorname{Tan}(S,p) = \operatorname{Tan}_{\mathbb{H}}(S,p)$$

for any point $p \in S$.

(3) $\mathcal{S}^k_\infty \llcorner S$ is comparable with $\mathcal{H}^k_E \llcorner S$.

In order to provide a full description of low-codimensional submanifolds, we must introduce here the notion of graph in a Lie group $\mathcal{G}$. To this end, assume the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is the direct sum of two subalgebras $\mathfrak{w}$ and $\mathfrak{v}$, that is

$$\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{v} \ .$$

Set now $\mathbb{G}_{\mathfrak{w}} := \exp \mathfrak{w}$, and $\mathbb{G}_{\mathfrak{v}} := \exp \mathfrak{v}$. We denote *system of coordinate planes* (i.e. left laterals) of $\mathbb{G}$ the double family $\mathcal{L}_{\mathfrak{v}}$ and $\mathcal{L}_{\mathfrak{w}}$ defined as

$$\begin{array}{lll} \mathcal{L}_{\mathfrak{v}}(p) & := p \cdot \mathbb{G}_{\mathfrak{v}} & \forall\, p \in \mathbb{G}_{\mathfrak{w}} \\ \mathcal{L}_{\mathfrak{w}}(q) & := q \cdot \mathbb{G}_{\mathfrak{w}} & \forall\, q \in \mathbb{G}_{\mathfrak{v}} \ . \end{array}$$

Observe that each $x \in \mathbb{G}$ belongs exactly to one leaf in $\mathcal{L}_{\mathfrak{v}}$ and to one in $\mathcal{L}_{\mathfrak{w}}$. Observe also that the leaves in $\mathcal{L}_{\mathfrak{v}}$ (or in $\mathcal{L}_{\mathfrak{w}}$) are invariant by translations, that is

$$x \in \mathcal{L}_{\mathfrak{v}}(p) \implies \tau_x \mathcal{L}_{\mathfrak{v}}(p) = \mathcal{L}_{\mathfrak{v}}(p) \ .$$

We propose the following definition of graphs in $\mathbb{G}$; this definition shares with the usual Euclidean notion many good features.

Definition 2.7. We say that a set $S \subset \mathbb{G}$ is a *graph along* $\mathbb{G}_{\mathfrak{v}}$ (or *along* $\mathfrak{v}$) if, for each $\xi \in \mathbb{G}_{\mathfrak{w}}$, $S \cap \mathcal{L}_{\mathfrak{v}}(\xi)$ contains at most one point. Equivalently if there is a function $\varphi : E \subset \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ such that

$$S = \{\xi \cdot \varphi(\xi) : \xi \in E\}$$

and we say that S *is the graph of* φ. The set $\mathbb{G}_{\mathfrak{w}}$ will be mentioned as the space of the parameters of the graph.

If we assume that $v_1, \cdots, v_k$ and $w_1, \cdots, w_{2n+1-k}$ are bases respectively of $\mathfrak{v}$ and $\mathfrak{w}$, then the function φ can be univocally associated with a k-uple of functions $(\varphi_1, \cdots, \varphi_k) : \tilde{E} \subset \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$, that makes the following diagram commutative

$$\begin{array}{ccc} \mathbb{G}_{\mathfrak{w}} & \xrightarrow{\ \varphi\ } & \mathbb{G}_{\mathfrak{v}} \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathbb{R}^{2n+1-k} & \xrightarrow[(\varphi_1, \cdots, \varphi_k)]{} & \mathbb{R}^k \end{array}$$

that is

$$\varphi\left(\exp\left(\sum_{l=1}^{2n+1-k} \xi_l w_l\right)\right) = \exp\left(\sum_{l=1}^{k} \varphi_l(\xi_1, \cdots, \xi_{d-k}) v_l\right)$$

when $\exp\left(\sum_{l=1}^{2n+1-k} \xi_l w_l\right) \in E$. Notice that one can always assume that $|v_1 \wedge \cdots \wedge v_k| = |w_1 \wedge \cdots \wedge w_{2n+1-k}| = 1$. Finally, we can restate once more Definition 2.7 in the following way: S is a graph (over $\mathbb{G}_{\mathfrak{w}}$ along $\mathbb{G}_{\mathfrak{v}}$) if

$$S = \left\{ \xi \cdot \exp\left(\sum_{l=1}^{k} \varphi_l(\xi_1, \cdots, \xi_{d-k}) v_l\right), \quad \xi \in E \right\} , \tag{12}$$

where $\xi := \exp\left(\sum_{l=1}^{d-k} \xi_l w_l\right)$.

Some of the systems of coordinate planes in $\mathbb{H}^n$ that we consider later satisfy the more restrictive assumption that $\mathfrak{w}$ is an ideal in $\mathfrak{g}$ and not simply a subalgebra. When this happens the graphs enjoy further useful properties. Hence we define

Definition 2.8 (Regular Graph)**.** Assume $\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{v}$, where $\mathfrak{v}$ and $\mathfrak{w}$ are subalgebras and $\mathfrak{w}$ is also an ideal. Let S be a subset of $\mathbb{G}$. We say that S is a *regular graph over* $\mathbb{G}_{\mathfrak{w}}$ *along* $\mathbb{G}_{\mathfrak{v}}$ if for each $p \in \mathbb{G}_{\mathfrak{w}}$, $S \cap \mathcal{L}_{\mathfrak{v}}(p)$ contains at most one point.

Definition 2.9 (Orthogonal Graph)**.** Suppose $\mathbb{G} \equiv \mathbb{H}^n$, with our previous notations, if $(w_1, \cdots, w_{2n+1-k})$ and $(v_1, \cdots, v_k)$ are basis respectively of $\mathfrak{w}$ and of $\mathfrak{v}$, if $|v_1 \wedge \cdots \wedge v_k| = |w_1 \wedge \cdots \wedge w_{2n+1-k}| = 1$ and if

$$w_1 \wedge \cdots \wedge w_{2n+1-k} = *(v_1 \wedge \cdots \wedge v_k)$$

we refer to S as an *orthogonal graph along* $\mathfrak{v}$.

As usual properties of the function φ are attributed to the graph of φ; in particular we say that the graph of φ is continuous exactly when the map $\varphi : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ is continuous.

We stress here that these intrinsic notions of graphs, adapted to the geometry of the group, are not a pointless generalization. From one side, the fact that a surface is locally a graph is, as usual, a powerful tool. On the other side, one could not have used the usual Euclidean notion. Indeed, as the following example shows, $\mathbb{H}$*-regular surfaces (of low codimension), in general, are not graphs in the usual Euclidean sense, while they are always, locally, graphs in the intrinsic Heisenberg sense.*

Example 2.10. In $\mathbb{H}^1$, with the notations of Definition 2.7, let $\mathfrak{v} = \mathrm{span}\,\{X\}$ and $\mathfrak{w} = \mathrm{span}\,\{Y, T\}$. Then $\mathbb{G}_{\mathfrak{v}} = \{(x, 0, 0) : x \in \mathbb{R}\}$ and $\mathbb{G}_{\mathfrak{w}} = \{(0, \eta, \tau) : \eta, \tau \in \mathbb{R}\}$. Then, fix $1/2 < \alpha < 1$, and take $\varphi : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ as $\varphi(0, \eta, \tau) = (|\tau|^{\alpha}, 0, 0)$. Define S as the graph of φ, precisely

$$S = \{\xi \cdot \varphi(\xi) : \xi \in \mathbb{G}_{\mathfrak{w}}\} = \{(|\tau|^{\alpha}, \eta, \tau + 2\eta|\tau|^{\alpha}) : \eta, \tau \in \mathbb{R}\}\ .$$

A non trivial theorem, proved in [2], states that if φ is sufficiently regular then its Heisenberg graph is a $\mathbb{H}$-regular surface. Our φ satisfies the hypotheses of that theorem hence S is a $\mathbb{H}$-regular surface. But, as one can easily check, S is not an Euclidean graph in any neighborhood of the origin.

If S is a graph in the sense of Definition 2.7, it is possible to write explicitly how S behaves under a generic translation. Assume S is the graph of φ over $\mathbb{G}_{\mathfrak{w}}$ and $p \in \mathbb{G}$, then $\tau_p(S)$ is still a graph over $\mathbb{G}_{\mathfrak{w}}$ of a new function φ_p. In the next proposition we show explicitly the form of the new function φ_p.

Proposition 2.11. *With the notations of Definition 2.7, assume that S is a graph over $\mathbb{G}_{\mathfrak{w}}$ along $\mathbb{G}_{\mathfrak{v}}$, that is $S = \{\Phi(\xi) := \xi \cdot \varphi(\xi) \ : \xi \in E\}$ and let $q \in \mathbb{G}$, $q = q_{\mathfrak{w}} \cdot q_{\mathfrak{v}}$ with $q_{\mathfrak{w}} \in \mathbb{G}_{\mathfrak{w}}$ and $q_{\mathfrak{v}} \in \mathbb{G}_{\mathfrak{v}}$. Then the translated set $\tau_q S$ is again a graph over $\mathbb{G}_{\mathfrak{o}w}$ along $\mathbb{G}_{\mathfrak{v}}$, precisely*

$$\tau_q S = \{\Phi_q(\eta) := \eta \cdot \varphi_q(\eta) \ : \ \eta \in E'\}\ ,$$

where $E' := q \cdot E \cdot (q_{\mathfrak{v}})^{-1} \subset \mathbb{G}_{\mathfrak{w}}$, $\varphi_q : E' \to \mathbb{G}_{\mathfrak{v}}$ is defined as

$$\varphi_q(\eta) := q_{\mathfrak{v}} \cdot \varphi(q^{-1} \cdot \eta \cdot q_{\mathfrak{v}})\ .$$

In addition

$$\Phi_q = \tau_{q^{-1}} \circ \Phi \circ \sigma_{q^{-1}}\ ,$$

where $\sigma_p : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{w}}$ is defined by

$$\sigma_p(\eta) = p \cdot \eta \cdot p_{\mathfrak{v}}^{-1}\ .$$

We assume now that $1 \le k \le n$ and that $v = v_1 \wedge \cdots \wedge v_k \in {}_H\bigwedge_k$, $v \neq 0$. That is $v_1, \cdots, v_k$ are linearly independent, left invariant vector fields in $\mathfrak{h}_1$ satisfying

$$[v_i, v_j] = 0\ \ ,\ \ \text{for all } 1 \le i, j \le k\ . \tag{13}$$

By definition, $*v \in {}_H\bigwedge_{2n+1-k}$ and we can assume $*v = w_1 \wedge \cdots \wedge w_{2n+1-k}$, with $w_1, \cdots, w_{2n-k} \in \mathfrak{h}_1$ and $w_{2n+1-k} = T$. We set

$$\mathfrak{v} := \text{span } \{v_1, \cdots, v_k\}, \quad \mathfrak{w} := \text{span } \{w_1, \cdots, w_{2n+1-k}\} .$$

Notice that both are subalgebras, $\mathfrak{w}$ is also an ideal and that

$$\mathfrak{w} \oplus \mathfrak{v} = \mathfrak{h} .$$

In what follows we prove that k-codimensional $\mathbb{H}$-regular surfaces are, locally, graphs in the Heisenberg sense. To this end, if the surface S is locally defined by the equation $S = \{p \in \mathcal{U} : f(p) = 0\}$, we have to check that if $\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k \neq 0$, then there exist k, linearly independent, horizontal vectors $v_1, \ldots, v_k$ such that

$$[v_i, v_j] = 0 \ , \ \text{ for } 1 \leq i, j \leq k \ , \tag{14}$$

$$\Delta \stackrel{\text{def}}{=} \left| \det \left([v_i f_j]_{1 \leq i,j \leq k} \right) \right| > 0 \ . \tag{15}$$

Notice that this problem does not appear when $k = 1$; indeed if $\nabla_H f \neq 0$ then there is at least one $i \in \{1, \ldots, 2n\}$ with $W_i f \neq 0$ and we can take $v_1 = W_i$.

When $k > 1$, condition $\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k \neq 0$ yields the existence of k vectors in $X_1, \ldots, Y_n$ such that (15) holds but not necessarily (14). For instance consider the following example

Example 2.12. Let $f = (f_1, f_2) : \mathbb{H}^2 \to \mathbb{R}^2$ be defined as

$$f(p_1, \ldots, p_5) = (p_1, p_3) .$$

Then S is the 2-codimensional plane $S = \{p_1 = p_3 = 0\}$. Writing explicitly the 2×4 matrix associated with $d_H f$, we see that all 2×2 minors vanish but for

$$\begin{bmatrix} X_1 f_1, & Y_1 f_1 \\ & \\ X_1 f_2, & Y_1 f_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ & \\ 0 & 1 \end{bmatrix} . \tag{16}$$

Clearly, the choice $v_1 = X_1$ and $v_2 = Y_1$ satisfy (15) but not (14). Hence we cannot foliate $\mathbb{H}^2$ using integral surfaces of v_1 and v_2, by Frobenius Theorem. Nevertheless an adapted foliation, satisfying both (14) and (15), exists: indeed it is enough to take

$$v_1 \stackrel{\text{def}}{=} X_1 + X_2, \qquad v_2 \stackrel{\text{def}}{=} Y_1 - Y_2 . \tag{17}$$

Clearly this is a typical non Euclidean phenomenon.

A crucial step in our result consists in proving that the procedure in (17) can be generalized. More precisely, we have:

Proposition 2.13. *For $2 < k \leq n$, let $f = (f_1, \ldots, f_k) : \mathbb{H}^n \to \mathbb{R}^k$, $f \in [\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)]^k$. If there is $p^0 \in \mathbb{H}^n$, such that*

$$\text{rank } \left[W_i f_j(p^0)\right]_{1 \leq i \leq 2n, 1 \leq j \leq k} = k ,$$

then there are an open $\mathcal{U} \ni p^0$ and a simple, integrable k-vector $v = v_1 \wedge \cdots \wedge v_k \in {}_H\bigwedge_k$ such that, for all $p \in \mathcal{U}$,

$$\det \, [v_i f_j(p)]_{1 \leq i,j \leq k} \neq 0 .$$

Thus, we can derive the result we were looking for: a $\mathbb{H}$-regular surface of low codimension is locally a graph. In fact, the following theorem holds.

Theorem 2.14. *Let $S \subset \mathbb{H}^n$ be a k-codimensional $\mathbb{H}$-regular surface, $1 \leq k \leq n$. Then S is locally a (normal) graph, that is, for each $p \in S$ it is possible to choose an open subset $\mathcal{U} \subset \mathbb{H}^n$, with $p \in \mathcal{U}$, a simple k-vector $v \in {}_H\bigwedge_k$, a simple $(2n+1-k)$-vector w and a function $\varphi : \mathbb{G}_{\mathfrak{w}} \to \mathbb{G}_{\mathfrak{v}}$ such that*

$$S \cap \mathcal{U} = \{\xi \cdot \varphi(\xi) : \ \xi \in \mathcal{V} \subset \mathbb{G}_{\mathfrak{w}}\} .$$

Moreover it is possible to choose v and w such that $|v| = 1$ and $w = *v$.

Coherently with our program, we can prove now that k-codimensional $\mathbb{H}$-regular surface, $1 \le k \le n$, have an intrinsic tangent group at any point.

Definition 2.15. Let $S = \{x : f(x) = 0\}$ be a k-codimensional $\mathbb{H}$-regular surface in $\mathbb{H}^n$ (with $1 \le k \le n$). The *tangent group* to S in p, indicated as $T^g_{\mathbb{H}}S(p)$, is the subgroup of $\mathbb{H}^n$ defined as

$$T^g_{\mathbb{H}}S(p) \stackrel{\text{def}}{=} \{x \in \mathbb{H}^n : d_H f_i(p)(x) = 0 \ \ , \ \ \text{for } i = 1, \cdots, k\} \ .$$

The *group normal* (or *horizontal normal*) $n_{\mathbb{H}}(p) \in \bigwedge_{k,p} \mathfrak{h}_1$ is defined by

$$n_{\mathbb{H}}(p) \stackrel{\text{def}}{=} \frac{\nabla_H f_1(p) \wedge \cdots \wedge \nabla_H f_k(p)}{|\nabla_H f_1(p) \wedge \cdots \wedge \nabla_H f_k(p)|} \ .$$

The $(2n+1-k)$-vector $t_{\mathbb{H}}(p) \in \bigwedge_{2n+1-k,p} \mathfrak{h}$ defined as

$$t_{\mathbb{H}}(p) \stackrel{\text{def}}{=} * n_{\mathbb{H}}(p)$$

will be said to be the *group tangent* to S in p.

Notice that, unlike the group normal vector, the group tangent vector is not horizontal. It can always be written in the form $t_{\mathbb{H}}(p) = \xi \wedge T$, where $\xi \in \bigwedge_{2n-k,p} \mathfrak{h}_1$. Moreover, if $t_{\mathbb{H}}(p) = v_1 \wedge \cdots \wedge v_{2n+1-k}$, then $T^g_{\mathbb{H}}S(p) = \exp(\text{span}\,\{v_1, \ldots, v_{2n+1-k}\})$.

As in the Euclidean setting, a $\mathbb{H}$-*orientation of* S will be identified with a continuous horizontal group vector field, or, equivalently, with a continuous group tangent vector field. If they exist, then S is said to be $\mathbb{H}$-orientable.

Finally notice that the definitions of $t_{\mathbb{H}}$ and of $n_{\mathbb{H}}$ are good ones. Indeed, as proved in the following Proposition, the notions of Heisenberg tangent group and of group normal to S do not depend on the defining function f.

Proposition 2.16. *If S is a k-codimensional $\mathbb{H}$-regular surface (with $1 \le k \le n$) and $p \in S$, then*

$$\text{Tan}_{\mathbb{H}}(S, p) = \tau_p T^g_{\mathbb{H}}S(p) \ . \tag{18}$$

Eventually, the intrinsic Hausdorff measure concentrated on a k-codimensional $\mathbb{H}$-regular surface can be written explicitly, as the following theorem shows.

Theorem 2.17. *Let S be a k-codimensional $\mathbb{H}$-regular surface, $1 \le k \le n$. By Theorem 2.14 and with the notations therein, we know that S is locally a normal graph, that is we can assume that there are an open subset $\mathcal{U} \subset \mathbb{H}^n$, a function $f = (f_1, \ldots, f_k) \in [\mathcal{C}^1_{\mathbb{H}}(\mathcal{U})]^k$, a simple k-vector $v = v_1 \wedge \cdots \wedge v_k \in {}_H\bigwedge_k$, with $|v| = 1$, a simple $(2n+1-k)$-vector $w \stackrel{\text{def}}{=} *v \in {}_H\bigwedge_{2n+1-k}$, a relatively open $\mathcal{V} \subset \mathbb{G}_{\mathfrak{w}}$ and a continuous function $\varphi : \mathcal{V} \to \mathbb{G}_{\mathfrak{v}}$ such that $S \cap \mathcal{U} = \{x \in \mathcal{U} : f(x) = 0\} = \{\Phi(\xi) \stackrel{\text{def}}{=} \xi \cdot \varphi(\xi), \quad \xi \in \mathcal{V}\}$. Now, if we put*

$$\Delta(p) \stackrel{\text{def}}{=} \left| \det\left[v_i f_j(p)\right]_{1 \le i,j \le k} \right| \ne 0 \quad \text{for } p \in \mathcal{U} \ ,$$

then

$$\mathcal{S}^{Q-k}_{\infty} \llcorner (S \cap \mathcal{U}) = \Phi_\sharp \left(\frac{|\nabla_H f_1 \wedge \cdots \wedge \nabla_H f_k|}{\Delta} \circ \Phi \right) \mathcal{H}^{2n+1-k}_E \llcorner \mathbb{G}_{\mathfrak{w}} \ . \tag{19}$$

Here, for a measure μ, $\Phi_\sharp \mu$ is the image measure of μ ([17], Definition 1.17). Notice also that, since $\mathbb{G}_{\mathfrak{w}}$ is a linear space, $\mathcal{H}^{2n+1-k}_E \llcorner \mathbb{G}_{\mathfrak{w}} = \mathcal{L}^{2n+1-k} \llcorner \mathbb{G}_{\mathfrak{w}}$, the $(2n+1-k)$-dimensional Lebesgue measure.

Corollary 2.18. *If S is k-codimensional $\mathbb{H}$-regular surface with $1 \le k \le n$, then the Hausdorff dimension of S with respect to the cc-distance d_c, or any other metric comparable with it, is $Q - k$.*

Recall that regular surfaces in general are not Euclidean regular. In fact, as we already stressed, recently Kirchheim and Serra Cassano provided an example of a 1-codimensional $\mathbb{H}$-regular surface S in $\mathbb{H}^1$ that has Euclidean Hausdorff dimension 2.5 and hence it is not a 2-dimensional Euclidean rectifiable set. Thus, the topological dimension of S equals 2, its Euclidean Hausdorff dimension equals 2.5 and its intrinsic Hausdorff dimension equals 3.

Nevertheless, if it happens that S is a k-codimensional Euclidean $\mathcal{C}^1$ submanifold of $\mathbb{R}^{2n+1} \equiv \mathbb{H}^n$, $1 \le k \le n$, then the surface measure $\mathcal{H}_E^{2n+1-k} \llcorner S$ is locally finite and its relation with the spherical Hausdorff measure $\mathcal{S}_\infty^{Q-k} \llcorner S$ takes a particularly simple form. This is the content of Theorem 2.20 below. In codimension 1, the formula has been proved by the authors in [7], and, with the $\mathbb{H}$-perimeter taking place of the Hausdorff measure, by Capogna, Danielli and Garofalo in [5].

Lemma 2.19. *Let S be an $\mathbb{H}$-regular surface of codimension k and suppose, in addition, that S is also an Euclidean $\mathcal{C}^1$-manifolds. With the notations of Theorem 2.17, we have*

$$\begin{aligned} &\mathcal{S}_\infty^{2n+2-k} \llcorner S = \\ &= \Big(\textstyle\sum_{1 \le i_1 < \dots < i_k \le 2n} \langle W_{i_1} \wedge \dots \wedge W_{i_k}, n \rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \Big)^{1/2} \ \mathcal{H}_E^{2n+1-k} \llcorner S \, , \end{aligned} \tag{20}$$

where

$$n = n_1 \wedge \dots \wedge n_k = \frac{\nabla f_1 \wedge \dots \wedge \nabla f_k}{|\nabla f_1 \wedge \dots \wedge \nabla f_k|_{\bigwedge_k \mathbb{R}^{2n+1}}} = \frac{\nabla f}{|\nabla f|_{\bigwedge_k \mathbb{R}^{2n+1}}}$$

is a continuous Euclidean unit normal k-vector field and $W_1 = X_1, \dots, W_{2n} = Y_n$.

Strictly speaking, an Euclidean regular surface S may be not $\mathbb{H}$-regular. Indeed, even if S is locally the zero set of a function $f \in [\mathcal{C}^1(\mathbb{R}^{2n+1})]^k \subset [\mathcal{C}^1_{\mathbb{H}}(\mathbb{H}^n)]^k$ with non-vanishing Euclidean gradient, nevertheless the non-degeneracy condition $\nabla_H f_1 \wedge \dots \wedge \nabla_H f_k \neq 0$ may fail to hold at some points. As in [14], a point p of an Euclidean $\mathcal{C}^1$ submanifold S is said to be a *characteristic point* of S if $\mathrm{Tan}(S,p) \subset H\mathbb{H}^n_p$ and, consequently, the non-degeneracy condition fails. We denote by $C(S)$ the set of these points.

When $k = 1$, it is known that $C(S)$ is *small* inside S. There are many results in this line, under various regularity hypotheses on the surfaces and using different surface measures (Euclidean versus intrinsic) to estimate the smallness. Balogh (see [4]) was the first one to prove that, in the Heisenberg groups, the intrinsic $(Q-1)$-Hausdorff measure of the characteristic set of an Euclidean $\mathcal{C}^1$ surface vanishes. Recently, Magnani ([14], 2.16) extended this result to Euclidean $\mathcal{C}^1$-submanifold of arbitrary codimension in general Carnot groups. Precisely, in the setting of the Heisenberg group, we have

$$\mathcal{S}_\infty^{Q-k}(C(S)) = 0 \tag{21}$$

if S is an Euclidean $\mathcal{C}^1$-submanifold of codimension k, $1 \le k \le n$ in $\mathbb{H}^n$. Since a $\mathcal{C}^1$-submanifold S in $\mathbb{H}^n$ can be written as $S = C(S) \cup (S \setminus C(S))$ and $S \setminus C(S)$ is a $\mathbb{H}$-regular surface, then, by Lemma 2.19, we have

Theorem 2.20. *If S is an Euclidean $\mathcal{C}^1$-submanifold of codimension k, $1 \le k \le n$ in $\mathbb{H}^n$, then*

$$\begin{aligned} &\mathcal{S}_\infty^{2n+2-k} \llcorner S = \\ &= \Big(\textstyle\sum_{1 \le i_1 < \dots < i_k \le 2n} \langle W_{i_1} \wedge \dots \wedge W_{i_k}, n \rangle^2_{\bigwedge_k \mathbb{R}^{2n+1}} \Big)^{1/2} \ \mathcal{H}_E^{2n+1-k} \llcorner S = \\ &= \Big(\textstyle\sum_{1 \le i_1 < \dots < i_k \le 2n} \Big(\det \left[\langle W_{i_\ell}, n_j \rangle_{\mathbb{R}^{2n+1}} \right]_{\ell, j = 1, \dots, k} \Big)^2 \Big)^{1/2} \ \mathcal{H}_E^{2n+1-k} \llcorner S \, , \end{aligned} \tag{22}$$

where $n = n_1 \wedge \dots \wedge n_k$ is a continuous Euclidean unit normal k-vector field and $W = (X_1, \dots, Y_n)$.

3. Appendix I: Federer-Fleming currents

We give here a natural definition of (Federer-Fleming) currents with respect to an intrinsic complex of differential forms on $\mathbb{H}^n$ and we also see that $\mathbb{H}$-regular surfaces can be naturally identified with currents defined in this way.

Let $\mathcal{U}$ be an open subset of $\mathbb{H}^n$ and let $\mathcal{D}^*(\mathcal{U}) = \mathcal{D}^0(\mathcal{U}) \oplus \cdots \oplus \mathcal{D}^{2n+1}(\mathcal{U})$ be the graded algebra of $\mathcal{C}^\infty$ differential forms on $\mathbb{R}^{2n+1}$ with compact support in $\mathcal{U}$.

Definition 3.1. Following Rumin [21] we denote by $\mathcal{D}^k_{\mathbb{H}}(\mathcal{U})$ (*Heisenberg k-differential forms*) the space of compactly supported smooth sections respectively of ${}_H\bigwedge_k \equiv \bigwedge_k \mathfrak{h}/\mathcal{I}^k$, when $1 \le k \le n$ and of ${}_H\bigwedge_k \equiv \mathcal{J}^k$ when $n+1 \le k \le 2n+1$. These spaces are endowed with the natural topology induced by that of $\mathcal{D}^k(\mathcal{U})$. We denote by $\mathcal{D}^*_{\mathbb{H}}(\mathcal{U}) = \mathcal{D}^0_{\mathbb{H}}(\mathcal{U}) \oplus \cdots \oplus \mathcal{D}^{2n+1}_{\mathbb{H}}(\mathcal{U})$ the graded algebra of all Heisenberg differential forms with compact support, where $\mathcal{D}^0_{\mathbb{H}}(\mathcal{U}) = \mathcal{C}^\infty(\mathcal{U})$.

The following Theorem is proved in [21].

Theorem 3.2 (Rumin). *There is a linear second order differential operator $D : \mathcal{D}^n_{\mathbb{H}}(\mathcal{U}) \to \mathcal{D}^{n+1}_{\mathbb{H}}(\mathcal{U})$ such that the following sequence has the same cohomology as the De Rham complex on $\mathcal{U}$:*

$$0 \to \mathcal{D}^0_{\mathbb{H}}(\mathcal{U}) \xrightarrow{d} \mathcal{D}^1_{\mathbb{H}}(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}^n_{\mathbb{H}}(\mathcal{U}) \xrightarrow{D}$$

$$\xrightarrow{D} \mathcal{D}^{n+1}_{\mathbb{H}}(\mathcal{U}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{D}^{2n+1}_{\mathbb{H}}(\mathcal{U}) \to 0$$

where d is the operator induced by the external differentiation from $\mathcal{D}^k(\mathcal{U}) \to \mathcal{D}^{k+1}(\mathcal{U})$, with $k \neq n$.

Definition 3.3. We call *Heisenberg k-current*, $1 \le k \le 2n+1$ any continuous linear functional on $\mathcal{D}^k_{\mathbb{H}}(\mathcal{U})$ and we denote by $\mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ the set of all Heisenberg k-currents.

Proposition 3.4. *If $1 \le k \le n$, any $T \in \mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ can be identified with an element $\tilde{T}$ of $\mathcal{D}_k(\mathcal{U})$, the space of all Euclidean k-currents by setting*

$$\tilde{T}(\omega) \stackrel{\text{def}}{=} T([\omega])$$

for any $\omega \in \mathcal{D}^k(\mathcal{U})$.

On the other hand, if $S \in \mathcal{D}_k(\mathcal{U})$ is such that $S(\alpha \wedge \theta) = 0$ for any $\alpha \in \mathcal{D}^{k-1}(\mathcal{U})$ and $S(\beta \wedge d\theta) = 0$ for any $\beta \in \mathcal{D}^{k-2}(\mathcal{U})$ if $k \ge 2$, then S induces a Heisenberg k-current $T \in \mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ by the identity

$$T([\omega]) \stackrel{\text{def}}{=} S(\omega)$$

for all $[\omega] \in \mathcal{D}^k_{\mathbb{H}}(\mathcal{U})$. Obviously, with our previous notations, $\tilde{T} = S$.

Definition 3.5. Let T be k-dimensional $\mathbb{H}$-current in an open set $\mathcal{U} \subset \mathbb{H}^n$, then the *mass* $\mathbf{M}_{\mathcal{V}}(T)$ of T in $\mathcal{V} \subset \mathcal{U}$, $\mathcal{V}$ open, is

$$\mathbf{M}_{\mathcal{V}}(T) \stackrel{\text{def}}{=} \sup\{T(\alpha) : \ \alpha \in \mathcal{D}^k_{\mathbb{H}}(\mathcal{V}), |\alpha| \le 1\}$$

We have:

Proposition 3.6. *Let $S \subset \mathcal{U}$ be a $\mathbb{H}$-regular surface as in Definitions 2.4 and 2.5. Assume S is oriented by a group tangent k-vector field $t_{\mathbb{H}}$. Then, if S is k-dimensional, $1 \le k \le n$, the map*

$$\alpha \to [\![S]\!](\alpha) \stackrel{\text{def}}{=} \int_S \langle \alpha | t_{\mathbb{H}} \rangle \, d\mathcal{S}^k_\infty$$

from $\mathcal{D}^k_{\mathbb{H}}$ to $\mathbb{R}$ is a Heisenberg k-current with locally finite mass. Precisely, if $\mathcal{V} \subset\subset \mathcal{U}$,

$$\mathbf{M}_{\mathcal{V}}([\![S]\!]) = \mathcal{S}^k_\infty(S \cap \mathcal{V}) \, .$$

Analogously, if S is k-codimensional, $1 \le k \le n$, the map

$$\alpha \to [\![S]\!]\,(\alpha) \stackrel{\text{def}}{=} \int_S \langle \alpha | \omega \rangle \, d\mathcal{S}_\infty^{Q-k}$$

from $\mathcal{D}_{\mathbb{H}}^{2n+1-k}$ to $\mathbb{R}$ is a Heisenberg $(2n+1-k)$-current with locally finite mass. Precisely, if $\mathcal{V} \subset\subset \mathcal{U}$,

$$\mathbf{M}_{\mathcal{V}}([\![S]\!]) = \int_{S\cap\mathcal{V}} |\mathrm{proj}_{{}_H\bigwedge_{2n+1-k}}(t_{\mathbb{H}})| \, d\mathcal{S}_\infty^{Q-k} \ , \tag{23}$$

where $\mathrm{proj}_{{}_H\bigwedge_{2n+1-k}} : \bigwedge_{2n+1-k} \mathfrak{h}_1 \to {}_H\bigwedge_{2n+1-k}$ is the orthogonal projection with respect to the Riemannian scalar product defined in $\bigwedge_{2n+1-k} \mathfrak{h}_1$.

Corollary 3.7. *There exists a geometric constant $c_{n,k} \in (0,1)$ such that for any k-codimensional $\mathbb{H}$-regular surface S, $1 \le k \le n$, we have*

$$c_{n,k}\, \mathcal{S}_\infty^{Q-k}(S \cap \mathcal{V}) \le \mathbf{M}_{\mathcal{V}}([\![S]\!]) \le \mathcal{S}_\infty^{Q-k}(S\cap\mathcal{V}) \ ,$$

for every Borel set $\mathcal{V}$.

Thanks to Rumin's result, the operators d and D act in the complex as external differentiation does in De Rham complex, and we can give the following (obvious) definition.

Definition 3.8. Let T be a Heisenberg k-current in an open set $\mathcal{U} \subset \mathbb{H}^n$ with $1 \le k \le d$. Then we define the Heisenberg $(k-1)$-current $\partial_{\mathbb{H}} T$, the *Heisenberg boundary of T*, by the identity

$$\partial_{\mathbb{H}} T(\alpha) = T(d\alpha) \quad \text{if } k \neq n+1$$

and

$$\partial_{\mathbb{H}} T(\alpha) = T(D\alpha) \quad \text{if } k = n+1 \ .$$

The following trivial statement says that - also when boundaries are concerned - low dimension $\mathbb{H}$-currents are but particular Euclidean currents.

Proposition 3.9. *If $1 \le k \le n$, the Heisenberg boundary $\partial_{\mathbb{H}} T$ of $T \in \mathcal{D}_{\mathbb{H},k}(\mathcal{U})$ can be identified as in Proposition 3.4 with the Euclidean $(k-1)$-current $\partial \tilde{T}$.*

When $k \ge n+1$, the structure of the boundary of a current is much more difficult to describe, even in the simplest situation of a current carried by a low codimensional $\mathbb{H}$-regular surface. As an example, consider the case $n = 1$, and let S be a 1-codimensional $\mathbb{H}$-regular (hyper)surface. We want to state here something similar to Stokes formula that yields that the boundary of a 2-dimensional current in $\mathbb{R}^3$ carried by a sufficiently regular portion of a 2-dimensional Euclidean differentiable manifold (a 2-dimensional oriented Euclidean differentiable manifold with boundary) is carried by the boundary itself, endowed with a suitable induced orientation.

First of all, we cannot think in general of a portion of $\mathbb{H}$-regular hypersurface - whatever regularity we assume for the boundary - as a differentiable manifold with boundary, since, as we pointed out repeatedly, $\mathbb{H}$-regular surfaces may be very "bad" from the Euclidean point of view ([13]). On the other hand, even when dealing with (Euclidean) smooth hypersurfaces with boundary, the mass of the boundary of the associated current may be not locally finite, unless the topological boundary is a horizontal curve.

Let us start by illustrating the last phenomenon: if $[\omega] \in \mathcal{D}_{\mathbb{H}}^1(\mathbb{H}^1)$ we can alway choose ω to be its horizontal representative $\omega = \omega_1 dp_1 + \omega_2 dp_2$. In this case, accordingly with Rumin's theorem ([21]), the operator D has the form

$$D[\omega] = d(\omega + \tilde{\omega}\theta) \ ,$$

where $\tilde{\omega} \in \mathcal{C}^\infty(\mathbb{H}^1)$, is chosen in order to have $d(\omega + \tilde{\omega}\theta) \in \mathcal{D}_{\mathbb{H}}^2(\mathbb{H}^1)$, i.e. such that $d(\omega + \tilde{\omega}\theta) \wedge \theta = 0$. An esplicit computation (see also [11], Section 6) shows that

$$\tilde{\omega} = \frac{1}{4}(W_2\omega_1 - W_1\omega_2) \ .$$

Consider now the 2-dimensional $\mathbb{H}$-current $[\![S]\!]$ carried by the hypersurface $S = \{p_1 = 0,\ p_2 > 0\}$ oriented by $W_2 \wedge T$. Let t_0 be the boundary of S, i.e. $t_0 = \{p_1 = p_2 = 0\}$. If $[\omega] \in \mathcal{D}^1_{\mathbb{H}}(\mathbb{H}^1)$, with $\omega = \omega_1 dp_1 + \omega_2 dp_2$ as above, by definition and by Stokes theorem (keeping also in mind that $\mathcal{S}^3_\infty \llcorner S = \mathcal{H}^2_E \llcorner S$, by (19)), we have

$$\partial_{\mathbb{H}} [\![S]\!] ([\omega]) \stackrel{\text{def}}{=} \int_S \langle D([\omega]) \,|\, W_2 \wedge T \rangle \, d\mathcal{H}^2_E = \int_S \langle d(\omega + \tilde{\omega}\theta) \,|\, W_2 \wedge T \rangle \, d\mathcal{H}^2_E =$$

$$= \int_{t_0} \langle \omega + \tilde{\omega}\theta | T \rangle \, d\mathcal{H}^1_E = \frac{1}{4} \int_{t_0} (\partial_2 \omega_1 - \partial_1 \omega_2) \, d\mathcal{H}^1_E \,.$$

Clearly, the above quantity can be made arbitrary large still keeping $|[\omega]| \leq 1$. This shows that $\partial_{\mathbb{H}} [\![S]\!]$, though being a well defined current in our sense, has no locally finite mass.

In fact, this phenomenon can be easily explained noticing that ∂S is not a regular 1-dimensional manifold in our sense, since it is not horizontal. On the contrary, if we assume that ∂S is a $\mathbb{H}$-regular 1-dimensional manifold, then, for instance, the following result can be derived from Theorem 5.4 in [11].

Theorem 3.10. *Let $S \subset \mathbb{H}^1$ be a $\mathbb{H}$-regular $\mathbb{H}$-oriented hypersurface, and let $V \subset S$ be the closure of a relatively open subset V_0 of S. We assume that V is a topological 2-manifold with boundary ∂V that is a finite union of disjoint simple closed $\mathbf{C}^1$-piecewise horizontal curves. Then $\partial [\![V_0]\!]$ is carried by ∂V and has finite mass.*

4. Appendix II: Lipschitz graphs

The notion of graph in a Lie groups yields naturally the following question: can we develop a theory of Lipschitz graphs in a Lie group (let us say in $\mathbb{H}^n$, for sake of simplicity)? So far, a coherent notion of Lipschitz graph in $\mathbb{H}^n$ is far from being well understood. Such a theory would have several applications, from geometric measure theory to boundary problems for pde's in groups. Here, we sketch some possible basic definitions.

Let $\mathbb{G} \equiv \mathbb{H}^n$ and let $\mathbb{W}$ and $\mathbb{V}$ be homogeneous subgroups of $\mathbb{G}$ such that $\mathbb{W} \cap \mathbb{V} = \{0\}$ and $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$, i.e. each$p \in \mathbb{G}$ can be written in a unique way as $p = p_{\mathbb{W}} \cdot p_{\mathbb{V}}$, where $p_{\mathbb{W}} \in \mathbb{W}$ and $p_{\mathbb{V}} \in \mathbb{V}$.

Assume that $S = \{\xi \cdot \varphi(\xi) : \xi \in E \subset \mathbb{W}\}$ is a graph over $\mathbb{W}$ (see Definition 2.7). It has been proved in [10] that if S is $\mathbb{H}$-regular and if $\varphi(0) = 0$ then there is $L > 0$ such that, $\forall \xi$ in a bounded neighborhood of 0,

$$|\varphi(\xi)| \leq L|\xi| \,. \tag{24}$$

Let $p = p_{\mathbb{W}} \cdot p_{\mathbb{V}} = p_{\mathbb{W}} \cdot \varphi(p_{\mathbb{W}}) \in S$, then the translated surface $\tau_{p^{-1}} S$ is the graph of

$$\varphi_{p^{-1}}(\eta) := (p^{-1})_{\mathbb{V}} \cdot \varphi(p \cdot \eta \cdot (p^{-1})_{\mathbb{V}})$$

Notice that

$$\varphi_{p^{-1}}(0) = 0$$

Indeed $\varphi_{p^{-1}}(0) = p_{\mathbb{V}}^{-1} \cdot \varphi(p \cdot p_{\mathbb{V}}^{-1}) = p_{\mathbb{V}}^{-1} \cdot \varphi(p_{\mathbb{W}}) = 0$ because $\varphi(p_{\mathbb{W}}) = p_{\mathbb{V}}$ by definition of graph. Hence, from (24) and because being $\mathbb{H}$-regular is invariant by group translations, for any $\nu \in \mathbb{W}$ we have

$$|\varphi_{p^{-1}}(\nu)| \equiv |p_{\mathbb{V}}^{-1} \cdot \varphi(p \cdot \nu \cdot p_{\mathbb{V}}^{-1})| \leq L|\nu| \,,$$

now recall that $p_{\mathbb{V}}^{-1} = \varphi(p_{\mathbb{W}})^{-1}$, and setting $\xi := p_{\mathbb{W}}$, $\eta := p \cdot \nu \cdot p_{\mathbb{V}}^{-1}$ and consequently $\nu = p^{-1} \cdot \eta \cdot p_{\mathbb{V}} = p_{\mathbb{V}}^{-1} \cdot (p_{\mathbb{W}})^{-1} \cdot \eta \cdot p_{\mathbb{V}}$, it follows that, $\forall \xi, \eta \in \mathbb{W}$,

$$|\varphi(\xi)^{-1} \cdot \varphi(\eta)| \leq L |\varphi(\xi)^{-1} \cdot (\xi^{-1} \cdot \eta) \cdot \varphi(\xi)| \,.$$

We use this property as a definition of intrinsically Lipschitz function:

Definition 4.1. Let $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$;
$\varphi : \mathbb{W} \to \mathbb{V}$ is $\mathbb{G}$-*Lipschitz*, with Lipschitz constant L, when

$$|\varphi(\xi)^{-1} \cdot \varphi(\eta)| \le L|\varphi(\xi)^{-1} \cdot (\xi^{-1} \cdot \eta) \cdot \varphi(\xi)|, \quad \forall \xi, \eta \in \mathbb{W}\ ; \tag{25}$$

$\varphi : \mathbb{V} \to \mathbb{W}$ is $\mathbb{G}$-*Lipschitz*, with Lipschitz constant L, when

$$|\eta^{-1} \cdot \xi \cdot \varphi(\xi)^{-1} \cdot \xi^{-1} \cdot \eta \cdot \varphi(\eta)| \le L|\xi^{-1} \cdot \eta|, \quad \forall \xi, \eta \in \mathbb{V}\ . \tag{26}$$

Hence, $\varphi : \mathbb{W} \to \mathbb{V}$ is $\mathbb{G}$-Lipschitz, with Lipschitz constant $L > 0$, exactly when, for each $x \in \mathbb{W}$, setting $p = x \cdot \varphi(x)$, the translated function $\varphi_{p^{-1}}$ satisfies

$$|\varphi_{p^{-1}}(y)| \le L|x|, \quad \forall y \in \mathbb{W}\ .$$

This can be restated in terms of cones.

Definition 4.2. Assume that $\mathbb{G} = \mathbb{B} \cdot \mathbb{N}$ is the product of two subgroups $\mathbb{B}$ and $\mathbb{N}$, with $\mathbb{B} \cap \mathbb{N} = \{0\}$. For $q \in \mathbb{G}$, $\alpha > 0$ we define the cones $C_{\mathbb{B},\mathbb{N}}(q, \alpha)$ with axis $\mathbb{N}$, base $\mathbb{B}$, vertex q as

$$C_{\mathbb{B},\mathbb{N}}(q, \alpha) := q \cdot C_{\mathbb{B},\mathbb{N}}(0, \alpha)$$

where

$$C_{\mathbb{B},\mathbb{N}}(0, \alpha) := \{p : |p_{\mathbb{B}}| \le \alpha |p_{\mathbb{N}}|\}\ .$$

Clearly, $C_{\mathbb{B},\mathbb{N}}(0, 0) = \mathbb{N}$. Moreover

$$\cup_{\alpha>0} C_{\mathbb{B},\mathbb{N}}(0, \alpha) = \big(\mathbb{G} \setminus \mathbb{B}\big) \cup \{0\}$$

If $S = \{x \cdot \varphi(x)\}$ is the graph of $\varphi : \mathbb{W} \to \mathbb{V}$ (or $\varphi : \mathbb{V} \to \mathbb{W}$) then $|\varphi(x)| < L|x|$ if and only if $C_{\mathbb{W},\mathbb{V}}(0, \alpha) \cap S = \{0\}$ for all α, $0 \le \alpha < 1/L$. In general $\varphi : \mathbb{W} \to \mathbb{V}$ is $\mathbb{G}$-Lipschitz, with Lipschitz constant L, if and only if S, the graph of φ, is such that

$$C_{\mathbb{W},\mathbb{V}}(q, \alpha) \cap S = \{q\}, \quad \forall q \in S, \forall \alpha : 0 \le \alpha < 1/L.$$

Indeed, if $q \in S$, $C_{\mathbb{W},\mathbb{V}}(0, \alpha) \cap \operatorname{graph} \varphi_{q^{-1}} = \{0\}$, hence $\{q\} = \tau_q(C_{\mathbb{W},\mathbb{V}}(0, \alpha) \cap \operatorname{graph} \varphi_{q^{-1}})$ $= \tau_q(C_{\mathbb{W},\mathbb{V}}(0, \alpha) \cap \tau_{q^{-1}} S) = C_{\mathbb{W},\mathbb{V}}(q, \alpha) \cap S$.

If $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$ and if we assume that $S = \{\Phi(\xi) := \xi \cdot \varphi(\xi) : \xi \in E \subset \mathbb{W}\}$ is the parametric image of a set $E \subset \mathbb{W}$ in $\mathbb{G}$. The same result proved in [10] says that, if S is $\mathbb{H}$-regular and if $\Phi(0) = 0$ then there is $L > 0$ such that, $\forall \chi \in E \subset \mathbb{W}$,

$$|\Phi(\chi)| \le |\chi| + |\varphi(\chi)| \le (1 + L)|\chi|\ .$$

Let us fix $\xi \in E \subset \mathbb{W}$ and let $p = \Phi(\xi)$, then $\Phi_{p^{-1}} : \mathbb{W} \to \mathbb{G}$ is such that $\Phi_{p^{-1}}(0) = 0$, hence

$$|\Phi_{p^{-1}}(\chi)| \le k|\chi| \qquad \forall \chi \in E' \subset \mathbb{G}_{\mathfrak{w}}\ .$$

Here and in the following the norms used are the norm of $\mathbb{G}$ restricted to $\mathbb{W}$ or to $\mathbb{V}$. Notice that

$$\begin{aligned} \Phi_{p^{-1}}(\chi) &= \Phi(\xi)^{-1} \cdot \Phi\left(\varphi(\xi) \cdot \varphi(\xi)^{-1} \cdot \xi \cdot \varphi(\xi) \cdot \chi \cdot \varphi(\xi)^{-1}\right) = \\ &= \Phi(\xi)^{-1} \cdot \Phi\left(\xi \cdot \varphi(\xi) \cdot \chi \cdot \varphi(\xi)^{-1}\right)\ . \end{aligned}$$

Hence, setting $\xi \cdot \varphi(\xi) \cdot \chi \cdot \varphi(\xi)^{-1} = \eta$, that is $\chi = \varphi(\xi)^{-1} \cdot \xi^{-1} \cdot \eta \cdot \varphi(\xi)$, we finally get

$$\begin{aligned} \left|\Phi(\xi)^{-1} \cdot \Phi(\eta)\right| &\le k\ \left|\varphi(\xi)^{-1} \cdot \xi^{-1} \cdot \eta \cdot \varphi(\xi)\right| = \\ &= \left|\Phi(\xi)^{-1} \cdot \xi^{-1} \cdot \eta \cdot \Phi(\xi)\right|. \end{aligned}$$

We use the last inequality as a definition.

Definition 4.3. Let $\mathbb{G} = \mathbb{W} \cdot \mathbb{V}$.

If $\Phi : E \subset \mathbb{W} \to \mathbb{G}$ we say that Φ is $\mathbb{G}$-*Lipschitz* in E if

$$\left|\Phi(\xi)^{-1} \cdot \Phi(\eta)\right| \le k\ \left|\Phi(\xi)^{-1} \cdot \xi^{-1} \cdot \eta \cdot \Phi(\xi)\right|, \quad \forall \xi, \eta \in E\ . \tag{27}$$

If $\Phi : F \subset \mathbb{V} \to \mathbb{G}$ we say that Φ is $\mathbb{G}$-*Lipschitz* in F if

$$\left|\Phi(\xi)^{-1} \cdot \Phi(\eta)\right| \le k \left|\xi^{-1} \cdot \eta\right|, \quad \forall \xi, \eta \in F \subset \mathbb{V}\ . \tag{28}$$

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