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Normal forms for Levi degenerate hypersurfaces

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Abstract[1]. Let $M \subset \mathbb{C}^4$ be real analytic hypersurface. Assume that it is 2-nondegenerate at the origin and that, after normalization, the diagonal components of the Levi form at the origin are $-1, 1, 0$. Normal forms and bounds of dimension of CR automorphism groups for such hypersurfaces are obtained.

1. Introduction

In this paper[2], we study the equivalence problem for CR manifolds. The problem asks for conditions on two CR manifolds M, M' at $p \in M, p' \in M'$ which guarantee that there exists a local biholomorphism defined near p such that it takes M to M' and p to p'. We shall construct a normal form for CR manifold to give an answer to the problem. S. S. Chern -J. K. Moser were the first to construct a normal form in the case of Levi nondegenerate hypersurfaces in [4]. For Levi degenerate hypersurfaces, several mathematicians have attacked to this problem. P. Wong [9] treated hypersurfaces whose Levi form vanishes at the reference point and is positive definite elsewhere on a neighborhood. After his result, N. Stanton [8] treated rigid hypersurfaces not necessarily pseudoconvex. Recently P. Ebenfelt [5] treated 2-nondegenerate hypersurfaces in $\mathbb{C}^3$ and classified the defining functions into eight types. In the case that the Levi form has precisely one non-zero eigenvalue, he constructed normal forms for such hypersurfaces. He also generalized his own results to higher dimensional cases [6].

Since this paper depends on P. Ebenfelt's results [6], let us explain his result precisely. Assume that the Levi form at the reference point has rank maximal minus 1 and is semi-definite. Then he classified defining functions into three types of partial normal forms. By partial normal form, we mean a normal form without any conditions on higher order terms. These three types of partial normal forms contain parameters and in case of real five-dimensional hypersurfaces, these parameters are specified and we can classify such hypersurfaces into five types. In this paper, we treat real seven-dimensional hypersurfaces whose Levi forms are neither definite nor semi-definite. In fact we assume that eigenvalues of Levi form are $-1, 1, 0$. We regard seven-dimensional hypersurface as five-dimensional one with one complex parameter. Then the five-dimensional hypersurface has semi-definite Levi form for a suitable choice of variables and a parameter. For such a choice of variables and a parameter, we can apply Theorem 1.2.10 $n = 2$ in [6]. Using his result, we shall prove the following. In the theorem, we denote $(z_1, z_2) = z'$, $(z', z_3) = z$ and $< z', \overline{z'} >= -|z_1|^2 + |z_2|^2$ and $N^j, \mathcal{N}^j$ are defined in §4.

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Theorem 1. *Let $M \subset \mathbb{C}^4$ be a real analytic hypersurface. Assume that it is 2-nondegenerate at the origin and that, after diagonalization if necessary, the components of Levi form at the origin are $-1, 1, 0$. Then the defining function of M is transformed into one of the following normal forms.*

$$\mathrm{Im}\, w = < z', \overline{z'} > + 2\mathrm{Re}\, \overline{z_3}(\lambda_1 z_1^2 + z_2 z_3) + N^1 \ , \tag{1}$$

$$\mathrm{Im}\, w = < z', \overline{z'} > + 2\mathrm{Re}\, \overline{z_3}(\lambda_2 z_1^2 + z_2 z_3 + z_1 z_3) + N^2 \ , \tag{2}$$

$$\mathrm{Im}\, w = < z', \overline{z'} > + 2\mathrm{Re}\, \overline{z_3}(\lambda_3 z_1^2 + \mu_3 z_2^2 + \nu_3 z_1 z_2) + N^3 \ , \tag{3}$$

$$\mathrm{Im}\, w = < z', \overline{z'} > + 2\mathrm{Re}\, \overline{z_3}(z_1^2 + \mu_4 z_2^2 + z_3^2 + \nu_4 z_1 z_2) + N^4 \ , \tag{4}$$

$$\mathrm{Im}\, w = < z', \overline{z'} > + 2\mathrm{Re}\, \overline{z_3}(\mu_5 z_2^2 + z_3^2 + \nu_5 z_1 z_2) + N^5 \ , \tag{5}$$

Here $\lambda_k, \mu_k \in \mathbb{R}, \nu_k \in \mathbb{C}$ and $N^j \in \mathcal{N}^j$ and at least one of λ_3 and ν_3 is non-zero. They are mutually nonequivalent, provided that

$$(\mu_4, \nu_4, \mu_5, \nu_5) \in \{\mu_4, \nu_5 \neq 0,\ \nu_4^2 = \mu_4\} \cup \{\nu_5 = 0,\ \nu_4^2 \neq \mu_4\} \cup \{\nu_5 = \mu_5 = 0\}$$
$$\cup \{\mu_4 = \mu_5 = 0\} \cup \{\mu_4 = 0,\ \nu_4 \neq \pm \frac{\nu_5}{|\mu_5|}\}$$

This theorem is proved in the category of formal power series. In view of [1], since our hypersurfaces are of finitely nondegenerate, formal equivalence implies, in fact, biholomorphic equivalence.

This paper is organized as follows. In §2, we introduce a transformation matrix of CR vector fields, which lead us to get transformation rules of coefficients of Taylor expansion of defining function and give a first step to get a partial normal form by using the argument of [5, 6]. In §3, we construct partial normal form. The conclusion of this section is summarized at the end of this section. For a normalization of higher order terms, we consider it in §4. We introduce a certain kind of linear operator between formal power series, and get partial differential equations and solve them. We need 2-nondegeneracy condition to solve them uniquely. We treat relatively easy cases in this section and we treat another case in §5. As a corollary, we have a bound on the dimension of stability group in §6. In the final section, some remarks are listed.

2. Transformation matrices for CR vector fields

In this section, we shall present transformation rules for coefficients of Taylor expansion of defining function and give a first step of construction of partial normal form . Let $M \subset \mathbb{C}^{n+1}$ be a real analytic hypersurface. Let $\{L_{\bar{\alpha}}\}_{\alpha=1,\dots,n}$ be a basis for the CR vector fields on M near $p \in M$ and θ a non-zero characteristic form near p. Denote by $g_{\bar{\alpha},\beta}$ the Levi form of M at p relative to the basis $\{L_{\bar{\alpha}}\}_{\alpha=1,\dots,n}$. Let $\{L'_{\bar{\beta}}\}_{\beta=1,\dots,n}$ and θ' be another CR vector fields and a non-zero characteristic form on M near p. Then there exist matrix $\{A^{\alpha}_{\beta}\} \in \mathrm{GL}(n, \mathbb{C})$ and $a \in \mathbb{R}_{\neq 0}$ such that

$$L'_{\bar{\beta},p} = \overline{A^{\alpha}_{\beta}} L_{\bar{\alpha},p}, \quad \theta'_p = a\theta_p \ .$$

Here we use the usual summation convention to raise and lower indices. Let $h_{\bar{\alpha},\bar{\beta},\gamma}$ be the coefficients of $\overline{z_\alpha}\ \overline{z_\beta}\ z_\gamma$ in the Taylor expansion of a defining function. We use $g'_{\bar{\alpha},\beta}$ and $h'_{\bar{\alpha},\bar{\beta},\gamma}$ when we work on $\{L'_{\bar{\beta}}\}_{\beta=1,\dots,n}$ and θ'. A change of bases as above yields transformation rules

$$g'_{\bar{\alpha},\beta} = a\overline{A^{\gamma}_{\alpha}} A^{\nu}_{\beta} g_{\bar{\gamma},\nu}, \quad h'_{\bar{\alpha},\bar{\beta},\gamma} = a\overline{A^{\sigma}_{\alpha}}\ \overline{A^{\mu}_{\beta}}\ A^{\nu}_{\gamma} h_{\bar{\sigma},\bar{\mu},\nu} \ .$$

We call $A = \{A^{\alpha}_{\beta}\}$ a transformation matrix. Now we go back to our case. Let $M \subset \mathbb{C}^4$ be a real analytic hypersurface. Assume that the matrix of the Levi form at the origin is diagonal and its elements are $-1, 1, 0$. Then by theorem 4.15 in [6], we may assume that the defining function for M is transformed to the form

$$\text{(6)} \qquad \operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re} z_3 \sum_{\alpha,\beta=1,2,3} h_{\bar{\alpha},\bar{\beta},3}\, \overline{z_\alpha}\, \overline{z_\beta} + \{\text{higher order terms}\} .$$

Since the transformation rule preserves the Levi form $< z', \overline{z'} >$, we have

$$\text{(7)} \qquad g'_{\bar{1},1} = a(\overline{A_1^1} A_1^1 g_{\bar{1},1} + \overline{A_1^2} A_1^2 g_{\bar{2},2}) = -1$$

$$\text{(8)} \qquad g'_{\bar{2},2} = a(\overline{A_2^1} A_2^1 g_{\bar{1},1} + \overline{A_2^2} A_2^2 g_{\bar{2},2}) = 1$$

$$\text{(9)} \qquad g'_{\bar{3},3} = a(\overline{A_3^1} A_3^1 g_{\bar{1},1} + \overline{A_3^2} A_3^2 g_{\bar{2},2}) = 0$$

$$\text{(10)} \qquad g'_{\bar{1},2} = a(\overline{A_1^1} A_2^1 g_{\bar{1},1} + \overline{A_1^2} A_2^2 g_{\bar{2},2}) = 0$$

$$\text{(11)} \qquad g'_{\bar{1},3} = a(\overline{A_1^1} A_3^1 g_{\bar{1},1} + \overline{A_1^2} A_3^2 g_{\bar{2},2}) = 0$$

$$\text{(12)} \qquad g'_{\bar{2},3} = a(\overline{A_2^1} A_3^1 g_{\bar{1},1} + \overline{A_2^2} A_3^2 g_{\bar{2},2}) = 0$$

Since $g_{\bar{1},1} = -1, g_{\bar{2},2} = 1$, we may choose the transformation matrix A as

$$\begin{pmatrix} \overline{A_1^1} & \overline{A_1^2} & \overline{A_1^3} \\ \overline{A_2^1} & \overline{A_2^2} & \overline{A_2^3} \\ 0 & 0 & \overline{A_3^3} \end{pmatrix} ,$$

where

$$\text{(13)} \qquad a(|A_1^1|^2 - |A_1^2|^2) = 1 ,$$

$$\text{(14)} \qquad a(-|A_2^1|^2 + |A_2^2|^2) = 1 ,$$

$$\text{(15)} \qquad \overline{A_1^1} A_2^1 - \overline{A_1^2} A_2^2 = 0 ,$$

$$\text{(16)} \qquad \overline{A_3^3}(\overline{A_1^1}\, \overline{A_2^2} - \overline{A_2^1}\, \overline{A_1^2}) \neq 0 .$$

Assume that $A_2^1, A_2^2 \neq 0$, then we have $\overline{A_1^1} = (A_2^2/A_2^1)\overline{A_1^2}$ from (10). Substitute this into (13), then we have $|A_1^2/A_2^1|^2 = 1$, which implies $\overline{A_1^2} = \overline{A_2^1} e^{i\alpha}$ and $\overline{A_1^1} = (\overline{A_2^1} A_2^2/A_2^1) e^{i\alpha}$ for some α. Therefore A can be written as

$$A = \begin{pmatrix} \dfrac{\overline{A_2^1}\, A_2^2}{A_2^1}\, e^{i\alpha} & \overline{A_2^1}\, e^{i\alpha} & \overline{A_1^3} \\ \overline{A_2^1} & \overline{A_2^2} & \overline{A_2^3} \\ 0 & 0 & \overline{A_3^3} \end{pmatrix} ,$$

where $a(-|A_2^1|^2 + |A_2^2|^2) = 1$. Denote the matrix of this type by $\mathcal{A}_1$.

Next assume $A_2^1 \neq 0$ and $A_2^2 = 0$. Then we have

$$\overline{A_2^1} = \frac{e^{i\alpha}}{\sqrt{-a}} \ , \ A_1^1 = 0 \ , \ \overline{A_1^2} = \frac{e^{i\beta}}{\sqrt{-a}} .$$

Therefore A can be written as

$$A = \begin{pmatrix} 0 & \dfrac{e^{i\beta}}{\sqrt{-a}} & \overline{A^3_1} \\ \dfrac{e^{i\alpha}}{\sqrt{-a}} & 0 & \overline{A^3_2} \\ 0 & 0 & \overline{A^3_3} \end{pmatrix}.$$

Denote the matrix of this type by $\mathcal{A}_2$.

In case of $A^1_2 = 0$, it follows from equations (13),..., (16) that

$$\overline{A^2_2} = \frac{e^{i\beta}}{\sqrt{a}} \ , \ A^2_1 = 0 \ , \ \overline{A^1_1} = \frac{e^{i\alpha}}{\sqrt{a}} \ ,$$

and therefore

$$A = \begin{pmatrix} \dfrac{e^{i\alpha}}{\sqrt{a}} & 0 & \overline{A^3_1} \\ 0 & \dfrac{e^{i\beta}}{\sqrt{a}} & \overline{A^3_2} \\ 0 & 0 & \overline{A^3_3} \end{pmatrix}.$$

Denote the matrix of this type by $\mathcal{A}_3$.

By the argument in [5] and [6], the defining function becomes one of the following,

$$\begin{aligned} \operatorname{Im} w =< z', \overline{z'} > +2\operatorname{Re} z_2|z_3|^2 \quad &+2\operatorname{Re} z_3 \textstyle\sum_{\beta=1,2,3} h_{\bar{1},\bar{\beta},3}\overline{z_1}\ \overline{z_\beta} + \\ &+\{\text{higher order terms}\} \end{aligned} \tag{17}$$

$$\begin{aligned} \operatorname{Im} w =< z', \overline{z'} > \quad &+2\operatorname{Re} z_3 \textstyle\sum_{\beta=1,2,3} h_{\bar{1},\bar{\beta},3}\overline{z_1}\ \overline{z_\beta} + \\ &+\{\text{higher order terms}\} \end{aligned} \tag{18}$$

$$\begin{aligned} \operatorname{Im} w =< z', \overline{z'} > +2\operatorname{Re} z_2^2\overline{z_3} \quad &+2\operatorname{Re} z_3 \textstyle\sum_{\beta=1,2,3} h_{\bar{1},\bar{\beta},3}\overline{z_1}\ \overline{z_\beta} + \\ &+\{\text{higher order terms}\} \end{aligned} \tag{19}$$

$$\begin{aligned} \operatorname{Im} w =< z', \overline{z'} > +2\operatorname{Re} z_3|z_3|^2 \quad &+2\operatorname{Re} z_3 \textstyle\sum_{\beta=1,2,3} h_{\bar{1},\bar{\beta},3}\overline{z_1}\ \overline{z_\beta} + \\ &+\{\text{higher order terms}\} \end{aligned} \tag{20}$$

$$\begin{aligned} \operatorname{Im} w =< z', \overline{z'} > +2\operatorname{Re} \overline{z_3}(z_2^2 + z_3^2) \quad &+2\operatorname{Re} z_3 \textstyle\sum_{\beta=1,2,3} h_{\bar{1},\bar{\beta},3}\overline{z_1}\ \overline{z_\beta} + \\ &+\{\text{higher order terms}\} \end{aligned} \tag{21}$$

3. Constructions of partial normal forms

In this section, we shall normalize coefficients of equations (17) , . . . , (21) to construct partial normal forms for M.

(I) The equation (17).

(I-i) $|h_{\bar{1},\bar{3},3}| \neq 0, 1$. Use the matrix in $\mathcal{A}_1$ with $\beta = 0$. By the transformation rule of $h_{\bar{3},\bar{3},3}$, we have $h'_{\bar{3},\bar{3},3} = a\overline{A^3_3}\ \overline{A^3_3}A^3_3 h_{\bar{3},\bar{3},3}$. Since the equation (17) does not contain $|z_3|^2\overline{z_3}$, we have $h_{\bar{3},\bar{3},3} = 0$, which implies $h'_{\bar{3},\bar{3},3} = 0$. Substituting $h_{\bar{2},\bar{3},3} = 1$ and components of $\mathcal{A}_1$ into the transformation rule of $h_{\bar{1},\bar{3},3}$, we have

$$h'_{\bar{1},\bar{3},3} = a|A^3_3|^2\, \overline{A^1_2} e^{i\alpha} \left\{ \frac{A^2_2}{A^1_2}\, h_{\bar{1},\bar{3},3} + 1 \right\} .$$

By putting $A^1_2 = -A^2_2\, h_{\bar{1},\bar{3},3}$, we have $h'_{\bar{1},\bar{3},3} = 0$. We can take

$$\overline{A^2_2} = \frac{1}{\sqrt{a(1 - |h_{\bar{1},\bar{3},3}|^2)}}$$

for some β by the equation (14).

Next consider $h_{\bar{1},\bar{2},3}$.

$$\begin{aligned} h'_{\bar{1},\bar{2},3} &= aA^3_3\Big(\overline{A^1_1}\,\overline{A^1_2}\, h_{\bar{1},\bar{1},3} + \overline{A^1_1}\,\overline{A^2_2}\, h_{\bar{1},\bar{2},3} + \overline{A^1_1}\,\overline{A^3_2}\, h_{\bar{1},\bar{3},3} + \overline{A^2_1}\,\overline{A^1_2}\, h_{\bar{1},\bar{2},3} + \\ &\qquad + \overline{A^2_1}\,\overline{A^3_2} + \overline{A^3_1}\,\overline{A^1_2}\, h_{\bar{1},\bar{3},3} + \overline{A^3_1}\,\overline{A^2_2}\Big) = \\ &= A^3_3\, \frac{e^{i\alpha}\overline{h_{\bar{1},\bar{3},3}}}{(1 - |h_{\bar{1},\bar{3},3}|^2) h_{\bar{1},\bar{3},3}} \left(-h_{\bar{1},\bar{1},3}\overline{h_{\bar{1},\bar{3},3}} + h_{\bar{1},\bar{2},3} + |h_{\bar{1},\bar{3},3}|^2 h_{\bar{1},\bar{2},3} \right) + \\ &\qquad + A^3_3\, \overline{A^3_1}\, \sqrt{a(1 - |h_{\bar{1},\bar{3},3}|^2)}\, . \end{aligned}$$

Therefore putting

$$\overline{A^3_1} = \frac{e^{i\alpha}\overline{h_{\bar{1},\bar{3},3}}}{(1 - |h_{\bar{1},\bar{3},3}|^2)^{3/2}\, h_{\bar{1},\bar{3},3}\, \sqrt{a}} \left\{ h_{\bar{1},\bar{1},3}\overline{h_{\bar{1},\bar{3},3}} - h_{\bar{1},\bar{2},3}(1 + |h_{\bar{1},\bar{3},3}|^2) \right\} ,$$

we have $h'_{\bar{1},\bar{2},3} = 0$.

The transformation rule of $h_{\bar{2},\bar{2},3}$ is

$$\begin{aligned} h'_{\bar{2},\bar{2},3} &= aA^3_3 \left(\overline{A^1_2}\,\overline{A^1_2}\, h_{\bar{1},\bar{1},3} + 2\overline{A^1_2}\,\overline{A^2_2}\, h_{\bar{1},\bar{2},3} + 2\overline{A^1_2}\,\overline{A^3_2}\, h_{\bar{1},\bar{3},3} + 2\overline{A^2_2}\,\overline{A^3_2} \right) = \\ &= \frac{A^3_3\, \overline{h_{\bar{1},\bar{3},3}}}{1 - |h_{\bar{1},\bar{3},3}|^2} \left(\overline{h_{\bar{1},\bar{3},3}}\, h_{\bar{1},\bar{1},3} - 2h_{\bar{1},\bar{2},3} \right) + \frac{2aA^3_3\overline{A^3_2}}{\sqrt{a(1 - |h_{\bar{1},\bar{3},3}|^2)}} (1 - |h_{\bar{1},\bar{3},3}|^2)\, . \end{aligned}$$

We can solve $h'_{\bar{2},\bar{2},3} = 0$ as

$$\overline{A^3_2} = -\frac{\overline{h_{\bar{1},\bar{3},3}}}{2\sqrt{a}(1 - |h_{\bar{1},\bar{3},3}|^2)^{3/2}} \left(h_{\bar{1},\bar{1},3}\, \overline{h_{\bar{1},\bar{3},3}} - 2h_{\bar{1},\bar{2},3} \right) .$$

Substituting $h_{\bar{2},\bar{3},3} = 1$ and components of $\mathcal{A}_1$ into a transformation rule of $h_{\bar{2},\bar{3},3}$, we have

$$h'_{\bar{2},\bar{3},3} = |A^3_3|^2\, \sqrt{a(1 - |h_{\bar{1},\bar{3},3}|^2)}\, .$$

By putting

$$a = \frac{1}{|A^3_3|^4(1 - |h_{\bar{1},\bar{3},3}|^2)} ,$$

we have $h'_{\bar{2},\bar{3},3} = 1$. Since $h'_{\bar{1},\bar{1},3}$ is calculated as

$$h'_{\bar{1},\bar{1},3} = \frac{(\overline{h_{\bar{1},\bar{3},3}})^2 e^{2i\alpha}}{(h_{\bar{1},\bar{3},3})^2(1 - |h_{\bar{1},\bar{3},3}|^2)} \left(h_{\bar{1},\bar{1},3} - 2h_{\bar{1},\bar{3},3}h_{\bar{1},\bar{2},3} \right) ,$$

we may take $h'_{\bar{1},\bar{1},3} \in \mathbb{R}$ for a suitable choice of α. Therefore we conclude that the equation (17) is transformed into

$$\operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re} \overline{z_3}(\lambda z_1^2 + z_2 z_3) + \{\text{higher order terms}\} \ , \ \ \lambda \in \mathbb{R}\, .$$

(I-ii) $|h_{\bar{1},\bar{3},3}| = 1$. In this case, we use the matrix in $\mathcal{A}_2$ with $\beta = \pi$. We list the transformation rules of $h_{\bar{\alpha},\bar{\beta},\gamma}$.

$$h'_{\bar{1},\bar{1},3} = a\overline{A^2_1}\,\overline{A^3_1}\, A^3_3\, h_{\bar{2},\bar{3},3} = -\frac{a}{\sqrt{-a}}\, \overline{A^3_1} A^3_3\, , \tag{22}$$

$$\begin{aligned} h'_{\bar{1},\bar{2},3} &= aA_3^3(\overline{A_1^2}\,\overline{A_2^1}\,h_{\bar{1},\bar{2},3} + \overline{A_1^2}\,\overline{A_2^3}\,h_{\bar{2},\bar{3},3} + \overline{A_1^3}\,\overline{A_2^1}\,h_{\bar{1},\bar{3},3}) = \\ &= aA_3^3\left\{\frac{e^{i\alpha}}{a}\,h_{\bar{1},\bar{2},3} - \frac{1}{\sqrt{-a}}\,\overline{A_2^3} + \frac{e^{i\alpha}}{\sqrt{-a}}\,\overline{A_1^3}\,h_{\bar{1},\bar{3},3}\right\}, \end{aligned} \tag{23}$$

$$h'_{\bar{1},\bar{3},3} = a\overline{A_1^2}\,\overline{A_3^3}\,A_3^3\,h_{\bar{2},\bar{3},3} = -\frac{a}{\sqrt{-a}}\,|A_3^3|^2\,, \tag{24}$$

$$\begin{aligned} h'_{\bar{2},\bar{2},3} &= aA_3^3(\overline{A_2^1}\,\overline{A_2^1}\,h_{\bar{1},\bar{1},3} + 2\overline{A_2^1}\,\overline{A_2^3}\,h_{\bar{1},\bar{3},3}) = \\ &= aA_3^3\,\overline{A_2^1}\left\{\frac{e^{i\alpha}}{\sqrt{-a}}\,h_{\bar{1},\bar{1},3} + 2\overline{A_2^3}\,h_{\bar{1},\bar{3},3}\right\}, \end{aligned} \tag{25}$$

$$h'_{\bar{2},\bar{3},3} = a\overline{A_2^1}\,\overline{A_3^3}\,A_3^3\,h_{\bar{1},\bar{3},3} = a\frac{e^{i\alpha}}{\sqrt{-a}}\,|A_3^3|^2\,h_{\bar{1},\bar{3},3}\,, \tag{26}$$

$$h'_{\bar{3},\bar{3},3} = a\overline{A_3^3}\,\overline{A_3^3}\,A_3^3\,h_{\bar{3},\bar{3},3}\,. \tag{27}$$

Since there is no term $\overline{z_3}\,\overline{z_3}\,z_3$ in the equation (17), we have $h'_{\bar{3},\bar{3},3} = 0$. Putting

$$\overline{A_2^3} = -\frac{e^{i\alpha}h_{\bar{1},\bar{1},3}}{2h_{\bar{1},\bar{3},3}\sqrt{-a}}\,,$$

we have $h'_{\bar{2},\bar{2},3} = 0$. Substitute $\overline{A_2^3}$ into (23) and put

$$\overline{A_1^3} = -\frac{1}{2(h_{\bar{1},\bar{3},3})^2\sqrt{-a}}\,(h_{\bar{1},\bar{1},3} - 2h_{\bar{1},\bar{2},3}\,h_{\bar{1},\bar{3},3})$$

to get $h'_{\bar{1},\bar{2},3} = 0$. By taking

$$a = -\frac{1}{|A_3^3|^4}\quad,\quad \alpha = \pi - \arg(h_{\bar{1},\bar{3},3})\,,$$

we have $h'_{\bar{2},\bar{3},3} = 1$. Substitute a into (24). Then we have $h'_{\bar{1},\bar{3},3} = 1$. Substitution a and $\overline{A_1^3}$ into (22) yields

$$h'_{\bar{1},\bar{1},3} = -\frac{A_3^3}{2(h_{\bar{1},\bar{3},3})^2}\,(h_{\bar{1},\bar{1},3} - 2h_{\bar{1},\bar{2},3}\,h_{\bar{1},\bar{3},3})\,.$$

For a suitable choice of A_3^3, we may take $h'_{\bar{1},\bar{1},3} \in \mathbb{R}$.

Therefore we conclude that the equation (17) is transformed into

$$\operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re}\overline{z_3}(\lambda z_1^2 + z_2 z_3 + z_1 z_3) + \{\text{higher order terms}\}\ ,\ \ \lambda \in \mathbb{R}\,.$$

(I-iii) $h_{\bar{1},\bar{3},3} = 0$. In this case we use the matrix in $\mathcal{A}_3$ with $\beta = 0$ and $\overline{A_2^3} = 0$. The transformation rules are the following.

$$h'_{\bar{1},\bar{1},3} = a\overline{A_1^1}\,\overline{A_1^1}\,A_3^3\,h_{\bar{1},\bar{1},3} = e^{2i\alpha}A_3^3\,h_{\bar{1},\bar{1},3}\,, \tag{28}$$

$$\begin{aligned} h'_{\bar{1},\bar{2},3} &= aA_3^3(\overline{A_1^1}\,\overline{A_2^2}\,h_{\bar{1},\bar{2},3} + \overline{A_1^3}\,\overline{A_2^2}\,h_{\bar{2},\bar{3},3}) = \\ &= A_3^3\sqrt{a}\left\{\frac{e^{i\alpha}}{\sqrt{a}}\,h_{\bar{1},\bar{2},3} + \overline{A_1^3}\right\}, \end{aligned} \tag{29}$$

$$h'_{\bar{1},\bar{3},3} = a\overline{A_1^2}\,\overline{A_3^3}\,A_3^3\,h_{\bar{2},\bar{3},3}\,, \tag{30}$$

$$h'_{\bar{2},\bar{2},3} = 2a\overline{A_2^2}\,\overline{A_2^3}\,A_3^3\,h_{\bar{2},\bar{3},3} = 0\,, \tag{31}$$

$$h'_{\bar{2},\bar{3},3} = a\overline{A_2^2}\,\overline{A_3^3}\,A_3^3\,h_{\bar{2},\bar{3},3} = \sqrt{a}\,|A_3^3|^2\,, \tag{32}$$

$$h'_{\bar{3},\bar{3},3} = a\overline{A_3^3}\,\overline{A_3^3}\,A_3^3\,h_{\bar{3},\bar{3},3}\ . \tag{33}$$

Substitute $h_{\bar{3},\bar{3},3} = 0$ and $A_1^2 = 0$, we have $h'_{\bar{3},\bar{3},3} = h'_{\bar{1},\bar{3},3} = 0$. Taking

$$\overline{A_1^3} = -e^{i\alpha}\,|A_3^3|^2\,h_{\bar{1},\bar{2},3}\ \ ,\ \ a = \frac{1}{|A_3^3|^4}\ ,$$

we have $h'_{\bar{2},\bar{3},3} = 1$, $h'_{\bar{1},\bar{2},3} = h'_{\bar{2},\bar{2},3} = 0$. For a suitable α in the transformation rule for $h_{\bar{1},\bar{1},3}$, we can take $h'_{\bar{1},\bar{1},3} \in \mathbb{R}$. Therefore we conclude that the equation (17) is transformed into

$$\operatorname{Im} w = < z', \overline{z'} > +2\operatorname{Re}\overline{z_3}(\lambda z_1^2 + z_2 z_3) + \{\text{higher order terms}\}\ \ ,\ \ \lambda \in \mathbb{R}\ .$$

(II) The equation (18).

(II-i) $h_{\bar{1},\bar{3},3} \neq 0$ In this case, we use the matrix in $\mathcal{A}_2$.

$$h'_{\bar{1},\bar{1},3} = a\overline{A_1^2}\,\overline{A_1^2}\,A_3^3\,h_{\bar{2},\bar{2},3}\ , \tag{34}$$

$$\begin{aligned} h'_{\bar{1},\bar{2},3} &= aA_3^3\,(\overline{A_1^2}\,\overline{A_2^1}\,h_{\bar{1},\bar{2},3} + \overline{A_1^3}\,\overline{A_2^1}\,h_{\bar{1},\bar{3},3}) = \\ &= a\overline{A_2^1}\,A_3^3\left\{\frac{e^{i\beta}}{\sqrt{-a}}\,h_{\bar{1},\bar{2},3} + \overline{A_1^3}\,h_{\bar{1},\bar{3},3}\right\}, \end{aligned} \tag{35}$$

$$h'_{\bar{1},\bar{3},3} = aA_3^3\,(\overline{A_1^2}\,\overline{A_3^3}\,h_{\bar{2},\bar{3},3} + \overline{A_1^3}\,\overline{A_3^3}\,h_{\bar{3},\bar{3},3})\ , \tag{36}$$

$$\begin{aligned} h'_{\bar{2},\bar{2},3} &= aA_3^3\,(\overline{A_2^1}\,\overline{A_2^1}\,h_{\bar{1},\bar{1},3} + 2\overline{A_2^1}\,\overline{A_2^3}\,h_{\bar{1},\bar{3},3}) = \\ &= a\overline{A_2^1}\,A_3^3\left\{\frac{e^{i\alpha}}{\sqrt{-a}}\,h_{\bar{1},\bar{1},3} + 2\overline{A_2^3}\,h_{\bar{1},\bar{3},3}\right\}, \end{aligned} \tag{37}$$

$$h'_{\bar{2},\bar{3},3} = a\overline{A_2^1}\,\overline{A_3^3}\,A_3^3\,h_{\bar{1},\bar{3},3} = a\frac{e^{i\alpha}}{\sqrt{-a}}\,|A_3^3|^2\,h_{\bar{1},\bar{3},3} \tag{38}$$

$$h'_{\bar{3},\bar{3},3} = a\overline{A_3^3}\,\overline{A_3^3}\,A_3^3\,h_{\bar{3},\bar{3},3}\ . \tag{39}$$

Substitute $h_{\bar{3},\bar{3},3} = h_{\bar{2},\bar{3},3} = h_{\bar{2},\bar{2},3} = 0$ into (39), (36) and (34). Taking

$$\overline{A_2^3} = \frac{|A_3^3|^2}{2}\,h_{\bar{1},\bar{1},3}\ \ ,\ \ \overline{A_1^3} = -\frac{e^{i\beta}h_{\bar{1},\bar{2},3}}{h_{\bar{1},\bar{3},3}}\,|A_3^3|^2\,|h_{\bar{1},\bar{3},3}|\ ,$$

$$a = -\frac{1}{|A_3^3|^4|\,h_{\bar{1},\bar{3},3}|^2}\ \ ,\ \ \alpha = \arg h_{\bar{1},\bar{3},3} - \pi\ ,$$

we have $h'_{\bar{2},\bar{3},3} = 1$ and the others are zero. This implies that the equation (18) can be transformed into

$$\operatorname{Im} w = < z', \overline{z'} > +2\operatorname{Re}\overline{z_3}z_2z_3 + \{\text{higher order terms}\}\ .$$

(II-ii) $h_{\bar{1},\bar{3},3} = 0$. In this case we use the matrix in $\mathcal{A}_3$.

By the transformation rule, we have

$$h'_{\bar{1},\bar{1},3} = a\overline{A_1^1}\,\overline{A_1^1}\,A_3^3\,h_{\bar{1},\bar{1},3} = e^{2i\alpha}A_3^3\,h_{\bar{1},\bar{1},3}\ , \tag{40}$$

$$h'_{\bar{1},\bar{2},3} = a\overline{A_1^1}\,\overline{A_2^2}\,A_3^3\,h_{\bar{1},\bar{2},3} = e^{i(\alpha+\beta)}A_3^3\,h_{\bar{1},\bar{2},3}\ , \tag{41}$$

and the others are zero. Taking

$$A_3^3 \in \mathbb{R}\ \ ,\ \ \alpha = -\frac{1}{2}\arg h_{\bar{1},\bar{1},3}\ \ ,\ \ \beta = \frac{1}{2}\arg h_{\bar{1},\bar{1},3} - \arg h_{\bar{1},\bar{2},3}\ ,$$

we have $h'_{\bar{1},\bar{1},3} \in \mathbb{R}$, $h'_{\bar{1},\bar{2},3} \in \mathbb{R}$, which allows us to write the equation (18) as

$$\operatorname{Im} w = < z', \overline{z'} > +2\operatorname{Re}\overline{z_3}(\lambda z_1^2 + \nu z_1 z_2) + \{\text{higher order terms}\}\ \ ,\ \ \lambda, \nu \in \mathbb{R}\ .$$

2-nondegeneracy implies that at least one of λ and ν is non-zero.

(III) The equation (19).

(III-i) $h_{\bar{1},\bar{3},3} \neq 0$. In this case, we use the matrix $\mathcal{A}_2$. By the transformation rule, we have $h'_{\bar{1},\bar{3},3} = h'_{\bar{3},\bar{3},3} = 0$ and

$$h'_{\bar{1},\bar{1},3} = a\overline{A_1^2}\,\overline{A_1^2}\,A_3^3\,h_{\bar{2},\bar{2},3} = -e^{2i\beta}A_3^3\ , \tag{42}$$

$$\begin{aligned} h'_{\bar{1},\bar{2},3} &= aA_3^3\,(\overline{A_1^2}\,\overline{A_2^1}\,h_{\bar{1},\bar{2},3} + \overline{A_1^3}\,\overline{A_2^1}\,h_{\bar{1},\bar{3},3}) = \\ &= aA_3^3\,\overline{A_2^1}\left\{\frac{e^{i\beta}}{\sqrt{-a}}\,h_{\bar{1},\bar{2},3} + \overline{A_1^3}\,h_{\bar{1},\bar{3},3}\right\}\ , \end{aligned} \tag{43}$$

$$\begin{aligned} h'_{\bar{2},\bar{2},3} &= aA_3^3\,(\overline{A_2^1}\,\overline{A_2^1}\,h_{\bar{1},\bar{1},3} + 2\overline{A_2^1}\,\overline{A_2^3}\,h_{\bar{1},\bar{3},3}) = \\ &= aA_3^3\,\overline{A_2^1}\left\{\frac{e^{i\alpha}}{\sqrt{-a}}\,h_{\bar{1},\bar{1},3} + 2\overline{A_2^3}\,h_{\bar{1},\bar{3},3}\right\}\ , \end{aligned} \tag{44}$$

$$h'_{\bar{2},\bar{3},3} = a\overline{A_2^1}\,\overline{A_3^3}\,A_3^3\,h_{\bar{1},\bar{3},3} = -\sqrt{-a}\,e^{i\alpha}\,|A_3^3|^2\,h_{\bar{1},\bar{3},3}\ . \tag{45}$$

By putting

$$A_3^3 = -e^{-2i\beta}\ \ ,\ \ \overline{A_1^3} = -\frac{e^{i\beta}\,h_{\bar{1},\bar{2},3}\,|h_{\bar{1},\bar{3},3}|}{h_{\bar{1},\bar{3},3}}\ \ ,\ \ \overline{A_2^3} = -\frac{e^{i\alpha}\,h_{\bar{1},\bar{1},3}\,|h_{\bar{1},\bar{3},3}|}{2h_{\bar{1},\bar{3},3}}\ ,$$

$$a = -\frac{1}{|h_{\bar{1},\bar{3},3}|^2}\ \ ,\ \ \alpha = \pi - \arg h_{\bar{1},\bar{3},3}\ ,$$

we have $h'_{\bar{1},\bar{1},3} = h'_{\bar{2},\bar{3},3} = 1$, $h'_{\bar{1},\bar{2},3} = h'_{\bar{2},\bar{2},3} = 0$. Therefore the equation (19) can be transformed into

$$\operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re}\overline{z_3}(z_1^2 + z_2z_3) + \{\text{higher order terms}\}\ .$$

(III-ii) $h_{\bar{1},\bar{3},3} = 0$. In this case, we use the matrix in $\mathcal{A}_2$. By the transformation rule, we have $h'_{\bar{1},\bar{3},3} = h'_{\bar{2},\bar{3},3} = h'_{\bar{3},\bar{3},3} = 0$ and

$$h'_{\bar{1},\bar{1},3} = a\overline{A_1^2}\,\overline{A_1^2}\,A_3^3\,h_{\bar{2},\bar{2},3} = -e^{2i\beta}A_3^3\ , \tag{46}$$

$$h'_{\bar{1},\bar{2},3} = a\overline{A_2^1}\,\overline{A_1^2}\,A_3^3\,h_{\bar{1},\bar{2},3} = -e^{i(\alpha+\beta)}A_3^3\,h_{\bar{1},\bar{2},3}\ , \tag{47}$$

$$h'_{\bar{2},\bar{2},3} = a\overline{A_2^1}\,\overline{A_2^1}\,A_3^3\,h_{\bar{1},\bar{1},3} = -e^{2i\alpha}A_3^3\,h_{\bar{1},\bar{1},3}\ . \tag{48}$$

By putting $A_3^3 = -e^{-2i\beta}$, $\alpha - \beta = -\arg h_{\bar{1},\bar{2},3}$, we have $h'_{\bar{1},\bar{1},3} = 1$, $h'_{\bar{1},\bar{2},3} \in \mathbb{R}$. Therefore the equation (19) is transformed into

$$\operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re}\overline{z_3}(z_1^2 + h'_{\bar{2},\bar{2},3}z_2^2 + h'_{\bar{1},\bar{2},3}z_1z_2) + \{\text{higher order terms}\}\ .$$

Denote $\theta = \arg h'_{\bar{2},\bar{2},3}$ and change z_2 to $z_2e^{i\theta/2}$. Then we may take $h'_{\bar{1},\bar{2},3} \in \mathbb{C}$, $h'_{\bar{2},\bar{2},3} \in \mathbb{R}$. Therefore we conclude that the equation (19) is transformed into

$$\begin{aligned} \operatorname{Im} w\ = < z', \overline{z'} > \ + \quad & 2\operatorname{Re}\overline{z_3}(z_1^2 + \mu z_2^2 + \nu z_1z_2) + \\ & + \{\text{higher order terms}\}\ \ ,\ \ \nu \in \mathbb{C},\ \mu \in \mathbb{R}\ . \end{aligned}$$

(IV) The equation (20).

In this case, we use the matrix in $\mathcal{A}_2$ with $\overline{A_1^3} = 0$. By the transformation rule, we have

$$h'_{\bar{1},\bar{1},3} = a\overline{A_1^3}\,\overline{A_1^3}\,A_3^3\,h_{\bar{3},\bar{3},3} = 0\ , \tag{49}$$

$$h'_{\bar{1},\bar{2},3} = a\overline{A_1^2}\,\overline{A_2^1}\,A_3^3\,h_{\bar{1},\bar{2},3} = -e^{i(\alpha+\beta)}\,A_3^3\,h_{\bar{1},\bar{2},3}\ , \tag{50}$$

$$h'_{\bar{1},\bar{3},3} = a\overline{A_1^3}\,\overline{A_3^3}\,A_3^3\,h_{\bar{3},\bar{3},3} = 0\ , \tag{51}$$

(52)
$$\begin{aligned} h'_{\bar 2,\bar 2,3} &= aA_3^3(\overline{A_2^1}\,\overline{A_2^1}\,h_{\bar 1,\bar 1,3} + 2\overline{A_2^1}\,\overline{A_2^3}\,h_{\bar 1,\bar 3,3} + \overline{A_2^3}\,\overline{A_2^3}\,h_{\bar 3,\bar 3,3}) = \\ &= aA_3^3\left\{-\frac{e^{2i\alpha}}{a}\,h_{\bar 1,\bar 1,3} + 2\,\frac{e^{i\alpha}\,\overline{A_2^3}}{\sqrt{-a}}\,h_{\bar 1,\bar 3,3} + (\overline{A_2^3})^2\right\}, \end{aligned}$$

(53)
$$\begin{aligned} h'_{\bar 2,\bar 3,3} &= aA_3^3\,(\overline{A_2^1}\,\overline{A_3^3}\,h_{\bar 1,\bar 3,3} + \overline{A_2^3}\,\overline{A_3^3}\,h_{\bar 3,\bar 3,3}) = \\ &= a|A_3^3|^2\left\{\frac{e^{i\alpha}}{\sqrt{-a}}\,h_{\bar 1,\bar 3,3} + \overline{A_2^3}\right\}, \end{aligned}$$

(54)
$$h'_{\bar 3,\bar 3,3} = a\overline{A_3^3}\,\overline{A_3^3}\,A_3^3\,h_{\bar 3,\bar 3,3}\,.$$

By putting

$$\overline{A_2^3} = -\frac{e^{i\alpha}}{\sqrt{-a}}\,h_{\bar 1,\bar 3,3}\ ,\ A_3^3\in\mathbb{R}\ ,\ a=\frac{1}{(A_3^3)^3}\ ,\ \alpha = -\frac{1}{2}\arg\left(-h_{\bar 1,\bar 1,3} + (h_{\bar 1,\bar 3,3})^2\right)\ ,$$

$$\beta = \frac{1}{2}\arg\left(-h_{\bar 1,\bar 1,3} + (h_{\bar 1,\bar 3,3})^2\right) - \arg h_{\bar 1,\bar 2,3}\ ,$$

we have $h'_{\bar 3,\bar 3,3}=1$, $h'_{\bar 1,\bar 1,3}=h'_{\bar 1,\bar 3,3}=h'_{\bar 2,\bar 3,3}=0$, $h'_{\bar 1,\bar 2,3}\in\mathbb{R}$, $h'_{\bar 2,\bar 2,3}\in\mathbb{R}$. Therefore the equation (20) is transformed into

$$\operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re}\overline{z_3}(\mu z_2^2 + z_3^2 + \nu z_1 z_2) + \{\text{higher order terms}\}\ ,\ \nu,\mu\in\mathbb{R}\,.$$

(V) The equation (21). In this case, we use the matrix in $\mathcal{A}_2$ with $\beta=0$, $\overline{A_1^3}=0$, $\overline{A_3^3}=-1$ and $a=-1$.

(55)
$$h'_{\bar 1,\bar 1,3} = a\overline{A_1^2}\,\overline{A_1^2}\,A_3^3\,h_{\bar 2,\bar 2,3} = -A_3^3 = 1\,,$$

(56)
$$h'_{\bar 1,\bar 2,3} = a\overline{A_1^2}\,\overline{A_2^1}\,A_3^3\,h_{\bar 1,\bar 2,3} = e^{i\alpha}h_{\bar 1,\bar 2,3}\,,$$

(57)
$$h'_{\bar 1,\bar 3,3} = a\overline{A_1^3}\,\overline{A_3^3}\,A_3^3\,h_{\bar 3,\bar 3,3} = 0\,,$$

(58)
$$\begin{aligned} h'_{\bar 2,\bar 2,3} &= aA_3^3\,(\overline{A_2^1}\,\overline{A_2^1}\,h_{\bar 1,\bar 1,3} + \overline{A_2^3}\,\overline{A_2^3}\,h_{\bar 3,\bar 3,3}) = \\ &= e^{2i\alpha}\,h_{\bar 1,\bar 1,3} + (\overline{A_2^3})^2\,, \end{aligned}$$

(59)
$$\begin{aligned} h'_{\bar 2,\bar 3,3} &= aA_3^3(\overline{A_2^3}\,\overline{A_3^3}\,h_{\bar 3,\bar 3,3} + \overline{A_2^1}\,\overline{A_3^3}\,h_{\bar 1,\bar 3,3}) = \\ &= -\overline{A_2^3} - e^{i\alpha}\,h_{\bar 1,\bar 3,3}\,, \end{aligned}$$

(60)
$$h'_{\bar 3,\bar 3,3} = a\overline{A_3^3}\,\overline{A_3^3}A_3^3\,h_{\bar 3,\bar 3,3} = h_{\bar 3,\bar 3,3}\,.$$

Putting

$$\overline{A_2^3} = -\frac{e^{i\alpha}}{\sqrt{-a}}\,h_{\bar 1,\bar 3,3}\ ,\ \alpha = -\arg h_{\bar 1,\bar 2,3}\,,$$

we have $h'_{\bar 3,\bar 3,3}=h'_{\bar 1,\bar 1,3}=1$, $h'_{\bar 2,\bar 3,3}=h'_{\bar 1,\bar 3,3}=0$, $h'_{\bar 1,\bar 2,3}\in\mathbb{R}$, $h'_{\bar 2,\bar 2,3}\in\mathbb{C}$. Therefore the equation (21) is transformed into

$$\begin{aligned} \operatorname{Im} w\ = < z', \overline{z'} > \quad &+ 2\operatorname{Re}\overline{z_3}(z_1^2 + h'_{\bar 2,\bar 2,3}\,z_2^2 + z_3^2 + h'_{\bar 1,\bar 2,3}\,z_1 z_2) + \\ &+\{\text{higher order terms}\}\,. \end{aligned}$$

Denote $\theta = \arg h'_{\bar 2,\bar 2,3}$ and change z_2 to $z_2 e^{i\theta/2}$. Then we have $h'_{\bar 1,\bar 2,3}\in\mathbb{C}$, $h'_{\bar 2,\bar 2,3}\in\mathbb{R}$. Therefore we conclude that the equation (21) is transformed into

$$\begin{aligned} \operatorname{Im} w\ = < z', \overline{z'} > \quad &+ 2\operatorname{Re}\overline{z_3}\,(z_1^2 + \mu z_2^2 + z_3^2 + \nu z_1 z_2) + \\ &+\{\text{higher order terms}\}\ ,\ \nu\in\mathbb{C}\,,\ \mu\in\mathbb{R}\,. \end{aligned}$$

We have proved that equations (17), ..., (21) are partially normalized into one of the following forms.

(I) $$\operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re} \overline{z_3}(\lambda z_1^2 + z_2 z_3) + \{\text{higher order terms}\}$$

(II) $$\operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re} \overline{z_3}(\lambda z_1^2 + z_2 z_3 + z_1 z_3) + \{\text{higher order terms}\}$$

(III) $$\operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re} \overline{z_3}(\lambda z_1^2 + \mu z_2^2 + \nu z_1 z_2) + \{\text{higher order terms}\}$$

(IV) $$\operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re} \overline{z_3}(z_1^2 + \mu z_2^2 + z_3^2 + \nu z_1 z_2) + \{\text{higher order terms}\}$$

(V) $$\operatorname{Im} w = < z', \overline{z'} > + 2\operatorname{Re} \overline{z_3}(\mu z_2^2 + z_3^2 + \nu z_1 z_2) + \{\text{higher order terms}\}$$

Here $\lambda, \mu \in \mathbb{R}$ and $\nu \in \mathbb{C}$.

4. Normalization in higher order terms, the cases (I), (II), (IV) and (V)

In this section, we normalize higher order terms in (I), (II), (IV) and (V). Denote the third order terms in equation numbered (j) by $2\operatorname{Re} \overline{z_3} p_j$, $j = I, \ldots, V$. For example, $p_I = \lambda z_1^2 + z_2 z_3$. Since p_{III} does not contain the variable z_3, we distinguish it from others. First, we define some notation. Let $f' = (f^1, f^2)$ and $f = (f', f^3)$. Let $F_{k,l}(z, \overline{z}, s)$ be a homogeneous formal power series of type (k, l), i.e. $F_{k,l}(t_1 z, t_2 \overline{z}, s) = t_1^k t_2^l F_{k,l}(z, \overline{z}, s)$. In what follows, $F_{k,l}, G_{k,l}, H_{k,l}, K_{k,l}, N_{k,l}$ denote formal power series of type (k, l). z and $\overline{z}$ are assigned the weight one and w and s are assigned two. Define the spaces of formal power series $\mathcal{F}$ and $\mathcal{G}^1$ by

$$\mathcal{F} = \Big\{ F(z, \overline{z}, s) = \sum_{k,l} F_{k,l}(z, \overline{z}, s) \Big| F = O(4),\ F_{k,l}(t_1 z, t_2 \overline{z}, s) = t_1^k t_2^l F_{k,l}(z, \overline{z}, s) \text{ and } F_{k,l}(z, \overline{z}, s) = \overline{F_{l,k}(z, \overline{z}, s)} \Big\},$$

$$\mathcal{G}^1 = \Big\{ (f, g)(z, w) \Big| f^l = O(3),\ f^3 = O(2),\ g = O(4), \quad \text{constant terms of } \frac{\partial^3 f^l}{\partial z^\alpha}, \frac{\partial^2 f^3}{\partial z^\beta}, \frac{\partial f^l}{\partial w}, \frac{\partial f^j}{\partial z_k}, \operatorname{Re} \frac{\partial^2 f^l}{\partial z_l \partial w}, \frac{\partial^2 f^1}{\partial z_2 \partial w} - \overline{\left(\frac{\partial^2 f^2}{\partial z_1 \partial w} \right)} \text{ vanish for } j, k = 1, 2, 3,\ l = 1, 2,\ |\alpha| = 3,\ |\beta| = 2 \Big\}.$$

Let $N^j(z, \overline{z}, s)$ be a formal power series which can be expressed by

$$N^j(z, \overline{z}, s) = \sum_{\min(k,l) \geq 1} N^j_{kl}(z, \overline{z}, s).$$

Let $\mathcal{N}^j \subset \mathcal{F}$, $j = I, \ldots, V$ be the spaces of formal power series whose elements are $N^j(z, \overline{z}, s)$ satisfying the following conditions.
For $j = I, II, IV, V$,

(61) $$N^j_{k,1} = \{ \overline{z_1} H_{k,0} = 0,\ \overline{z_2} G_{k,0} = 0,\ \overline{z_2} \frac{\partial p_j}{\partial z_3} K_{k-1,0} = 0,\ k \geq 4 \}$$

(62) $$N^j_{4,2} = \{ \overline{z_1} < z', \overline{z'} > \quad H_{3,0} = 0\ ,\ \overline{z_2} < z', \overline{z'} > G_{3,0} = 0\ , \quad \overline{z_2} \frac{\partial p_j}{\partial z_3} < z', \overline{z'} > K_{2,0} = 0 \}$$

(63) $$N^j_{3,1} = \{ < z', \overline{z'} > p_j H_{0,0} = 0 \}$$

(64) $$N^j_{3,2} = \{ z_1 < z', \overline{z'} >^2 \quad H_{0,0} = 0\ ,\ z_2 < z', \overline{z'} >^2 G_{0,0} = 0\ , \quad < z', \overline{z'} >^2 p_j K_{0,0} = 0 \}$$

$$N^j_{2,1} = \{\overline{z_1} H_{2,0} = 0 \ , \ \overline{z_2} G_{2,0} = 0\} \tag{65}$$

$$N^j_{1,1} = \{< z', \overline{z'} > H_{0,0} = 0\} \tag{66}$$

$$N^j_{2,2} = \{\overline{z_1} < z', \overline{z'} > H_{1,0} = 0 \ , \ \overline{z_2} < z', \overline{z'} > G_{1,0} = 0\} \tag{67}$$

$$N^j_{3,3} = \{\overline{z_1} < z', \overline{z'} >^2 H_{1,0} = 0 \ , \ \overline{z_2} < z', \overline{z'} >^2 G_{1,0} = 0\} \ . \tag{68}$$

For $j = III$,

$$N^{III}_{k,2} = \{\overline{p_{III}} H_{k,0} = 0 \ , \ k \geq 5\} \tag{69}$$

$$N^{III}_{k,1} = \{\overline{z_1} H_{k,0} = 0 \ , \ \overline{z_2} G_{k,0} = 0 \ , \ k \geq 3\} \tag{70}$$

$$N^{III}_{5,3} = \{\overline{p_{III}} < z', \overline{z'} > H_{4,0} = 0\} \tag{71}$$

$$N^{III}_{4,2} = \{p_{III} < z', \overline{z'} >^2 H_{0,0} = 0\} \tag{72}$$

$$N^{III}_{5,4} = \{\overline{p_{III}} < z', \overline{z'} >^2 H_{3,0} = 0\} \tag{73}$$

$$N^{III}_{4,3} = \{p_{III} < z', \overline{z'} >^2 H_{0,1} = 0\} \tag{74}$$

$$N^{III}_{3,2} = \{\overline{z_1} < z', \overline{z'} > H_{2,0} = 0 \ , \ \overline{z_2} < z', \overline{z'} > G_{2,0} = 0\} \tag{75}$$

$$N^{III}_{2,1} = \{z_1 < z', \overline{z'} > H_{0,0} = 0 \ , \ z_2 < z', \overline{z'} > G_{0,0} = 0\} \tag{76}$$

$$\begin{aligned} N^{III}_{5,5} = \{|z_1|^2 < z', \overline{z'} >^4 \quad H_{0,0} = 0 \ , \ |z_2|^2 < z', \overline{z'} >^4 G_{0,0} = 0 \ , \\ \overline{z_1} z_2 < z', \overline{z'} >^4 K_{0,0} = 0\} \end{aligned} \tag{77}$$

$$N^{III}_{4,4} = \{\overline{z_1} < z', \overline{z'} >^3 H_{1,0} = 0 \ , \ \overline{z_2} < z', \overline{z'} >^3 G_{1,0} = 0\} \tag{78}$$

$$N^{III}_{3,3} = \{\overline{p_{III}} < z', \overline{z'} > H_{2,0} = 0\} \tag{79}$$

$$N^{III}_{1,1} = \{< z', \overline{z'} > H_{0,0} = 0\} \ , \tag{80}$$

For example, $N^j_{1,1}$, $j = I, II, IV, V$, do not contain the terms of the form $(-|z_1|^2 + |z_2|^2) \times$ $\{$a function of $s\}$. First, we consider the equations (I), (II), (IV) and (V). We follow an argument in [5, 6]. We decompose $(f, g) \in \mathcal{G}^1$ and $F, F' \in \mathcal{F}$ into weighted homogeneous parts as

$$f^l(z, w) = \sum_{\mu=3}^{\infty} f^l_\mu(z, w) \ , \quad l = 1, 2 \ ,$$

$$f^3(z, w) = \sum_{\mu=2}^{\infty} f^3_\mu(z, w) \ , \quad g(z, w) = \sum_{\mu=4}^{\infty} g_\mu(z, w) \ ,$$

$$F(z, \overline{z}, s) = \sum_{\mu=4}^{\infty} F_\mu(z, \overline{z}, s) \ , \quad F'(z, \overline{z}, s) = \sum_{\mu=4}^{\infty} F'_\mu(z, \overline{z}, s) \ .$$

The formal power series $F, F' \in \mathcal{F}$ are related as

$$\begin{aligned} \mathrm{Im}(w + g(z, s + i\phi)) = &< z' + f'(z, s + i\phi), \overline{z'} + \overline{f'(z, s + i\phi)} > + \\ &+ 2\mathrm{Re}\left\{(\overline{z_3} + \overline{f^3(z, s + i\phi)}) p_j(z + f(z, s + i\phi))\right\} + \\ &+ F'\left(z + f(z, s + i\phi), \overline{z} + \overline{f(z, s + i\phi)}, \mathrm{Re}(w + g(z, s + i\phi))\right) \ , \end{aligned}$$

where

$$\phi = \phi(z, \overline{z}, s) = < z', \overline{z'} > + 2\mathrm{Re}\, \overline{z_3} p_j(z) + F(z, \overline{z}, s) \ .$$

Identifying terms of weighted degree $\mu \geq 4$, we obtain

$$
\begin{aligned}
F_\mu + \operatorname{Im} g_\mu = \quad & -z_1\overline{f^1_{\mu-1}} - \overline{z_1} f^1_{\mu-1} + z_2\overline{f^2_{\mu-1}} + \overline{z_2} f^2_{\mu-1} + \\
& +2\operatorname{Re}\{p_j(z)\overline{f^3_{\mu-2}} + \overline{z_3} f^3_{\mu-2}\frac{\partial p_j}{\partial z_3}\} + F'_\mu + \dots,
\end{aligned} \tag{81}
$$

where $F_\mu = F_\mu(z,\overline{z},s)$ and $F'_\mu = F'_\mu(z,\overline{z},s+i < z',\overline{z'} >)$ Define the linear operator $L_j : \mathcal{G}^1 \to \mathcal{F}$,

$$
\begin{aligned}
L_j(f',f^3,g) = \quad & -\operatorname{Im} g - z_1\overline{f^1} - \overline{z_1} f^1 + z_2\overline{f^2} + \overline{z_2} f^2 + \\
& +2\operatorname{Re}\{p_j(z)\overline{f^3} + \overline{z_3}\,\frac{\partial p_j}{\partial z_3} f^3\}\Big|_{(z,s+i<z',\overline{z'}>)} .
\end{aligned} \tag{82}
$$

If the equation

$$L_j(f',f^3,g) = F \ \ , \ \ \operatorname{mod} \mathcal{N}^j$$

has a unique solution $(f',f^3,g) \in \mathcal{G}^1$ for any $F \in \mathcal{F}$, then, given any $F' \in \mathcal{F}$, the equation (81) allows us to inductively determine F_μ, and $(f'_{\mu-1}, f^3_{\mu-2}, g_\mu)$. It is sufficient to prove the following.

Lemma 1. *For any $F \in \mathcal{F}$, the equation*

$$L_j(f',f^3,g) = F \ \ , \ \ \operatorname{mod} \mathcal{N}^j$$

has a unique solution $(f',f^3,g) \in \mathcal{G}^1$ for $j = I, II, IV, V$.

Proof. The proof contains three steps. The first step is to pick up terms of type (k,l) from the equation (82). The second step is to get partial differential equations for homogeneous parts of f', f^3, g with initial conditions. Initial conditions come from the fact that $f', f^3, g \in \mathcal{G}^1$. Then, the third step, we solve partial differential equations obtained in the second step.

In what follows, we use the notation

$$f'(z,w) = \frac{\partial f}{\partial w} \ \ , \dots , \ \ f^{(m)}(z,w) = \frac{\partial^m f}{\partial w^m} \ ,$$

$$F_{k,l} = F_{k,l}(z,\overline{z},s) \ , \ \ f_{k,l} = f_{k,l}(z,\overline{z},s) \ , \ \ \overline{f_{k,l}} = \overline{f_{k,l}}(\overline{z},z,s) \ , \ \ \text{etc} .$$

For example,

$$f^1_{1,1} = i < z',\overline{z'} > (f^1_{0,0})' \ \ , \ \ \overline{f^1_{1,1}} = -i < z',\overline{z'} > (\overline{f^1_{0,0}})' \ ,$$

$$f^1_{3,2} = -\frac{< z',\overline{z'} >^2}{2}(f^1_{1,0})'' \ \ , \ \ \text{etc} .$$

Collecting terms of equal types in (82), we obtain the following system of equations for $k \geq 3$.

$$-\frac{1}{2i}\, g_{k,0} = F_{k,0} \tag{83}$$

$$-\frac{1}{2i}\, g_{k+1,1} - \overline{z_1} f^1_{k+1,0} + \overline{z_2} f^2_{k+1,0} + \overline{z_3}\,\frac{\partial p_j}{\partial z_3} f^3_{k,0} = F_{k+1,1} \ \operatorname{mod} N^j_{k+1,1} \tag{84}$$

$$-\frac{1}{2i}\, g_{2,0} + p_j\overline{f^3_{0,0}} = F_{2,0} \tag{85}$$

$$-\frac{1}{2i}\, g_{3,1} - \overline{z_1} f^1_{3,0} + \overline{z_2} f^2_{3,0} + \overline{z_3}\,\frac{\partial p_j}{\partial z_3} f^3_{2,0} + p_j\overline{f^3_{1,1}} = F_{3,1} \ \operatorname{mod} N^j_{3,1} \tag{86}$$

$$-\frac{1}{2i}\, g_{4,2} - \overline{z_1} f^1_{4,1} + \overline{z_2} f^2_{4,1} + \overline{z_3}\,\frac{\partial p_j}{\partial z_3} f^3_{3,1} + p_j\overline{f^3_{2,2}} + \overline{p_j} f^3_{4,0} = F_{4,2} \ \operatorname{mod} N^j_{4,2} \tag{87}$$

$$-\frac{1}{2i}\,g_{1,0}-z_1\overline{f^1_{0,0}}+z_2\overline{f^2_{0,0}}=F_{1,0} \tag{88}$$

$$\begin{aligned}-\frac{1}{2i}\,g_{2,1}-z_1\overline{f^1_{1,1}}-\overline{z_1}f^1_{2,0}\quad &+z_2\overline{f^2_{1,1}}+\overline{z_2}f^2_{2,0}+p_j\overline{f^3_{1,0}}+\overline{z_3}\,\frac{\partial p_j}{\partial z_3}\,f^3_{1,0}=\\ &=F_{2,1}\mod N^j_{2,1}\end{aligned} \tag{89}$$

$$\begin{aligned}-\frac{1}{2i}\,g_{3,2}-z_1\overline{f^1_{2,2}}\quad &-\overline{z_1}f^1_{3,1}+z_2\overline{f^2_{2,2}}+\overline{z_2}f^2_{3,1}+p_j\overline{f^3_{2,1}}+\overline{p_j}f^3_{3,0}+\\ &+\overline{z_3}\,\frac{\partial p_j}{\partial z_3}f^3_{2,1}=F_{3,2}\mod N^j_{3,2}\end{aligned} \tag{90}$$

$$-\mathrm{Im}\,g_{0,0}=F_{0,0} \tag{91}$$

$$-\mathrm{Im}\,g_{1,1}-z_1\overline{f^1_{1,0}}-\overline{z_1}f^1_{1,0}+z_2\overline{f^2_{1,0}}+\overline{z_2}f^2_{1,0}+\overline{z_3}\,\frac{\partial p_j}{\partial z_3}f^3_{0,0}=F_{1,1}\mod N^j_{1,1} \tag{92}$$

$$\begin{aligned}-\mathrm{Im}\,g_{2,2}\quad &-z_1\overline{f^1_{2,1}}-\overline{z_1}f^1_{2,1}+z_2\overline{f^2_{2,1}}+\overline{z_2}f^2_{2,1}+\overline{z_3}\,\frac{\partial p_j}{\partial z_3}\,f^3_{1,1}+p_j\overline{f^3_{2,0}}+\\ &+\overline{p_j}f^3_{2,0}+z_3\overline{\frac{\partial p_j}{\partial z_3}}\overline{f^3_{1,1}}=F_{2,2}\mod N^j_{2,2}\end{aligned} \tag{93}$$

$$\begin{aligned}-\mathrm{Im}\,g_{3,3}-\quad &z_1\overline{f^1_{3,2}}-\overline{z_1}f^1_{3,2}+z_2\overline{f^2_{3,2}}+\overline{z_2}f^2_{3,2}+\overline{z_3}\,\frac{\partial p_j}{\partial z_3}\,f^3_{2,2}+p_j\overline{f^3_{3,1}}+\\ &+\overline{p_j}f^3_{3,1}+z_3\overline{\frac{\partial p_j}{\partial z_3}}\overline{f^3_{2,2}}=F_{3,3}\mod N^j_{3,3}\end{aligned} \tag{94}$$

$g_{k,0}$ is uniquely determined from the equation (83). Since we have

$$-\frac{1}{2i}\,g_{k+1,1}=(-\frac{1}{2i}g_{k,0})'i<z',\overline{z'}>=iF'_{k,0}<z',\overline{z'}>\,,$$

the equation (84) becomes

$$-\overline{z_1}f^1_{k+1,0}+\overline{z_2}f^2_{k+1,0}+\overline{z_3}\,\frac{\partial p_j}{\partial z_3}f^3_{k,0}=F_{k+1,1}+\dots\mod N^j_{k+1,1}\,. \tag{95}$$

We solve this equation mod $N^j_{k+1,1}$, which means that we may assume that $F_{k+1,1}$ is of the form

$$F_{k+1,1}=\overline{z_1}H^1_{k+1,0}+\overline{z_2}H^2_{k+1,0}+\overline{z_3}\,\frac{\partial p_j}{\partial z_3}H^3_{k,0}\,.$$

Then we can solve (95) uniquely for $f^1_{k+1,0}$, $f^2_{k+1,0}$ and $f^3_{k,0}$ for all $k\geq 3$.

Substitute

$$-\frac{1}{2i}\,g_{4,2}=(-\frac{1}{2i}\,g_{2,0})''(-\frac{<z',\overline{z'}>^2}{2})=\frac{<z',\overline{z'}>^2}{2}\,p_j(\overline{f^3_{0,0}})''\,,$$

$$f^3_{3,1}=i<z',\overline{z'}>(f^3_{2,0})'\,,$$

$$f^l_{4,1}=i<z',\overline{z'}>(f^l_{3,0})'\ ,\ \ l=1,2\,,$$

$$\overline{f^3_{2,2}}=-\frac{<z',\overline{z'}>^2}{2}\,(\overline{f^3_{0,0}})''$$

into (87) and calculate mod $N^j_{4,2}$. Since $f^3_{4,0}$ is known from the previous step, the resulting equation is

$$<z',\overline{z'}>\{-\overline{z_1}(f^1_{3,0})'+\overline{z_2}(f^2_{3,0})'+\overline{z_3}\,\frac{\partial p_j}{\partial z_3}\,(f^3_{2,0})'\}=F_{4,2}+\dots\mod N^j_{4,2}\,. \tag{96}$$

By the definition of N^j_{42}, homogeneous parts $(f^1_{3,0})'$, $(f^2_{3,0})'$, and $(f^3_{2,0})'$ are uniquely determined by (96). $f^1_{3,0}$ is uniquely determined given a choice of $f^1_{3,0}(z,0)$. Since $f^1_{3,0}(z,0)$

denotes the part of $f^1(z,0)$ that is homogeneous of degree 3, the choice of $f^1_{3,0}(z,0)$ is equivalent to the choice of the constant term of

$$\frac{\partial^3 f^1}{\partial z^\alpha}\ , \quad |\alpha| = 3\ .$$

Since $f^1,\ f^2$ and f^3 are in $\mathcal{G}^1$, the constant terms of

$$\frac{\partial^3 f^2}{\partial z^\alpha}\ , \quad |\alpha| = 3\ , \quad \frac{\partial^2 f^3}{\partial z^\beta}\ , \quad |\beta| = 2$$

are 0, which implies that $f^1_{3,0},\ f^2_{3,0}$ and $f^3_{2,0}$ are uniquely determined.

Substitute $f^1_{3,0},\ f^2_{3,0}$ obtained above and

$$-\frac{1}{2i}\, g_{3,1} = (-\frac{1}{2i}\, g_{2,0})' i < z', \overline{z'} >= -i < z', \overline{z'} > p_j(\overline{f^3_{0,0}})'\ ,$$

$$\overline{f^3_{1,1}} = -i < z', \overline{z'} > (\overline{f^3_{0,0}})'$$

into (86), then the resulting equation is

$$(97) \qquad < z', \overline{z'} > p_j(\overline{f^3_{0,0}})' = F_{3,1} + \dots \mod N^j_{3,1}\ .$$

It follows from the definition of $N^j_{3,1}$ that $(f^3_{0,0})'$ is uniquely determined by (97). Since the constant term of f^3 is 0, $f^3_{0,0}$ is uniquely determined. Substitute $f^3_{0,0}$ so obtained into (85), then $g_{2,0}$ is uniquely determined.

Next we turn to the equation (89) and (90). We have

$$-\frac{1}{2i}\, g_{3,2} = \frac{< z', \overline{z'} >^2}{2}\,(-z_1(\overline{f^1_{0,0}})'' + z_2(\overline{f^2_{0,0}})'')\ , \quad \overline{f^l_{2,2}} = -\frac{< z', \overline{z'} >^2}{2}\,(\overline{f^l_{0,0}})''\ ,$$

$$f^l_{3,1} = i < z', \overline{z'} > (f^l_{2,0})'\ , \quad \overline{f^3_{2,1}} = -i < z', \overline{z'} > (\overline{f^3_{1,0}})'\ , \quad f^3_{2,1} = i < z', \overline{z'} > (f^3_{1,0})'\ ,$$

$$-\frac{1}{2i}\, g_{2,1} = i < z', \overline{z'} > (z_1(\overline{f^1_{0,0}})' - z_2(\overline{f^2_{0,0}})')\ , \quad \overline{f^l_{1,1}} = -i < z', \overline{z'} > (\overline{f^l_{0,0}})'$$

for $l = 1, 2$. Substitute these and known functions into (89) and (90), then they become

$$(98) \qquad \begin{aligned} &2i < z', \overline{z'} > \{z_1(\overline{f^1_{0,0}})' \quad - z_2(\overline{f^2_{0,0}})'\} - \overline{z_1}(f^1_{2,0}) + i\overline{z_2}(f^2_{2,0}) + \\ &\qquad + p_j \overline{f^3_{1,0}} + \overline{z_3}\, \frac{\partial p_j}{\partial z_3}\, f^3_{1,0} = F_{2,1} + \dots \mod N^j_{2,1}\ , \end{aligned}$$

$$(99) \qquad \begin{aligned} &-i\overline{z_1} < z', \overline{z'} > \quad (f^1_{2,0})' + i\overline{z_2} < z', \overline{z'} > \ (f^2_{2,0})' - i < z', \overline{z'} > p_j\ (\overline{f^3_{1,0}})' + \\ &\qquad + i\overline{z_3} < z', \overline{z'} > \frac{\partial p_j}{\partial z_3}\,(f^3_{1,0})' = F_{3,2} + \dots \mod N^j_{3,2}\ . \end{aligned}$$

Differentiate (98), multiply it by $i < z', \overline{z'} >$ and subtract from (99). Then we obtain

$$(100) \quad < z', \overline{z'} >^2 \{z_1(\overline{f^1_{0,0}})'' - z_2(\overline{f^2_{0,0}})''\} - i < z', \overline{z'} > p_j(\overline{f^3_{1,0}})' = F_{3,2} + \dots \mod N^j_{3,2}\ .$$

By definition of $N^j_{3,2}$, the functions $(f^1_{0,0})''$, $(f^2_{0,0})''$ and $(f^3_{1,0})'$ are uniquely determined. The assumption that $f^1,\ f^2,\ f^3$ are in $\mathcal{G}^1$ implies that the constant terms of

$$\frac{\partial f^l}{\partial w}\ , \quad l = 1, 2 \ \text{ and } \ \frac{\partial f^3}{\partial z_k}\ , \quad k = 1, 2, 3$$

are 0, which also imply that $f^1_{0,0},\ f^2_{0,0}$, and $f^3_{1,0}$ are uniquely determined. Substitute these into (88), then $g_{1,0}$ is determined.

Since we have known $g_{2,1},\ \overline{f^l_{1,1}}$ and $f^3_{1,0}$, the equation (98) becomes

$$-\overline{z_1} f^1_{2,0} + \overline{z_2} f^2_{2,0} = F_{2,1} + \dots \mod N^j_{2,1}\ ,$$

which implies that $f^1_{2,0}$ and $f^2_{2,0}$ are uniquely determined.

We proceed to solve the equations from (91) to (94). Since we have

$$-\mathrm{Im}\, g_{2,2} = -\frac{< z', \overline{z'} >^2}{2} F''_{0,0} \ , \ f^l_{2,1} = i < z', \overline{z'} > (f^l_{1,0})' \ ,$$

$$\overline{f^l_{2,1}} = -i < z', \overline{z'} > (\overline{f^l_{1,0}})' \ , \ l = 1, 2 \ ,$$

the equation (93) becomes

$$< z', \overline{z'} > \{\mathrm{Im}\, \overline{z_1}(f^1_{1,0})' - \mathrm{Im}\, \overline{z_2}(f^2_{1,0})'\} = F_{2,2} + \dots \mod N^j_{2,2} \ . \tag{101}$$

The power series $f^1_{1,0}$, $f^2_{1,0}$ can be written

$$f^1_{1,0}(z,s) = a_1(s)z_1 + a_2(s)z_2 + a_3(s)z_3 \ ,$$

$$f^2_{1,0}(z,s) = b_1(s)z_1 + b_2(s)z_2 + b_3(s)z_3 \ ,$$

and hence

$$\mathrm{Im}(\overline{z_1}(f^1_{1,0})') = |z_1|^2 \mathrm{Im}\, a'_1 + \mathrm{Im}(a'_2 \overline{z_1} z_2) + \mathrm{Im}(a'_3 \overline{z_1} z_3) \ .$$

$\mathrm{Im}\, \overline{z_2}(f^2_{1,0})'$ has a similar expression. Substitute these into (93), then the resulting equation is

$$< z', \overline{z'} > \Big\{ |z_1|^2 \mathrm{Im}\, a'_1 - |z_2|^2 \mathrm{Im}\, b'_2 + \mathrm{Im}(a'_3 \overline{z_1} z_3) - \mathrm{Im}(b'_3 \overline{z_2} z_3) +$$

$$+ \mathrm{Im}\{(a'_2 + \overline{b'_1}) \overline{z_1} z_2\} \Big\} = F_{2,2} + \dots \mod N^j_{2,2} \ .$$

The definition of $N^j_{2,2}$ and the fact that the constant terms of

$$\mathrm{Im}\, \frac{\partial f^1}{\partial z_1} \ , \ \mathrm{Im}\, \frac{\partial f^2}{\partial z_2} \ , \ \frac{\partial f^1}{\partial z_3} \ , \ \frac{\partial f^2}{\partial z_3} \ , \ \frac{\partial f^1}{\partial z_2} + \overline{\frac{\partial f^2}{\partial z_1}}$$

are 0 imply that

$$\mathrm{Im}\, a_1 \ , \ \mathrm{Im}\, b_2 \ , \ a_3 \ , \ b_3 \ , \ a_2 + \overline{b_1} \tag{102}$$

are uniquely determined.

Next turn to the equations (92) and (94). Substitute

$$\mathrm{Im}\, g_{1,1} = - < z', \overline{z'} > \mathrm{Re}\, g'_{0,0} \ , \ -\mathrm{Im}\, g_{3,3} = \frac{< z', \overline{z'} >^3}{6} \mathrm{Re}\, g'''_{0,0} \ ,$$

$$f^l_{3,2} = -\frac{< z', \overline{z'} >^2}{2} (f^l_{1,0})'' \ , \ \overline{f^l_{3,2}} = -\frac{< z', \overline{z'} >^2}{2} (\overline{f^l_{1,0}})''$$

and known functions into (92) and (94). Differentiate the resulting equation (92) twice, multiply it by $< z', \overline{z'} >^2 /6$ and subtract from the resulting equation (94), then we obtain

$$< z', \overline{z'} >^2 \Big\{ |z_1|^2 \mathrm{Re}\, a''_1 - |z_2|^2 \mathrm{Re}\, b''_2 + \mathrm{Re}(a''_3 \overline{z_1} z_3) - \mathrm{Re}(b''_3 \overline{z_2} z_3) +$$

$$+ \mathrm{Re}\{(a''_2 + \overline{b''_1}) \overline{z_1} z_2\} \Big\} = F_{3,3} + \dots \mod N^j_{3,3} \ .$$

For example, by the definition of $N^j_{3,3}$, $\mathrm{Re}\, a''_1$ is determined uniquely. Therefore the chices of the constant terms of

$$\mathrm{Re}\, \frac{\partial^2 f^1}{\partial z_1 \partial w} \ , \ \mathrm{Re}\, \frac{\partial f^1}{\partial z_1}$$

determine $\mathrm{Re}\, a_1$ uniquely. Analogously, $\mathrm{Re}\, b_2$, $a_2 - \overline{b_1}$ are also uniquely determined by the choices of constant terms of some differentials. Combine with (102), we conclude that $a_1, \dots b_3$ are uniquely determined. Consequently, $f^1_{1,1}$, and $f^2_{1,1}$ are uniquely determined.

Substitute $-\mathrm{Im}\, g_{1,1} = - < z', \overline{z'} > (\mathrm{Re}\, g_{0,0})'$ and known functions into the equation (92), then we have

$$(\mathrm{Re}\, g_{0,0})' < z', \overline{z'} >= F_{1,1} + \dots \mod N^j_{1,1} \ .$$

It follows from the same argument as above that $\mathrm{Re}\, g_{0,0}$ is uniquely determined. Since $\mathrm{Im}\, g_{0,0}$ is determined from (91), $g_{0,0}$ is determined uniquely.

This completes the proof of Lemma 1. □

5. Normalization in higher order terms, the case (III)

The difference between the equation (III) and the other equations is, as mentioned before, that p_{III} does not contain z_3, which implies that, for example, the equation (84) does not contain $f^3_{k,0}$. Therefore, we have to add more equations. It follows from 2-nondegeneracy condition that $p_{III} \not\equiv 0$. In case of $p_{III} \equiv 0$, f^3 bellow is not determined uniquely. We use the following system of equations instead of from (84) to (94).

$$-\frac{1}{2i}\, g_{k,0} = F_{k,0} \tag{103}$$

$$-\frac{1}{2i}\, g_{k+1,1} - \overline{z_1} f^1_{k+1,0} + \overline{z_2} f^2_{k+1,0} = F_{k+1,1} \mod N^j_{k+1,1} \tag{104}$$

$$-\frac{1}{2i}\, g_{k+2,2} - \overline{z_1} f^1_{k+2,1} + \overline{z_2} f^2_{k+2,1} + p_{III} f^3_{k+2,0} = F_{k+2,2} \mod N^j_{k+2,2} \tag{105}$$

$$-\frac{1}{2i}\, g_{2,0} + p_{III}\overline{f^3_{0,0}} = F_{2,0} \tag{106}$$

$$-\frac{1}{2i}\, g_{3,1} - \overline{z_1} f^1_{3,0} + \overline{z_2} f^2_{3,0} + p_{III}\overline{f^3_{1,1}} = F_{3,1} \mod N^j_{3,1} \tag{107}$$

$$-\frac{1}{2i}\, g_{4,2} - \overline{z_1} f^1_{4,1} + \overline{z_2} f^2_{4,1} + p_{III}\overline{f^3_{2,2}} + \overline{p_{III}} f^3_{4,0} = F_{4,2} \mod N^j_{4,2} \tag{108}$$

$$-\frac{1}{2i}\, g_{5,3} - \overline{z_1} f^1_{5,2} + \overline{z_2} f^2_{5,2} + p_{III}\overline{f^3_{3,3}} + \overline{p_{III}} f^3_{5,1} = F_{5,3} \mod N^j_{5,3} \tag{109}$$

$$-\frac{1}{2i}\, g_{1,0} - z_1\overline{f^1_{0,0}} + z_2\overline{f^2_{0,0}} = F_{1,0} \tag{110}$$

$$-\frac{1}{2i}\, g_{2,1} - z_1\overline{f^1_{1,1}} - \overline{z_1} f^1_{2,0} + z_2\overline{f^2_{1,1}} + \overline{z_2} f^2_{2,0} + p_{III}\overline{f^3_{1,0}} = F_{2,1} \mod N^j_{2,1} \tag{111}$$

$$\begin{aligned} -\frac{1}{2i}\, g_{3,2} - z_1\overline{f^1_{2,2}} - \overline{z_1} f^1_{3,1} \quad &+ z_2\overline{f^2_{2,2}} + \overline{z_2} f^2_{3,1} + p_{III}\overline{f^3_{2,1}} + \overline{p_{III}} f^3_{3,0} = \\ &= F_{3,2} \mod N^j_{3,2} \end{aligned} \tag{112}$$

$$\begin{aligned} -\frac{1}{2i}\, g_{4,3} - z_1\overline{f^1_{3,3}} - \overline{z_1} f^1_{4,2} \quad &+ z_2\overline{f^2_{3,3}} + \overline{z_2} f^2_{4,2} + p_{III}\overline{f^3_{3,2}} + \overline{p_{III}} f^3_{4,1} = \\ &= F_{4,3} \mod N^j_{4,3} \end{aligned} \tag{113}$$

$$\begin{aligned} -\frac{1}{2i}\, g_{5,4} - z_1\overline{f^1_{4,4}} - \overline{z_1} f^1_{5,3} \quad &+ z_2\overline{f^2_{4,4}} + \overline{z_2} f^2_{5,3} + p_{III}\overline{f^3_{4,3}} + \overline{p_{III}} f^3_{5,2} = \\ &= F_{5,4} \mod N^j_{5,4} \end{aligned} \tag{114}$$

$$-\operatorname{Im} g_{0,0} = F_{0,0} \tag{115}$$

$$-\operatorname{Im} g_{1,1} - z_1\overline{f^1_{1,0}} - \overline{z_1} f^1_{1,0} + z_2\overline{f^2_{1,0}} + \overline{z_2} f^2_{1,0} = F_{1,1} \mod N^j_{1,1} \tag{116}$$

$$-\operatorname{Im} g_{2,2} - z_1\overline{f^1_{2,1}} - \overline{z_1} f^1_{2,1} + z_2\overline{f^2_{2,1}} + \overline{z_2} f^2_{2,1} + p_{III}\overline{f^3_{2,0}} + \overline{p_{III}} f^3_{2,0} = F_{2,2} \tag{117}$$

$$\begin{aligned} -\operatorname{Im} g_{3,3} - z_1\overline{f^1_{3,2}} - \overline{z_1} f^1_{3,2} \quad &+ z_2\overline{f^2_{3,2}} + \overline{z_2} f^2_{3,2} + p_{III}\overline{f^3_{3,1}} + \overline{p_{III}} f^3_{3,1} = \\ &= F_{3,3} \mod N^j_{3,3} \end{aligned} \tag{118}$$

(119)
$$\begin{aligned}-\mathrm{Im}\, g_{4,4} - z_1\overline{f^1_{4,3}} - \overline{z_1} f^1_{4,3} \quad & +z_2\overline{f^2_{4,3}} + \overline{z_2} f^2_{4,3} + p_{III}\overline{f^3_{4,2}} + \overline{p_{III}} f^3_{4,2} = \\ & = F_{4,4} \mod N^j_{4,4}\end{aligned}$$

(120)
$$\begin{aligned}-\mathrm{Im}\, g_{5,5} - z_1\overline{f^1_{5,4}} - \overline{z_1} f^1_{5,4} \quad & +z_2\overline{f^2_{5,4}} + \overline{z_2} f^2_{5,4} + p_{III}\overline{f^3_{5,3}} + \overline{p_{III}} f^3_{5,3} = \\ & = F_{5,5} \mod N^j_{5,5}\end{aligned}$$

Furthermore we have to replace $\mathcal{G}^1$ by $\mathcal{G}^2$,

$$\mathcal{G}^2 = \Big\{(f,g)(z,w)\Big| f^l = O(3),\ f^3 = O(2),\ g = O(4),$$
$$\text{constant terms of } \frac{\partial^2 f^l}{\partial z^\alpha},\ |\alpha| = 2,\ \frac{\partial^4 f^3}{\partial z^\beta},\ |\beta| = 4,\ \frac{\partial f^3}{\partial w},$$
$$\frac{\partial^{|\alpha|+\beta} f^3}{\partial z^\alpha \partial w^\beta},\ |\alpha| = 1,3,\ \beta = 0,1,\ \mathrm{Re}\,\frac{\partial^4 f^l}{\partial z_l \partial w^3},\ \frac{\partial^{1+\beta} f^l}{\partial z_k \partial w^\beta},\ \beta = 0,1,2,\ k = 1,2,3,$$
$$\frac{\partial^4 f^1}{\partial z_2 \partial w^3} - \overline{\left(\frac{\partial^4 f^2}{\partial z_1 \partial w^3}\right)} \text{ vanish for } l = 1,2\Big\}.$$

The proof of this case proceeds along the same line as the proof in the previous section. By the definition of $\mathcal{N}^{III}$, we can find solutions for equations above in the space $\mathcal{G}^2$. This completes the proof of this case. The proof of Theorem 1 also ends by §4 and §5.

6. Dimensions of the groups of biholomorphic transformations

We say that the coordinate (z,w) are regular for M at $(0,0)$ if the defining function for M at the point is of the form $\mathrm{Im}\, w = \phi(z,\overline{z},\mathrm{Re}\, w)$ with $\phi(z,0,s) \equiv \phi(0,\overline{z},s) \equiv 0$. In this section, we shall find transformations which preserve regular coordinates and the forms (1),..., (5). For stating the proposition, we prepare some notation. Denote $I = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Recall that p_j be a polynomial of order two definined in section 4. For such a polynomial and power series f_1, f_2 with variables z_1, z_2, z_3, denote $p(f_1,f_2)_{jk}$ by a polynomial whose variables are $z_j z_k$.

Proposition 1. *A transformation (f^1,f^2,f^3,g) which preserve regular coordinates and the forms (1), ..., (5) are the following.*

$$\begin{pmatrix} f^1 \\ f^2 \end{pmatrix} = A\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + w\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} + \begin{pmatrix} z_k z_l E^1_{lk} \\ z_k z_l E^2_{lk} \end{pmatrix} + \mathcal{O}(3),$$
$$f^3(z,w) = K_1 z_1 + K_2 z_2 + d z_3 + \mathcal{O}(3),$$
$$g(z,w) = cw + w(\alpha z_1 + \beta z_2) + \mathcal{O}(4),$$

where $c \in \mathbb{R}\setminus\{0\}$, $d \in \mathbb{C}\setminus\{0\}$, $a_1, a_2 \in \mathbb{C}$. A, B_j, E^j_{lk} satisfy the following relations.

$$A^* I A = cI,\quad \begin{pmatrix} B_1 \\ -B_2 \end{pmatrix} = A^{*-1}\begin{pmatrix} \overline{a_1} \\ \overline{a_2} \end{pmatrix},$$
$$\begin{pmatrix} -E^1_{11} \\ E^2_{11} \end{pmatrix} = A^{*-1}\begin{pmatrix} -a_1 - \overline{K}_1 p(f^1,f^2)_{11} \\ -K_2 p(f^1,f^2)_{11} \end{pmatrix},$$
$$\begin{pmatrix} -E^1_{22} \\ E^2_{22} \end{pmatrix} = A^{*-1}\begin{pmatrix} -\overline{K}_1 p(f^1,f^2)_{22} \\ a_2 - \overline{K}_2 p(f^1,f^2)_{22} \end{pmatrix},$$

$$\begin{pmatrix} -E^1_{12} - E^1_{21} \\ \\ E^2_{12} + E^2_{21} \end{pmatrix} = A^{*-1} \begin{pmatrix} a_2 - \overline{K}_1 p(f^1, f^2)_{12} \\ \\ -a_1 - \overline{K}_2 p(f^1, f^2)_{12} \end{pmatrix}$$

Proof. The proof is almost same as the one for Proposition 1.1.9 in [6].

Using this proposition, we can compute a bound on the dimension of the stability group $\mathrm{Aut}(M, p_0)$ of a hypersurface M as in Theorem 1. Since our hypersurface M is 2-nondegenerate at p_0, the stability group of M is a real, finite dimensional Lie group by the result in [1]. Counting the number of parameters in $\mathcal{G}^1$, $\mathcal{G}^2$ and f^1, f^2, f^3, g, we find the following estimation.

Corollary 1. *Let $M \subset \mathbb{C}^4$ be a real analytic hypersurface with the same conditions as in Theorem 1. Then the following holds.*

(1) *If the defining function of M is one of* (1), (2), (4) *and* (5), *then* $\dim_{\mathbb{R}} \mathrm{Aut}(M, 0) \leq 61$.
(2) *If the defining function of M is of the form* (3)), *then* $\dim_{\mathbb{R}} \mathrm{Aut}(M, 0) \leq 118$.

7. Some remarks

(1) For hypersurfaces in $\mathbb{C}^4$ with degenerate Levi forms at the origin, they can be classified into five types in terms of Levi forms. Namely, after diagonalization, the diagonal components are $(1, 1, 0), (1, -1, 0), (1, 0, 0), (0, 0, 0)$. The first case is studied in [6]. This article treats the second case with 2-nondegeneracy condition. By the same argument in this article, normal forms of hypersurfaces of the third case may be constructed with assumption of 2-nondegeneracy.
(2) In case that M is k-nondegenerate, $k \geq 3$, as far as I know, we can not determine (f', f^3, g) uniquely in Lemma 1. The reason is that p_{III} may be constantly zero.

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