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The twisted tensor product representation of finite classical groups and some related geometry

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Abstract[1]. We survey some recent results on twisted tensor product group embeddings identifying their geometry and in some cases addressing maximality.

Introduction

Let G be a finite classical group with natural module V_0 of dimension $n \geq 2$ over the Galois field $GF(q^t)$. Let V^{ψ^i} denote the G-module V_0 with group action given by $v \cdot g = vg^{\psi^i}$, where g^{ψ^i} denotes the matrix g with its entries raised to the q^i-th power, $i = 0, \ldots, t-1$. Then one can form the tensor product module $V_0 \otimes V_0^{\psi} \otimes \cdots \otimes V_0^{\psi^{t-1}}$, a module which can be realized over the field $GF(q)$. This gives rise to an embedding of the group G in a classical group having an n^t-dimensional natural module over $GF(q)$, yielding an absolutely irreducible representation of the group G. Also let V^* denote the G-module with group action given by $v \cdot g = vg^*$, where g^* is the inverse-transpose of g. For t even, there is a similar module given by $V \otimes V^{*\psi} \otimes V^{\psi^2} \otimes \cdots \otimes V^{*\psi^{t-1}}$, realizable over $GF(q^2)$. Such representations are given by Steinberg ([10]) and further studied by Seitz ([9]). As Seitz observed, the normalizers of such "twisted tensor product groups" might easily be considered a ninth Aschbacher class [1].

The geometry of maximal subgroups in the Aschbacher classes is well understood (with the possible exception of the class $\mathcal{C}_6$). Our main purpose is to describe the geometry of subgroups lying outside the Aschbacher classes, little being known at present.

In the first part we concentrate on classical groups of low dimension, namely with $t = 2$ and $n = 3$, and study the embeddings $PGL(3, q^2)$ in $PGL(9, q)$, $PGL(3, q^2)$ in $PU(9, q^2)$ and $\Omega(3, q^2)$ in $\Omega(9, q)$; in the last case q is odd. We identify the normalizers of the embedded groups as (in most cases) maximal subgroups and stabilizers of geometrical configurations: hermitian veroneseans, twisted hermitian veroneseans and rational curves.

In the second part we study the geometry of two other classes of twisted tensor product groups: $PSL(2, q^t) \leq P\Omega^+(2^t, q)$; and $PSp(2m, q^t) \leq P\Omega^{\epsilon}((2m)^t, q)$. Throughout this second part we shall assume that q is even and that $t \geq 2$. An *ovoid* $\mathcal{O}$ in a classical polar space [4, Chapter 26] is a set of singular points such that every maximal totally singular subspace contains just one point of $\mathcal{O}$. The points of $\mathcal{O}$ are pairwise non-orthogonal. More

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generally a *partial ovoid* is a set of pairwise non-orthogonal singular points. A partial ovoid is said to be *complete* if it is maximal with respect to set-theoretic inclusion.

The possibility of the existence of ovoids in polar spaces of various dimensions has been studied extensively, for both odd and even q (although the results referred to here are solely for even q). On the one hand there are known to be ovoids in $PG(7,q)$, both infinite families such as the unitary ovoids and the Desarguesian ovoids and individual ovoids such as Dye's ovoid, and although the 2-transitive ovoids have been classified by Kleidman in [19], there is no general classification of ovoids. On the other hand Thas ([20]) has shown that quadrics in $PG(2n,q)$ and elliptic quadrics in $PG(2n+1,q)$ have no ovoids if $n \geq 4$ and Kantor ([18]) has shown that hyperbolic quadrics in $PG(2n+1,2)$ have no ovoids if $n \geq 4$. Further, Blokhuis and Moorhouse ([11]) established an upper bound for the size of a partial ovoid of a polar space, a consequence of which is the non-existence of ovoids of hyperbolic quadrics in $PG(2n+1,q)$ if $n \geq 4$. A survey on ovoids in classical polar space is included by Thas in [22], and further information can be found in [20], [21], [4]. There are known to be examples of partial ovoids on quadrics in $PG(4n+3,8)$ whose size meets the Blokhuis-Moorhouse bound ([15]) for all values of n; also in [15] there are examples of complete partial ovoids on quadrics in $PG(4n+1,8)$ whose size falls just short of the Blokhuis-Moorhouse bound.

We find that our embedding of $PSL(2,q^t)$ is associated with an embedding of $PG(1,q^t)$ as a partial ovoid of a quadric in $PG(2^t-1,q)$; if $t \geq 3$, then the quadric is hyperbolic. In $PG(2^t-1,q)$ with q even, the Blokhuis-Moorhouse bound is given by q^t+1. We thus have a family of partial ovoids whose size attains the Blokhuis-Moorhouse bound. In particular when $t=3$ and $q \geq 4$ the embedding yields a nice description of a Desarguesian ovoid of the hyperbolic quadric of $PG(7,q)$ ([16], [17]) as the image of a projective line in much the same way as an elliptic quadric of $PG(3,q)$ is the image of a projective line. Similarly our embedding of $PSp(2m,q^t)$ in $P\Omega^{\epsilon}((2m)^t,q)$ has a particular application when $m=2$ in the embedding of ovoids of $PG(3,q^t)$ as partial ovoids of $PG(4^t-1,q)$ again with size attaining the Blokhuis-Moorhouse bound. The families of complete partial ovoids arising from Suzuki-Tits ovoids are not equivalent to those arising from elliptic quadrics or projective lines; the partial ovoids given by Dye in [15] are different again.

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1. $n=3, t=2$ and some generalisations

1.1. The Hermitian Veronesean of $PG(2,q^2)$.

1.1.1. *Tensored spaces.* Let V_i, $1 \leq i \leq t$ be vector spaces of dimension n_i over the Galois field $GF(q)$. Then $V = V_1 \otimes \cdots \otimes V_t$ is a vector space of dimension $\prod_{i=1}^{t} n_i = n$.

Assuming that $m_i = n_i - 1 \geq 1$ for each i, let $PG(m_1,q)$, $PG(m_2,q)$, ..., $PG(m_t,q)$ be the projective spaces over $GF(q)$ corresponding to $V_1, V_2, ..., V_t$. The set of all vectors in V of the form $v_1 \otimes \cdots \otimes v_t$ with $0 \neq v_i \in V_i$ corresponds to a set of points in $PG(n-1,q)$ known as the *Segre variety*, $S_{m_1,\ldots,m_r}$, of $PG(m_1,q),\ldots,PG(m_r,q)$, [4, 25.5].

1.1.2. *A representation of* $GL(3,q^2)$. Let $G = GL(3,q^2)$ and let $\psi : GF(q^2) \to GF(q^2)$ be the Frobenius automorphism of $GF(q^2)$ given by $x \mapsto x^q$; we sometimes write $\bar{x}$ for x^q . Let V_0 be the natural module for $GL(3,q^2)$ over $GF(q^2)$. Let V_0^{ψ} be the G-module with group action given by $v \cdot g = vg^{\psi}$, where vg^{ψ} denotes the matrix g with its entries raised to the q-th power and let $V = V_0 \otimes V_0^{\psi}$. Then we have a representation $\rho : G \to GL(3^2,q^2)$ with $\rho(g) = g \otimes g^{\psi} \in GL(3,q^2) \otimes GL(3,q^2)$. This representation of $GL(3,q^2)$ is absolutely irreducible (c.f. [10]). The two representations ρ and $\rho\psi$ are isomorphic, so this representation of G on V can be written over $GF(q)$ (c.f. [1, 26.3]). Moreover if ψ_0

is the Frobenius automporphism of $GF(q^2)$ given by $x \mapsto x^{q_0}$ for any $q_0 \le q$, then ρ and $\rho\psi_0$ are not isomorphic (c.f. [10]) and so ρ cannot be written over $GF(q_0)$.

We can give a concrete construction of a $GF(q)$-subspace of V fixed by $\rho(G)$. If v_1, v_2, v_3 is a basis for V_0 and $\alpha \in GF(q^2) \setminus GF(q)$ is fixed, then the vectors $v_i \otimes v_i$, $v_i \otimes v_j + v_j \otimes v_i$ and $\alpha v_i \otimes v_j + \alpha^q v_j \otimes v_i$ $(i \le j)$ form a basis for an 3^2- dimensional $GF(q)$-subspace V_q of V fixed by G. There is an involution $\theta \in GL(3^2, q^2)$ on V that takes $v_i \otimes v_j$ to $v_j \otimes v_i$ for each i, j. We see that θ fixes V_q and normalizes $\rho(G)$; it is not difficult to show that θ does not lie in $\rho(G)$. Factoring out scalars we get an embedding of $PGL(3, q^2)$ in $PGL(3^2, q)$. Restricting to matrices with determinant one, we find $\rho(SL(3, q^2)) \le SL(3^2, q)$ so that $PSL(3, q^2)$ is embedded in $PSL(3^2, q)$. The involution $-\theta$ lies in $SL(3^2, q)$ and normalizes $\rho(SL(3, q^2))$.

The realization over $GF(q)$ can be seen in another way. Let $\phi : V \to V$, $\lambda u_1 \otimes u_2 \to \lambda^q u_2 \otimes u_1$, with each u_i being one of v_1, v_2, v_3, extended linearly over $GF(q)$. Then ϕ is a semi-linear map that commutes with $\rho(G)$. Let W be the set of all vectors in V that are fixed by ϕ. Then for all $u \in W, g \in G$, $\phi(g(u)) = g(\phi(u)) = g(u)$, and so $g(u) \in W$. Thus the set W is fixed by G and it is a $GF(q)$-subspace of V. We observe that W contains all the vectors in V_q above. Moreover $GF(q)$-linearly independent vectors in W are linearly independent over $GF(q^2)$. For otherwise, consider a minimally-sized counterexample: $w_1, \dots, w_r$ are linearly independent over $GF(q)$ but not over $GF(q^2)$. Then, there are scalars $\mu_1, \dots, \mu_r \in GF(q^2)$ such that $\sum_{i=1}^r \mu_i w_i = 0$, with not all μ_i in $GF(q)$, and we may assume, without loss of generality that $\mu_r = 1$. Now $\sum_{i=1}^r \mu_i^q w_i = 0$ and so $\sum_{i=1}^{r-1} (\mu_i^q - \mu_i) w_i = 0$. We get a contradiction to r minimal. Given the absolute irreducibility of $\rho(G)$ we conclude that W has dimension 3^2 over $GF(q)$. Thus $W = V_q$.

1.2. **The Hermitian embedding and its automorphism group.** Every element $z \in GF(q^2)$ has a unique representation as $x + \alpha y$ with $x, y \in GF(q)$ and $\bar{z} = x + \bar{\alpha} y$. Let $PG(2, q^2)$ denote the projective plane over $GF(q^2)$ and consider the map $\varphi : PG(2, q^2) \to PG(8, q^2)$ defined as follows:

$$(X_0, X_1, X_2) \to (X_0^{q+1}, X_1^{q+1}, X_2^{q+1}, X_0 X_1^q, X_0^q X_1, X_0 X_2^q, X_0^q X_2, X_1 X_2^q, X_1^q X_2).$$

The map φ is well-defined and injective. φ is called the *Hermitian embedding of* $PG(2, q^2)$ and we denote by $\widehat{H}$ the image of such a correspondence in $PG(8, q^2)$. We note that $\widehat{H}$ is contained in the Segre variety $S_{2,2} \simeq PG(2, q^2) \times PG(2, q^2)$. In fact $\widehat{H} = \{(P, \bar{P})f : P \in PG(2, q^2)\}$, where f is the Segre map sending $PG(2, q^2) \times PG(2, q^2)$ onto $S_{2,2}$. Indeed, the co-ordinate system for $PG(8, q^2)$ corresponds to the basis $v_i \otimes v_j$ $(1 \le i \le 3, 1 \le j \le 3)$ for V and the points of $\widehat{H}$ all lie in the Baer subgeometry of $PG(8, q^2)$ determined by the subset $V_q = W$ of V. The point-set $\widehat{H}$ is a variety of the Baer subgeometry known as *the Hermitian Veronesean of* $PG(2, q^2)$ [8], [13]. We denote the variety $\mathcal{H}$ when regarding it as a variety in $PG(8, q)$.

The variety $\mathcal{H}$ can also be described in terms of a normal line spread of $PG(5, q)$ [8]. If $\tau : PG(5, q^2) \to PG(5, q^2)$ is the map sending the point $P(X_0, \dots, X_5)$ to $P(\bar{X}_3, \bar{X}_4, \bar{X}_5, X_0, X_1, X_2)$, then the points fixed by τ form a subgeometry $\mathcal{G}$ of $PG(5, q^2)$ isomorphic to $PG(5, q)$. If π is the plane with equations $X_3 = X_4 = X_5 = 0$, then the plane $\bar{\pi}$ with equations $X_0 = X_1 = X_2$ is disjoint from π. The set of lines of $PG(5, q^2)$ joining a point $P \in \pi$ with the point $\bar{P} \in \bar{\pi}$ is a normal line spread of $\mathcal{G}$ which can be represented on the Grassmannian $G_{1,5}$ of lines of $PG(5, q)$ by the variety $\mathcal{H}$. The variety $\mathcal{H}$ is a $(q^4 + q^2 + 1)$-cap of $PG(8, q)$ and it is not contained in any proper subspace of $PG(8, q)$ [8], [13].

Let $G(\mathcal{H}) = \{\zeta \in PGL(9, q) : \zeta(\mathcal{H}) = \mathcal{H}\}$. The group $G(\mathcal{H})$ is a subgroup of

$PGL(9,q)$ containing $PGL(3,q^2)$ [8], [13]. Given a projectivity ξ of $PG(2,q^2)$, the corresponding projectivity of $G(\mathcal{H}) \leq PGL(9,q)$, denoted by $\xi^{\mathcal{H}}$, is called *the Hermitian lifting of* ξ, or briefly the *$\mathcal{H}$-lifting of* ξ [13].

Let ξ be a linear collineation of $PG(2,q^2)$ with matrix representation $A = (a_{ij})$, $i,j = 0,1,2$. The matrix representation of the $\mathcal{H}$-lifting $\xi^{\mathcal{H}}$ of ξ is the matrix whose generic column is

$$(\bar{a}_{0i}a_{0j}, \bar{a}_{0i}a_{1j}, \bar{a}_{0i}a_{2j}, \bar{a}_{1i}a_{0j}, \bar{a}_{1i}a_{1j}, \bar{a}_{1i}a_{2j}, \bar{a}_{2i}a_{0j}, \bar{a}_{2i}a_{1j}, \bar{a}_{2i}a_{2j})$$

with $0 \leq i,j \leq 2$. In particular, $\xi^{\mathcal{H}}$ is the collineation induced by the Kronecker product $A \otimes A^{\psi}$. Hence, the embedding $PGL(3,q^2) \leq PGL(9,q)$ gives the representation of the group $PGL(3,q^2)$ as an automorphism group of the Hermitian Veronesean $\mathcal{H}$. Notice that the involutory Frobenius automorphism of $GF(q^2)$ induces a collineation of $PG(8,q)$ fixing $\mathcal{H}$ (actually, it interchanges the planes π and $\bar{\pi}$).

We briefly recall Aschbacher's Theorem for Classical groups over $GF(q)$ [1]. Eight classes of "large" subgroups of a given classical group G are defined: $\mathcal{C}_1$, reducible subgroups; $\mathcal{C}_2$, imprimitive subgroups; $\mathcal{C}_3$, stabilizers of field extensions of $GF(q)$; $\mathcal{C}_4$, $\mathcal{C}_7$, stabilizers of various tensor product decompositions; $\mathcal{C}_5$, classical groups over subfields of $GF(q)$; $\mathcal{C}_6$, symplectic-type groups; $\mathcal{C}_8$, other classical groups over $GF(q)$. Aschbacher's Theorem states that any subgroup of G, not containing the socle of G, is either contained in a member of one of $\mathcal{C}_1$ - $\mathcal{C}_8$ or is almost simple and is induced by an absolutely irreducible subgroup modulo scalars. A full discussion of the theorem is given in [6]. Moreover, the same source gives tables with details of the stucture of maximal subgroups in each class. In the following we make extensive use of Table 3.5.A ($SL(n,q)$). We remark that a complete list of maximal subgroups of $SL(9,q)$ is given by P.B. Kleidman in his Ph. D. Thesis [5]. However no proof is given there, nor have the proofs been subsequently published elsewhere. A.S Kondratiev has results that give information on subgroups not contained in a maximal subgroup of classes $\mathcal{C}_1$ - $\mathcal{C}_8$ but they do not apply to the subgroups we are interested in (c.f. [7] for a survey).

Theorem 1.1. *The full stabilizer H of the Hermitian Veronesean $\mathcal{H}$ in $PSL(9,q)$ is almost simple and is induced by an absolutely irreducible subgroup of $SL(9,q)$ modulo scalars.*

Corollary 1.2. *If Kleidman's list in* [5] *is correct, then H is isomorphic to $PSL(3,q^2) \cdot [(q-1,3)^2/(q-1,9)] \cdot C_2$ and is a maximal subgroup of $PSL(9,q)$.*

1.2.1. *Generalizations.* Here we discuss a generalization of the ideas above in which we consider mappings from $GL(n,q^t)$ to $GL(n^t,q)$.

Remark 1.3. The concrete realization over $GF(q)$ described above can be extended to a more general setting. Let $G = GL(n,q^t)$ and let $\psi : GF(q^t) \to GF(q^t)$ be the Frobenius automorphism of $GF(q^t)$ given by $x \mapsto x^q$. Let V_0 be the natural module for $GL(n,q^t)$ over $GF(q^t)$ with $V_0^{\psi^i}$ the G-module with group action given by $V \cdot g = vg^{\psi^i}$, and let $V = V_0 \otimes V_0^{\psi} \otimes V_0^{\psi^2} \cdots \otimes V_0^{\psi^{t-1}}$. Then we have a representation $\rho : G \to GL(n^t,q^t)$ with $\rho(g) = g \otimes g^{\psi} \otimes \cdots \otimes g^{\psi^{t-1}}$. As with the specific case above, this representation of $GL(n,q^t)$ is absolutely irreducible, can be written over $GF(q)$ but over no subfield of $GF(q)$. This time let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V_0 and let $\phi : V \to V$, $\lambda u_1 \otimes u_2 \otimes \cdots \otimes u_t \to \lambda^q u_t \otimes u_1 \otimes \cdots \otimes u_{t-1}$, with each u_i being one of $v_1, v_2, \ldots, v_n$, extended linearly over $GF(q)$. The set W of all vectors in V that are fixed by ϕ is fixed by G and is a $GF(q)$-subspace of V. Moreover $GF(q)$-linearly independent vectors in W are linearly independent over $GF(q^t)$ and we conclude that W has dimension n^t over $GF(q)$. We return to this later.

1.3. **The Twisted Hermitian Veronesean of $PG(2,q^2)$.**

1.3.1. *Embedding $PGL(3,q^2)$ in $PU(9,q^2)$.* The notation here is similar to that used earlier, with $G = GL(3,q^2)$, ψ the Frobenius automorphism of $GF(q^2)$ and V_0 the natural module for $GL(3,q^2)$ over $GF(q^2)$. Let V_0^* be the dual module of V_0 (with group action given by $v \cdot g = vg^* = v(g^T)^{-1}$) and let $V = V_0^* \otimes V_0^{\psi}$. Then we have an absolutely irreducible representation $\rho^* : G \to GL(3^2,q^2)$ with $\rho^*(g) = g^* \otimes g^{\psi} \in GL(3,q^2) \otimes GL(3,q^2)$ [10]. The module presented here is dual to $V_0 \otimes V_0^{\psi *}$ but is a more convenient setting from our point of view. The modules $V^* = V_0 \otimes (V_0^{\psi *})$ and $V^{\psi} = (V_0^{\psi *}) \otimes V_0$ are isomorphic and so $\rho^*(G)$ fixes a Hermitian form on V. In general such a representation cannot be realized over a subfield of $GF(q^2)$ (see [2], [6, Theorem 5.4.5]). Indeed, suppose $V_0^* \otimes V_0^{\psi}$ can be realized over a proper subfield $GF(q_0)$ of $GF(q^2)$. Then $V_0^* \otimes V_0^{\psi} \simeq V_0^{\psi_0 *} \otimes V_0^{\psi\psi_0}$, where ψ_0 is the automorphism $x \mapsto x^{q^0}$ of $GF(q^2)$. By [10] these two representations are equivalent if and only if, either $V_0^* \simeq V_0^{\psi_0 *}$ (i.e., $V_0 \simeq V_0^{\psi_0}$), which is not possible, or $V_0^* \simeq V_0^{\psi\psi_0}$ and $V_0^{\psi} \simeq V_0^{\psi_0 *}$. The latter can happen if and only if $\psi_0 = \psi$ and $V_0 \simeq V_0^*$, which in turn is possible if and only if $GL(3,q^2)$ fixes a symmetric or symplectic bilinear form on V_0. As $GL(3,q^2)$ fixes no such form on V_0, its representation on V cannot be realized over a proper subfield of $GF(q^2)$. The same applies to $SL(3,q^2)$.

The representation of $GL(3,q^2)$ may be stated explicitly as follows. Assume that we have a fixed basis v_1, v_2, v_3 for V_0 as in the previous section. A non-degenerate Hermitian form is defined by $(u \otimes v, w \otimes z) = (uz^{\psi T}).(w^{\psi} v^T)$ and this is preserved by $\rho^*(g) = (g^T)^{-1} \otimes g^{\psi}$ for all $g \in G$. It follows that $PGL(3,q^2)$ can be embedded in $PU(9,q^2)$. Recall that the involution θ of $V(9,q^2)$ takes $v_i \otimes v_j$ to $v_j \otimes v_i$ for each i,j; we now observe that θ lies in $U(9,q^2)$ and normalizes (but does not lie in) $\rho^*(G)$. We find that $\rho^*(SL(3,q^2)) \leq SU(9,q)$ with $PSL(3,q^2)$ embedded in $PSU(9,q^2)$; $-\theta \in SU(9,q^2)$ and normalizes $\rho^*(SL(3,q^2))$. We shall shortly see that the image of $PGL(3,q^2)$ is an automorphism group of a variety that we call the *Twisted Hermitian Veronesean* of $PG(2,q^2)$ and denote by $\mathcal{H}^*$.

1.3.2. *The Twisted Hermitian Veronesean.* In considering the action of $G = GL(3,q^2)$ on $V(9,q^2)$, we see that one orbit is given by $\{(v_1 \otimes v_2)\rho^*(g) : g \in GL(3,q^2)\}$ and this orbit consists of singular vectors. The corresponding orbit in $PG(8,q^2)$ is preserved by (the image of) $PGL(3,q^2)$. Let $\mathcal{R}$ be the set of non-zero singular vectors of the form $u \otimes v$. For any $u \otimes v \in \mathcal{R}$ and any $g \in G$ we see that $(u \otimes v)g = ug^* \otimes vg^{\psi}$ is singular and so lies in $\mathcal{R}$. It is straightforward to calculate that $u \otimes v$ is singular if and only if $u.w^{\psi T} = 0$, so singular vectors of the form $v_1 \otimes w$ are precisely the vectors given by $w = \lambda v_2 + \mu v_3$ where $\lambda, \mu \in GF(q^2)$; such a singular vector is mapped to $v_1 \otimes v_2$ by the inverse of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^q & \mu^q \\ 0 & \nu & \zeta \end{pmatrix},$$

where $\nu, \zeta \in GF(q^2)$ such that the matrix is non-singular. Thus G is transitive on $\mathcal{R}$, i.e., $\mathcal{R}$ is precisely the orbit that we initially identified. The involution $-\theta$ preserves the Hermitian form and preserves the tensor product $V_0 \otimes V_0$ so it preserves $\mathcal{R}$. Hence the stabilizer in $U(9,q^2)$ of $\mathcal{R}$ has a subgroup isomorphic to $GL(3,q^2) \cdot C_2$.

Let $\mathcal{H}^*$ be the set of points in $PG(8,q^2)$ corresponding to $\mathcal{R}$. We call this the *Twisted Hermitian Veronesean* of $PG(2,q^2)$. This set is the intersection of the Hermitian variety corresponding to the given Hermitian form and the Segre variety $S_{2,2}$. As we have seen above, the points of $\mathcal{H}^*$ corresponding to $v_1 \otimes w$ for various w are just $P(v_1 \otimes (\lambda v_2 + \mu v_3))$, i.e., are the points on a line. It follows that $\mathcal{H}^*$ consists of $q^4 + q^2 + 1$ disjoint lines of the form $u \otimes L$. At the same time $\mathcal{H}^*$ can be expressed as the disjoint union of lines of the form $L \otimes u$.

Theorem 1.4. *The full stabilizer H^* of the Twisted Hermitian Veronesean $\mathcal{H}^*$ in $PSU(9,q^2)$ is almost simple and is induced by an absolutely irreducible subgroup of $SU(9,q^2)$ modulo scalars.*

Corollary 1.5. *If Kleidman's list in* [5] *is correct, then H^* is isomorphic to $PSL(3,q^2)[(q+1,3)^2/(q+1,9)]\cdot C_2$ and is a maximal subgroup of $PSU(9,q^2)$.*

1.3.3. *Generalizations.* As before, we discuss a generalization of the ideas above and consider mappings from $GL(n,q^2)$ to $U(n^2,q^2)$.

Remark 1.6. As with the Section 1.2, the situation we have described is a part of a more general picture. From [6, Lemma 2.10.15 ii, Theorem 5.4.5], there is an absolutely irreducible representation ρ^* of the group $G = GL(n,q^2)$ on $V = V_0^* \otimes V_0^\psi$ over $GF(q^2)$ that fixes a Hermitian form, not generally realizable over a subfield of $GF(q^2)$. As argued above, ρ^* can be realized over a subfield of $GF(q^2)$ if and only if $GL(n,q^2)$ fixes a symmetric or symplectic bilinear form on V_0, and this can never happen. However, when we consider $SL(n,q^2)$, we find that it fixes a non-degenerate symplectic bilinear form precisely when $n=2$. In this one case, $\rho^*(SL(2,q^2))$ can be realized over $GF(q)$, effectively we have the well known isomorphism between $PSL(2,q^2)$ and $\Omega^-(4,q)$. The non-degenerate Hermitian form defined by $(u\otimes v, w\otimes z) = (uz^{\psi T}).(w^\psi v^T)$ is preserved by $\rho^*(G)$. It now follows that $PGL(n,q^2)$ can be embedded in $PU(n^2,q^2)$. The involution θ lies in $U(n^2,q^2)$ and normalizes (but does not lie in) $\rho^*(G)$. We find that for $n \geq 3$ the image of $PGL(n,q^2)$ acts transitively on the intersection of a Hermitian variety and a Segre variety, the automorphism group of this intersection contains $PGL(n,q^2)\cdot C_2$ and so the full automorphism group is absolutely irreducible. This intersection can be expressed as the disjoint union of subspaces of (projective) dimension $n-2$ in two ways.

1.4. $PSL(2,q^2) \simeq \Omega(3,q^2) \leq \Omega(9,q)$, q odd, as the stabilizer of a rational curve.

1.4.1. *Embedding $\Omega(3,q^2)$ in $\Omega(9,q)$.* Now suppose that q is odd, that $H \leq GL(3,q^2)$ and that H fixes a non-degenerate symmetric bilinear form f_0 on V_0. Then one can define a non-degenerate symmetric bilinear form $f = f_0 \otimes f_0$ on V by $f(u_1\otimes u_2, w_1\otimes w_2) = f_0(v_1,w_1).f_0(v_2,w_2)$, fixed by $\rho(H)$. Assume that the basis $\{v_1,v_2,v_3\}$ chosen for V_0 is such that $f_0(v_i,v_j) \in GF(q)$ for each i,j. Recall the semilinear map ϕ introduced in Section 1.1.2 (with W its space of fixed vectors). Then for any $u,v \in W = V_q$ we have $f(u,v) = f(\phi(u),\phi(v)) = f(u,v)^q$. Hence $f(u,v) \in GF(q)$ for all $u,v \in W$. If $H = O(3,q^2)$, then $\rho(H)$ is absolutely irreducible on V and therefore the restriction of f to W is non-degenerate. Thus $\rho(O(3,q^2)) \leq O(9,q)$. Indeed (considering commutator subgroups) $\rho(\Omega(3,q^2)) \leq \Omega(9,q)$ and the restriction of ρ to $\Omega(3,q^2)$ is injective.

Let us specifically choose the basis v_1,v_2,v_3 for V_0 so that the quadratic form corresponding to f_0 is given by $Q_0(\lambda_1v_1+\lambda_2v_2+\lambda_3v_3) = \lambda_3^2 - \lambda_1\lambda_2$. Then the points on the conic $\mathcal{C}_0$ of Q_0 can be represented by $(1,t^2,t) : t\in GF(q^2)$ together with $(0,1,0)$. The image $\mathcal{X}$ of $\mathcal{C}_0$ in the Hermitian Veronesean $\mathcal{H}$ is then given by

$$\{P(1,t^{2q+2},t^{q+1},t^{2q},t^2,t^q,t,t^{2+q},t^{2q+1}) : t \in GF(q^2)\} \cup \{P(0,1,0,\ldots,0)\}.$$

Thus $\mathcal{X}$ is a rational curve, all of whose points lie in a Baer subgeometry. Put another way, $\mathcal{X}$ is just the orbit of $\rho(SO(3,q^2))$ on $PG(8,q^2)$ given by $\{P(v_1g\otimes v_1g^\psi) : g \in SO(3,q^2)\}$. A point $x\otimes x^\psi$ of $\mathcal{H}$ is singular precisely when x is singular. Hence if $\mathcal{Q}$ is the quadric corresponding to the bilinear form f the points of $\mathcal{Q}$ lying on $\mathcal{H}$ are precisely the points of $\mathcal{X}$, i.e., $\mathcal{X}$ is the intersection of $\mathcal{H}$ and $\mathcal{Q}$. No two points of $\mathcal{X}$ are orthogonal so $\mathcal{X}$ is a partial ovoid.

There is a further geometric description. Using the geometric setting of Section 1.2, take a conic C in π and $\bar{C}$ in $\bar{\pi}$. The lines joining a point on C with its conjugate on $\bar{C}$ form a set $\mathcal{Y}$ of q^2+1 lines defined over $GF(q)$, and it lies in the subgeometry $\mathcal{G}$ of $PG(5,q^2)$.

The image of $\mathcal{Y}$ on the Grassmannian $G_{1,5}$ of lines of $PG(5,q)$, under the Plücker map, is the curve $\mathcal{X}$.

Proposition 1.7. *Let X be the full stabilizer of the rational curve $\mathcal{X}$ in $\Omega(9,q)$ (q odd), then X contains a subgroup isomorphic to $PSL(2,q^2)\cdot C_2$.*

Theorem 1.8. *The full stabilizer X of the rational curve $\mathcal{X}$ is almost simple and is an absolutely irreducible subgroup of $\Omega(9,q)$.*

Corollary 1.9. *Assume that $q \neq 3$. If Kleidman's list in* [5] *is correct, then X is isomorphic to $PSL(2,q^2)\cdot C_2$ and is maximal in $P\Omega(9,q)$.*

In considering the case $q = 3$ we find the following.

Corollary 1.10. *If $q = 3$ and Kleidman's list in* [5] *is correct, then X is isomorphic to A_{10} and is maximal in $P\Omega(9,3)$.*

1.4.2. *Generalizations.* Let us consider possible generalizations of the ideas above. On this occasion we consider different forms as well as mappings from subgroups of $GL(n,q^t)$ to $GL(n^t,q)$, and we consider possible embeddings of alternating groups.

If $O(n,q^t)$ is the orthogonal group of a non-degenerate symmetric bilinear form f_0 on $V(n,q^t)$ (with q odd) and if ρ is the representation of $GL(n,q^t) \to GL(n^t,q^t)$ described in Subsection 1.2.1, then $\rho(O(n,q^t)$ preserves a non-degenerate symmetric bilinear form f on $V(n^t,q)$. If f_0 can be given by a matrix with entries in $GF(q)$, then $f = f_0 \otimes \cdots \otimes f_0$ (t copies of f_0); in other cases some care is required in writing down f. If an appropriate basis is chosen for V_0, then f is defined on $V_q = W$ over $GF(q)$ and $\rho(O(n,q^t)) \leq O(n^t,q)$. If we assume $n \geq 3$ and exclude the case $O^+(4,q^t)$, the subgroup $\rho(\Omega(n,q^t))$ is absolutely irreducible and cannot be written over a subfield of $GF(q)$.

If $Sp(n,q^t)$ is the symplectic group of a non-degenerate alternating form f_0 on $V(n,q^t)$ (with n even but q odd or even), then $\rho(Sp(n,q^t))$ preserves the tensor product form f. If t is odd, then f is an alternating form and we find that $\rho(Sp(n,q^t))$ is a subgroup of $Sp(n^t,q)$. If t is even and q is odd, then f is a symmetric bilinear form and $\rho(Sp(n,q^t))$ is a subgroup of $O(n^t,q)$. If q is even (and n must then be even), then $O(n,q^t)$ maybe regarded as a subgroup of $Sp(n,q^t)$ so $\rho(O(n,q^t)) \leq Sp(n^t,q)$, but more than this $\rho(Sp(n,q^t))$ preserves a quadratic form on $V_q = W$ so $\rho(O(n,q^t)) \leq \rho(Sp(n,q^t)) \leq O(n^t,q)$. If $U(n,q^t)$ is the unitary group of a non-degenerate Hermitian form f_0 on $V(n,q^t)$ (with q square and t odd), then the tensor product form f is an Hermitian form preserved by $\rho(U(n,q^t))$ and $\rho(U(n,q^t)) \leq U(n^t,q)$. [Except in the case of $O^+(4,q^t)$, the image under ρ is absolutely irreducible and cannot be written over a subfield of $GF(q)$.]

It is worth noting that the restrictions on n mean that there is no irreducible subgroup $\rho(Sp(3,q^2))$ of $SL(9,q)$ and thus, for q even, no irreducible subgroup $\rho(O(3,q^2))$ of $SL(9,q)$. The restriction on t for $U(n,q^t)$ is more subtle. Steinberg's Tensor Product Theorem leads us to believe that for t even $\rho(U(n,q^t))$ is not absolutely irreducible. Indeed for the case $t = 2$ it is known that $\rho(U(n,q^2))$ is reducible, for it follows from [3, Theorem 43.14] that $\rho(U(n,q^2))$ fixes all vectors in a 1-dimensional subspace of $V(n^2,q^2)$; moreover the restriction of the Hermitian form f to $V_q = W$ is actually a symmetric bilinear form so $\rho(U(n,q^2))$ is a subgroup of $O(n^2,q)$ (for q odd) or $Sp(n^2,q)$ (for q even).

2. Embedding $Sp(2m,q^t)$ in $\Omega^\epsilon((2m)^t,q)$

2.1. **Introduction.** Let V_0 now be a $2m$-dimensional vector space over $GF(q^t)$. Then $V = V_0 \otimes V_0 \otimes \cdots \otimes V_0$ (t copies of V_0) is a vector space of dimension $(2m)^t$. If f_0 is a non-degenerate alternating form on V_0 then one can define a non-degenerate alternating form on V as follows:

$$f(u_1 \otimes \cdots \otimes u_t, w_1 \otimes \cdots \otimes w_t) = \prod_{i=1}^{t} f_0(u_i, w_i).$$

Moreover there exists a unique quadratic form Q on V such that $Q(u_1 \otimes \cdots \otimes u_t) = 0$ for all $u_i \in V_0$ and such that f is the bilinear form associated with Q ([1]). If U is an m-dimensional totally isotropic subspace of V_0, then $U \otimes V_0 \otimes \cdots \otimes V_0$ is a totally singular subspace of V of dimension $(2m)^t/2$, so Q is a quadratic form of maximal Witt index.

Let G be the group $Sp(2m, q^t)$ acting on V_0 and preserving f_0. Let $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_m$ be a symplectic basis for V_0 (so $f_0(x_i, x_j) = f_0(y_i, y_j) = 0$ and $f_0(x_i, y_j) = \delta_{ij}$). The action of G on V, as described in the introduction, is given by $u_1 \otimes \cdots \otimes u_t \mapsto u_1 g \otimes u_2 g^{\psi} \otimes \cdots \otimes u_t g^{\psi^{t-1}}$ with g^{ψ^i} preserving f_0 for each i. Thus g preserves Q on V. Taking note of the fact that $Sp(2m, q^t)$ is perfect, we thus have a representation $\rho : G \to \Omega^+((2m)^t, q^t)$.

In Section 1.1.2 we introduced a semi-linear map ϕ on V and the subset W of V consisting of all vectors fixed by ϕ. Here we prove a number of properties of ϕ and W that we shall need to refer to. We recall that $\phi(\lambda u_1 \otimes u_2 \otimes \cdots \otimes u_t) = \lambda^q u_t \otimes u_1 \otimes \cdots \otimes u_{t-1}$, whenever each u_i is one of $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ and $\lambda \in GF(q^t)$ and then ϕ is extended linearly over $GF(q)$. If we write $v^{\psi} = \sum(\lambda_i^q x_i + \mu_i^q y_i)$ when $v = \sum(\lambda_i x_i + \mu_i y_i) \in V_0$, then $\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_t) = v_t^{\psi} \otimes v_1^{\psi} \otimes \cdots \otimes v_{t-1}^{\psi}$. We have not explicitly referred to the case: $m = 1$ here. However, given any basis x_1, y_1 for V_0 when $m = 1$ we can define an alternating form f_0 on V_0 such that $f_0(x_1, y_1) = 1$ and the symplectic group $Sp(2, q^t)$ is just $SL(2, q^t)$.

Lemma 2.1. (i) *The semilinear map ϕ commutes with $\rho(G)$ on V;*

- (ii) *The set $W = \{w \in V : \phi(w) = w\}$ is a $GF(q)$-subspace of V, (globally) stabilized by $\rho(G)$;*
- (iii) *Any vectors in W that are linearly independent over $GF(q)$ are linearly independent over $GF(q^t)$;*
- (iv) *W has dimension $(2m)^t$ over $GF(q)$ and spans V over $GF(q^t)$.*
- (v) *$Q(w) \in GF(q)$ for each $w \in W$ and Q is non-degenerate on restriction on W.*

Since $Sp(2m, q^t)$ has a trivial centre and is simple, ρ is injective. Thus we may regard $Sp(2m, q^t)$ as a subgroup of $\Omega^{\epsilon}(2^t, q)$, and equivalently in the projective context, $PSp(2m, q^t)$ as a subgroup of $P\Omega^{\epsilon}((2m)^t, q)$.

2.2. **The nature of the quadrics.** It is of some interest to know the nature of the quadratic form on W described above. The investigation is complex but yields the following.

Theorem 2.2. *If t is odd, then Q has maximal Witt index on W. If t is even, then Q has maximal Witt index on W except when $t = 2$ and $m = n/2$ is odd. In the exceptional case, Q has non-maximal Witt index on W.*

2.3. **Embedding $PG(2m-1, q^t)$ in $PG((2m)^t - 1, q)$.**

2.3.1. *Embedding the projective line and partial ovoids.* Given $0 \neq v \in V_0$ let us denote by $\underline{v}$ the vector $v \otimes v^{\psi} \otimes \ldots \otimes v^{\psi^{t-1}} \in V$; recall that all such vectors lie in W and are singular. Observe that for $0 \neq \lambda \in GF(q^t)$ we have $\underline{\lambda v} = (\lambda\lambda^q \ldots \lambda^{q^{t-1}})\underline{v}$ with $\lambda\lambda^q \ldots \lambda^{q^{t-1}} \in GF(q)$. Hence the injective map: $V_0 \to W$, $v \to \underline{v}$ leads to an injective map $\varphi : PG(2m-1, q^t) \to PG((2m)^t - 1, q)$. Suppose that $x, y \in V_0$ such that $f_0(x, y) \neq 0$. Then

$$\begin{aligned} f(\underline{x}, \underline{y}) &= f_0(x, y) \cdot f_0(x^{\psi}, y^{\psi}) \cdot \cdots \cdot f_0(x^{\psi^{t-1}}, y^{\psi^{t-1}}) \\ &= f_0(x, y) \cdot f_0(x, y)^q \cdot \cdots \cdot f_0(x, y)^{q^{t-1}} \neq 0 \,. \end{aligned}$$

This leads to the following theorem:

Theorem 2.3. *Suppose that $\mathcal{L}$ is either the projective line $PG(1,q^t)$ or a partial ovoid of a symplectic polarity of $PG(2m-1,q^t)$ with $m \geq 2$, and that $\mathcal{P} = \varphi(\mathcal{L})$ in $PG((2m)^t-1,q)$. Then $\mathcal{P}$ is a partial ovoid of a non-degenerate quadric in $PG((2m)^t-1,q)$ having the same size as $\mathcal{L}$. The quadric is hyperbolic unless $t=2$ and $m=n/2$ is odd, in which case it is elliptic.*

□

2.3.2. *The Blokhuis-Moorhouse bound.* In their 1995 paper [11] Blokhuis and Moorhouse give an upper bound for size of a partial ovoid of a classical polar space in $PG(k,q)$. If $q = p^e$ where p is prime, then the size of a partial ovoid is no greater than $\binom{k+p-1}{k}^e + 1$. If $p=2$, then this bound becomes simply $(k+1)^e+1$. If, in addition, $k+1=2^t$ for some t, then the bound is $2^{te}+1 = q^t+1$. In particular we get the same value for the bound in $PG(2^a-1, 2^{et})$ and $PG(2^{at}-1, 2^e)$ for any $a \geq 1$.

The following theorem is a corollary to Theorem 2.3. It demonstrates that, for $p=2$, the Blokhuis-Moorhouse bound is sharp for arbitrarily large dimension of the form 2^t-1.

Theorem 2.4. *If $\mathcal{L}$ is the projective line $PG(1,q^t)$, then $\varphi(\mathcal{L})$ is a partial ovoid of a non-degenerate quadric in $PG(2^t-1,q)$ whose size attains the Blokhuis-Moorhouse bound. If $\mathcal{L}$ is a partial ovoid of a symplectic polarity of $PG(2^a-1,q^t)$ (with $a \geq 2$ and $q = 2^e$) whose size attains the Blokhuis-Moorhouse bound, then $\varphi(\mathcal{L})$ is a partial ovoid of a non-degenerate quadric in $PG(2^{at}-1,q)$ whose size attains the Blokhuis-Moorhouse bound.*

2.3.3. *Complete partial ovoids from embeddings of $PG(1,q^t)$ and ovoids of $PG(3,q^t)$.* Now let us consider complete partial ovoids in $PG(2^t-1,q)$ and $PG(4^t-1,q)$ arising as images of projective lines and of elliptic quadrics or Suzuki-Tits ovoids of $PG(3,q^t)$ respectively. We identify the stabilizers of these partial ovoids using the the classification of finite 2-transitive groups. In turn this relies on the Classification of Finite Simple Groups. Our source for the list of finite 2-transtive groups is [12] where the groups are listed in Tables 7.3 and 7.4. Our interest is in groups having permutation degree 2^k+1 for some $k \geq 1$.

Result 2.5. *A 2-transitive permutation group of degree 2^k+1 for some $k \geq 1$ is almost simple with unique minimal normal subgroup one of the following: A_M with $M = 2^k+1$, $SL(2,2^k)$, $PSU(3,2^{2k/3})$, $Sz(2^{k/2})$ where $k/2$ an odd integer.*

We consider three possibilities:

- Case PL: $\mathcal{L}$ is the projective line $PG(1,q^t)$ with $t \geq 3$, $G = SL(2,q^t)$, $2^k = q^t$, we write $\Omega = \Omega^+(2^t,q)$ and $PW = PG(2^t-1,q)$.
- Case EQ: $\mathcal{L}$ is an elliptic quadric of $PG(3,q^t)$, $G = \Omega^-(4,q^t)$, $2^k = q^{2t}$, we write $\Omega = \Omega^+(4^t,q)$ and $PW = PG(4^t-1,q)$.
- Case STO: $\mathcal{L}$ is a Suzuki-Tits ovoid of $PG(3,q^t)$ with q an odd power of 2 and t odd, $G = Sz(q^t)$, $2^k = q^{2t}$, we write $\Omega = \Omega^+(4^t,q)$ and $PW = PG(4^t-1,q)$.

We write $\mathcal{P} = \varphi(\mathcal{L})$, $\tilde{F} = \rho(F)$ for any subgroup F of G and let $\tilde{H}$ be the stabilizer of $\mathcal{P}$ in Ω. Then $\tilde{H}$ contains $\tilde{G}$ and acts 2-transitively on $\mathcal{P}$. The action of $\tilde{G}$ on the vector space W is irreducible by Steinberg's Tensor Product Theorem ([10, Theorem 7.4, Theorem 12.2]) and so the points of $\mathcal{P}$ span the corresponding projective space. As the number of points exceeds the vector space dimension and $\tilde{G}$ acts 2-transitively, the only transformations fixing each point of $\mathcal{P}$ are scalar maps, here the identity is the only possibility. Thus the action of $\tilde{H}$ is faithful and we have:

Proposition 2.6. *$\tilde{H}$ is almost simple with unique minimal normal subgroup X being isomorphic to one of the following: A_M with $M = 2^k+1$, $SL(2,2^k)$, $PSU(3,2^{2k/3})$, $Sz(2^{k/2})$ where $k/2$ is an odd integer.*

The subgroup $X \cap \tilde{G}$ is normal in $\tilde{G}$, but $\tilde{G}$ is simple so either $X \cap \tilde{G} = 1$ or $\tilde{G} \leq X$.

Proposition 2.7. $\tilde{G} \leq X$.

Proposition 2.8. *$X = \tilde{G}$ except when $q = 2$, in which case $X = A_M$ with $M = 2^k + 1$.*

Proposition 2.9. *The normalizer of X stabilizes $\mathcal{P}$.*

In summary, the results above give:

Theorem 2.10. (a) *Suppose that $q \geq 2$ is even and that $t \geq 3$. If $\mathcal{L} = PG(1, q^t)$, then the stabilizer of $\mathcal{P}$ in $\Omega^+(2^t, q)$ is the normalizer of $\rho(SL(2, q^t))$.*

(b) *Suppose that $q \geq 2$ is even and that $t \geq 2$. If $\mathcal{L}$ is an elliptic quadric in $PG(3, q^t)$, then the stabilizer of $\mathcal{P}$ in $\Omega^+(4^t, q)$ is the normalizer of $\rho(\Omega^-(4, q^t))$.*

(c) *Suppose that $q \geq 2$ is an odd power of 2 and that $t \geq 3$ is odd. If $\mathcal{L}$ is a Suzuki-Tits ovoid in $PG(3, q^t)$, then the stabilizer of $\mathcal{P}$ in $\Omega^+(4^t, q)$ is the normalizer of $\rho(Sz(q^t))$.*

(d) *If $q = 2$, then in all three cases $\mathcal{P}$ is a polygon whose stabilizer is the alternating group on the points.*

Remark 2.11. Comparing the partial ovoids arising as images of Suzuki-Tits ovoids with those arising as the images of elliptic quadrics, the difference in the structure of the stabilizers shows that the two families of partial ovoids are not equivalent. The cases where $\mathcal{L} = PG(1, q^t)$ (with $t \geq 2$ even) and where $\mathcal{L}$ is an elliptic quadric of $PG(3, q^t)$ lead to stabilizers with the same structure, but it is not clear whether or not the partial ovoids are equivalent. Indeed, the embedding procedure allows for the possibility of a stepped embedding of $PG(1, q^{abcd\cdots})$ as a partial ovoid in $PG(2^a - 1, q^{bcd\cdots})$ that is then embedded in $PG(2^{ab} - 1, q^{cd\cdots})$ and so on, and it is not clear that the resulting partial ovoid is necessarily equivalent to that obtained from a single step embedding of $PG(1, q^{abcd\ldots})$ in $PG(2^{abcd\ldots} - 1, q)$.

2.3.4. *Dye's partial ovoids.* In [14], R.H. Dye constructed new ovoids on a hyperbolic quadric in $PG(7, 8)$. Later, in [15], he constructed partial ovoids on quadrics (sometimes elliptic, sometimes hyperbolic) in $PG(2m + 1, 8)$ whose size attains the Blokhuis-Moorhouse bound. The construction of ovoids in $PG(7, 8)$ involved an initial construction of nine points of a polygon, followed by the addition of points lying on conics formed by the intersection of various planes with the quadric. This approach formed the basis of the constructions of partial ovoids in the later paper. We are led to ask whether the partial ovoids we have constructed in $PG(2^t - 1, 8)$ could be those constructed by Dye. The answer is no for the reason that Dye's partial ovoids, by construction, meet some planes in conics, whereas our partial ovoids never meet a plane in a conic.

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