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Multicontact maps: results and conjectures

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Abstract[1]. A multicontact structure on a manifold is a choice of several distinguished mutually transversal subbundles of the tangent space. A smooth map preserving these subbundles is a multicontact map. This notion arises naturally in parabolic geometries, and a result of M. Cowling, F. de Mari, H. M. Reimann, and the author shows (among other things) that the local group of all multicontact maps is finite dimensional in these cases. There is some further evidence to show that this may be true in all cases that are non-degenerate in a suitable sense. The article[2] contains a discussion of existing results and methods with some further speculations.

1. Introduction

The usual notion of contact structure on a manifold has been extended, and much studied, in the following direction: a distribution, more exactly a subbundle of the tangent bundle which is in an appropriate sense non-degenerate, is fixed on a manifold. In recent joint work with M. Cowling, F. de Mari and H. M. Reimann [5] there appears a further generalization which does not seem to have been studied, at least not explicitly. This is the notion of a *multicontact structure*, where several distributions satisfying a joint non-degeneracy condition are fixed on a manifold. A *multicontact map* is then defined as a smooth map preserving all these distributions or possibly permuting them.

In the following we will use the word "multicontact" as implying that there are more than one distributions in the structure (even though the case of a single distribution could be considered to be a special case). In [5] it was found that all the multicontact structures that arise naturally in parabolic geometry have finite dimensional Lie groups as their automorphism groups (actually the groups governing the geometry themselves). This is true in the strong sense that any diffeomorphism between two connected open sets which preserves the structure is the restriction of an element of the group. In the case of one single distribution (a generalized contact structure), which occurs when the parabolic subgroup under consideration is maximal, the (local) automorphism group is sometimes finite dimensional, sometimes not. One can raise interesting questions here too (cf. the discussion in [12]), but in the present article I will discuss only properly multicontact maps.

The results in [5] are about structures defined with the aid of of the theory of semisimple Lie groups, so from the general point of view they are very special. Nevertheless, together with some further evidence that will be mentioned later, they make it not unreasonable

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to conjecture that quite generally every multicontact structure which is non-degenerate in an appropriate sense has finite dimensional local automorphism group.

The purpose of this article is to make some of this more precise and to discuss briefly the existing results about the subject. The article is a considerable improvement over my talk at the Levico meeting: this is due to the numerous comments made by L. Boutet de Monvel, M. Eastwood, R. Graham, C.-K. Han, G. Schmalz and D. Zaitsev. I am grateful to all of them for the permission to incorporate their remarks in this article.

2. The simplest special case and a possible general statement

The simplest non-trivial multicontact structure is gotten by taking an open set in $\mathbb{R}^3$ with two one-dimensional distributions on it. The distributions can be thought of as being generated by vector fields X, Y (determined only up to non-zero scalar factors). The only reasonable non-degeneracy condition in this case is that X, Y and $[X, Y]$ are linearly independent at all points. This amounts to the same as saying that the sum of the two distributions is direct and is a contact structure in the classical sense.

Before continuing let us note that the most convenient way to talk about local automorphisms is to say that a vector field V is an *infinitesimal automorphism* of a structure if the local flow $\{\exp tV\}$ generated by V consists of automorphisms. When a multicontact structure is given by the distributions $D_1, \cdots, D_k$, this means that if X is a vector field belonging to some D_j, then so is $[V, X]$. These vector fields V always form a Lie algebra; the question is whether this algebra, which we will call the *automorphism algebra* is finite dimensional or not.

Returning to the 3-dimensional example we see that the infinitesimal automorphisms are the vector fields V such that

$$[V, X] = \lambda X \ , \qquad [V, Y] = \mu Y$$

with some non-zero functions λ, μ.

In this case, i.e. the most general 3-dimensional case, it is "well-known" (even though I did not know it until the conference) that the automorphism algebra can have at most 8 dimensions. At least one version of the proof is explicitly in the literature and will be discussed in Section 4. Other essentially equivalent statements are known and will be mentioned in what follows. The proofs are all in the same spirit although with fairly widely differing details. A very short and simple proof is due to C.-K. Han and is to be found in this volume.

Even though the 3-dimensional case is far from being typical, it is interesting to see what the problem looks like after making some elementary reductions. As well known, one can make a variable change such that $X = \partial/\partial x$. With a further variable change (cf. [7] in the complex case which is no different from the real case) one can arrange that $Y = \partial/\partial y + x\partial/\partial z + B(x, y, z)\partial/\partial x$. (A geometric implication of this is a strengthening of Darboux's theorem: a contact structure in $\mathbb{R}^3$ with a distinguished direction X given in each contact plane can be brought to the normal form span $\{\partial/\partial x, \partial/\partial y + x\partial/\partial z\}$ in such a way that $X = \partial/\partial x$. For further geometric remarks cf. [1], pp. 42-56). In the case $B \equiv 0$ the system can easily be solved by hand; one finds that the solution space is 8-dimensional: The general result, that the dimension of the solution space is at most 8, is more difficult.

Perhaps it appears already from the normal form above, and it will be very clear from Section 4, that in higher dimensions the situation can be much more complicated. It is already not clear what the correct non-degeneracy conditions are. (Here I am particularly grateful to D. Zaitsev for his remarks). Of course, it has to be true that the repeated brackets of vector fields belonging to the various distributions D_j span the full tangent space at all points (a Hörmander condition), but this is far from enough. A simple example to show this is in four variables x, y, z, t with three one-dimensional distributions spanned by the vector fields $\partial/\partial x$, $\partial/\partial y + x\,\partial/\partial z$, $\partial/\partial t$, respectively. This is the product on an $\mathbb{R}^3$ example as above and of the t-space: clearly the automorphism algebra in infinite dimensional.

One can, of course, take the sum D of all the distributions D_j $(1 \leq j \leq k)$ and impose a non-degeneracy condition on D. This approach is too crude because it ignores the multicontact structure and deals only with a generalized contact structure. Perhaps the following condition would be sufficient: at each point p, for each i, j we define the form $L_{ij}(\xi, \eta)$ on $D_{i,p} \times D_{j,p}$ by extending ξ and η to vector fields X, Y belonging to D_i respectively D_j and then taking $[X, Y]_p$ modulo $D_i + D_j$. For every $\xi \neq 0$ in $D_{i,p}$ there should exist j and $\eta \in D_{j,p}$ such that $L_{ij}(\xi, \eta) \neq 0$.

Finding the right non-degeneracy condition is certainly part of the problem. One should not really talk of a conjecture, but rather a question: is there a good general condition that guarantees that the automorphism algebra of a multicontact structure is finite-dimensional? Put in this form it makes sense also if the "contact" case (one single distribution) is included. But the contact case (cf. Section 3 and [12]) has a different flavour and seems to be more difficult. To guarantee finiteness of the automorphism algebra may require a condition that is much too strong for the properly multicontact case.

3. The results in [5]

Let G be a real simple Lie group, $\mathfrak{g} = \mathfrak{k} + \mathfrak{q}$ a Cartan decomposition of its Lie algebra. One chooses a maximal Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{q}$ and one considers the joint eigenspaces of $ad(\mathfrak{a})$ in $\mathfrak{g}$. The corresponding eigenvalues are linear functions on $\mathfrak{a}$; those that are not identically zero are called the $\mathfrak{a}$-roots. They can be given a linear order; the minimal positive ones are called the simple $\mathfrak{a}$-roots. Fixing a subset of the simple $\mathfrak{a}$-roots and denoting their joint kernel by $\mathfrak{a}_p$ one looks now at the joint eigenspaces of $ad(\mathfrak{a}_p)$. The corresponding joint eigenvalues are the $\mathfrak{a}_p$-roots (they are just restrictions of the $\mathfrak{a}$-roots), to be called roots in what follows. They inherit the ordering. The joint eigenspace corresponding to a root α will be denoted $\mathfrak{n}_\alpha$, the set Δ of minimal positive roots will again be called simple roots. Now

$$\mathfrak{n}_p = \sum_{\alpha>0} \mathfrak{n}_\alpha \quad , \quad \overline{\mathfrak{n}}_p = \sum_{\alpha>0} \mathfrak{n}_{-\alpha}$$

are nilpotent subalgebras and there is a vector direct sum decomposition

$$\mathfrak{g} = \overline{\mathfrak{n}}_p + \mathfrak{a}_p + \mathfrak{m}_p + \mathfrak{n}_p$$

where $\mathfrak{m}_p$ is a reductive subalgebra centralizing $\mathfrak{a}_p$. $\mathfrak{p} = \mathfrak{m}_p + \mathfrak{a}_p + \mathfrak{n}_p$ is now a parabolic subalgebra, its normalizer P in G is a parabolic subgroup. One has the unique decomposition $P = M_p A_p N_p$ as a product of subgroups (corresponding to the subalgebras above). The homogeneous space G/P is the model space of a "parabolic geometry". Its tangent space at the base point is identified with $\overline{\mathfrak{n}}_p$, so it is the direct sum of the subspaces $\overline{\mathfrak{n}}_{-\alpha}$ $(\alpha > 0)$. It can also be thought of as the direct sum of the subspaces $\mathfrak{g}_{-i} = \sum_{\ell(\alpha)=i} \mathfrak{n}_{-\alpha}$ where $\ell(\alpha)$, the level of α, is the number of roots appearing in the representation of α as a sum of simple roots.

A remarkable fact, not difficult to prove, is that under the action of G the subspaces $\mathfrak{n}_{-\delta}$ for $\delta \in \Delta$ propagate to form subbundles of $T(G/P)$, thus defining a G-invariant multicontact structure on G/P. (On the natural imbedding of $\overline{N}$ into G/P as an open dense orbit this structure is, of course, just the one given by left $\overline{N}$-translations.) The sum of these subbundles gives a (generalized) contact structure, determined by $\mathfrak{g}_{-1}$. The structure is properly multicontact when there are at least two simple roots, i.e. exactly when P is not a maximal parabolic subgroup. Otherwise there is only a contact structure present.

In either case there is a further structure. The action of G on the tangent space identified with $\overline{\mathfrak{n}}_p$ is such that at each point it acts as an element of the group $M_p A_p$ via the adjoint representation of G. One can express this by saying that G/P has a *conformal* structure, and G acts by conformal maps.

When $G = O(n+1,1)$, $n \geq 3$, then $dim\,\mathfrak{a} = 1$, so there is only one parabolic subgroup P (with $\mathfrak{a}_p = \mathfrak{a}$). Now $G/P \cong S^n$. There is only one positive root, so the "contact" condition is vacuous. On $\overline{\mathfrak{n}}_p \cong \mathbb{R}^n$ the group M_pA_p acts as $O(n) \cdot \mathbb{R}^+$ giving the classical conformal structure. When $G = SU(n,1)$, $n \geq 2$, there is still only one P but two positive roots, α and 2α. G/P is S^{2n-1} with the standard contact structure, M_pA_p acts on $\mathfrak{g}_{-1} \cong \mathbb{C}^{n-1}$ as $SU(n-1) \cdot \mathbb{R}^+$, giving CR geometry. When $G = SL(3,\mathbb{R})$, there is a non-maximal parabolic P, namely the triangular subgroup. The corresponding $\overline{\mathfrak{n}}_p$ is the 3-dimensional Heisenberg group; $\mathfrak{g}_{-1}$ gives its standard contact structure. There are two simple roots, they give a multicontact structure of the type described in the Introduction.

The situation considered in [5] is that of G/P where P is any fixed non-maximal parabolic. It is also assumed that G is the largest centerless version of itself, i.e. $G = Aut(\mathfrak{g})$. The main result states that if f is a C^2 multicontact diffeomorphism of a connected open subset of G/P onto another, then f is the restriction of some element in G. (So the automorphism algebra of the multicontact structure is $\mathfrak{g}$.) For the same class of spaces it is also shown that with a certain number of exceptions the same conclusion already follows from the hypothesis that f is contact (i.e. preserves $\mathfrak{g}_{-1}$).

In the proof first it is shown that every infinitesimal automorphism V is determined by a (vector-valued) function p satisfying a differential system which is easy to write down in a canonical coordinate system of $\overline{N}_p$. Then, using properties of the root structure it is shown that all solutions of the system are polynomial. Next, with an intrinsic notion of degree on $\overline{N}_p$, one quickly sees that if p is of homogeneous degree less than L, the highest possible level of a root, then the corresponding V belongs to the Lie algebra $\overline{\mathfrak{n}}_p$. There is an involution in the Weyl group of G which maps every solution p of homogenous degree $> L$ onto one of degree $< L$. From this one sees then that every V arises as the action of some element of $\mathfrak{g}$. In the last step the result is globalized, it is shown that every automorphism comes from some element of $G = Aut(\mathfrak{g})$.

It is to be emphasized that the result on multicontact maps does not involve any case-by-case checking, and is true without any exceptions. Neither of these statements is true for the result on contact maps. This may be taken as an indication that, even in the absence of any group homogeneity, multicontact structures are easier to handle and have stronger properties than contact structures.

4. The Tanaka-Yamaguchi results

The results to be discussed here do not mention the notion of multiconformality, yet they contain many of the results of [5], and in certain directions much more.

In the 1960's N. Tanaka started the modern study of generalized contact structures (one subbundle D of the tangent bundle). The classical theory of prolongations of G-structures is not of much use here: it only records that a subspace of the tangent space is preserved. Whether the automorphism algebra of such a structure is finite dimensional depends on non-degeneracy conditions to which the G-structure is insensitive. Tanaka's theory is meant to remedy this. In [10] he considers the following setup.

There is a sequence of subbundles of the tangent bundle $T(M)$:

$$D = D^{-1} \subset D^{-2} \subset \cdots \subset D^{-m} = T(M)$$

such that

$$\mathcal{D}^p = \mathcal{D}^{p+1} + [\mathcal{D}^{p+1}, \mathcal{D}^{-1}] \ , \qquad (p < -1)$$

where $\mathcal{D}^p$ denotes the sheaf of local cross-sections on D^p. At every $x \in M$,

$$\mathfrak{m}(x) = \bigoplus_{p<0} D_x^p / D_x^{p+1}$$

has a natural graded nilpotent Lie algebra structure defined in the obvious way (taking representatives, extending them to vector fields, taking brackets, etc.). It is assumed throughout that for all $x \in M$, $\mathfrak{m}(x)$ is isomorphic to one Lie algebra, which is then denoted by $\mathfrak{m}$.

For $\mathfrak{m}$, indeed for any graded nilpotent Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ with $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ the *algebraic prolongation* $\mathfrak{g}(\mathfrak{m})$ is defined as the largest $\mathbb{Z}$-graded algebra $\bigoplus \mathfrak{g}_p(\mathfrak{m})$ such that $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$ and such that no element in $\mathfrak{g}_p(\mathfrak{m})$ for $p \geq 0$ centralizes $\mathfrak{m}$. (Note that $\mathfrak{g}_0(\mathfrak{m})$ is then the Lie algebra of the group of graded automorphisms of $\mathfrak{m}$. There is also a characterization of $\mathfrak{g}(\mathfrak{m})$ in terms of Lie algebra cohomology.) Tanaka proves that if $\mathfrak{g}(\mathfrak{m})$ is finite dimensional then so is the automorphism algebra of the contact structure.

In [13] Yamaguchi applies this result to the case where $\mathfrak{m}$ is the algebra $\overline{\mathfrak{n}}_p$ obtained from a simple group and a parabolic subgroup, with $\mathfrak{g}_{-1} = \bigoplus_{\delta \in \Delta} \mathfrak{n}_\delta$. Using Kostant's results on Lie algebra cohomology and some classification theory he proves that, with the exception of a certain list of cases, the automorphism algebra of the contact structure is finite (in fact isomorphic with the Lie algebra of the initial simple group). This contains, up to the exceptions, the results of [5], together with results about the maximal parabolic case, on the infinitesimal level (i.e. up to statements about the actual automorphism group).

The 3-dimensional "simplest case" is not covered by this result. It is among the exceptions, as it has to be, since the automorphism algebra of an ordinary contact structure is well known to be infinite-dimensional. There is, however, more in Tanaka and Yamaguchi: given a fixed subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}_0(\mathfrak{m})$ in the setup above, Tanaka defines the relative prolongation $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ of $\mathfrak{m}$ by defining inductively for $k > 0$, $\mathfrak{g}_k = \{X \in \mathfrak{g}_k(\mathfrak{m}) \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{k-1}\}$ and taking the sum of these with $\mathfrak{m}$ and $\mathfrak{g}_0$. (This is an analogue of the $\mathfrak{g}_0$-reduction of the frame bundle. We should refer here to the more general framework in [8] and [4] where the analogy with G-structures is brought out more clearly.) Finiteness of $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ implies that those infinitesimal automorphisms of the contact structure whose action on the contact plane belongs to $ad(\mathfrak{g}_0)$ form a finite-dimensional Lie algebra. In [13] Yamaguchi considers the case of $\mathfrak{m} = \overline{\mathfrak{n}}_p$ with $\mathfrak{g}_0$ defined to be $\mathfrak{m}_p + \mathfrak{a}_p$ (in the notation of Section 3), i.e. the case of "conformal maps". He finds that with very few exceptions, in each of which P is a maximal parabolic, this choice of $\mathfrak{g}_0$ gives a finite $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$.

Of course, "conformal" is a stronger hypothesis than "multicontact" (which in turn is stronger than "contact"). But the multicontact case can also be fitted into Tanaka's theory. One has to choose $\mathfrak{g}_0$ to be the centralizer of the subalgebra $\mathfrak{a}_p$ in $\mathfrak{g}_0(\mathfrak{m})$: this corresponds exactly to those contact maps which preserve each simple root space individually. The (infinitesimal version of the) [5] results described in Section 3 can then undoubtedly be extracted from [13] by checking the list of exceptions from the contact result.

This is particularly easy where $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ and P is the non-maximal parabolic. Then $\mathfrak{g}_0(\mathfrak{m}) = \mathfrak{a}_p = \mathfrak{a}$ (and $\mathfrak{m}_p = 0$), so the multicontact and conformal maps are the same. In the 3-dimensional simplest example of multiconformal structure the algebra $\mathfrak{m}(x)$ is automatically isomorphic, at every x, with the $\overline{\mathfrak{n}}_p$ here. Therefore, for this example, the finiteness of the automorphism algebra of the multicontact structure follows from the Tanaka-Yamaguchi results.

5. Final remarks

The methods described in Section 4 originate in S. Lie's work on differential equations. Lie did not use geometric language; the geometric reformulation and further development of his results is due to E. Cartan. Fundamental in the theory is the notion of a Cartan connection which permits the reformulation of the question of equivalence of the differential equations as a question of equivalence of connections, which can then be studied by geometric methods.

This seems at the moment to be the only really general theory with which the problem of multicontact maps can be attacked. For it to be applicable a certain number of geometric hypotheses seem to be necessary, e.g. the Lie algebra $\mathfrak{m}(x)$ described in Section 4 must be independent of x. Even with such limitations, however, the Cartan methods can accomplish much, and there are a number of results that go beyond the situations described in Sections 3 and 4 where the geometry is governed by semisimple groups. In [6] it is shown that to a very large class of differential equations (the "holonomic systems", which include all systems of ODE-s) viewed up to contact transformations one can associate a Cartan connection.

Beside Tanaka's articles there is a lot more on the general theory, developing it further, and taking somewhat different approaches to it. Most important are perhaps the works of Morimoto [8] and Čap and Schichl [4]. The subject being vast and difficult it may be useful to mention here that the introductory discussions in [6] explain much of the basics in a very clear way. A much more comprehensive introduction to the subject can be found in [2]. In an appendix to this our "simplest case", the multicontact structure on $\mathbb{R}^3$, is discussed in some detail.

The "simplest case", even if it is not typical because of the low dimensions, is actually quite interesting. M. Eastwood calls it "real CR geometry" because of the close analogy with 3-dimensional CR geometry where it is the complexification of the contact plane that has two distinguished subspaces: those of the holomorphic and antiholomorphic tangent vectors. CR geometry has been much studied, and some of its results are of interest also for real CR geometry. In [3], which is a quick proof of a general result on bounding the dimension of the automorphism algebra, 3-dimensional CR geometry is discussed as an example, and the possible automorphism algebras are studied (for this cf. also [7]). As Eastwood pointed out to me, the real analogue of the number 5 in this context is 6, in the sense that if the automorphism algebra has less than 8 dimensions then it can have at most 6.

Returning to the general conjecture of the automorphism algebra of non-degenerate multicontact structures being finite, it must be admitted that the evidence for it is not exactly overwhelming. The examples in Sections 3 and 4 come from semisimple Lie groups and are therefore very special, with many symmetries. To strengthen our case, here is another example, not directly related to semisimple groups.

Let $J^k(\mathbb{R},\mathbb{R})$ be the manifold of k-jets of $\mathbb{R}$-valued functions on $\mathbb{R}$. (This is the same thing as a coordinate neighborhood in the k-jets of a 2-dimensional manifold.) It has its standard generalized contact structure; the horizontal space has 2 dimensions. As well known (cf. also [11], [12]) the automorphism algebra of this structure is infinite. Now let us define a multicontact structure by fixing two one-dimensional subbundles of the contact bundle. For this structure the automorphism algebra is finite. This is quite easy to prove either by further refining the computations on p. 336 of [11] or by using Bäcklund's Theorem (clearly explained in [12]) to reduce the question to the case of $J^1(\mathbb{R},\mathbb{R})$ where it amounts exactly to our "simplest 3-dimensional case".

After all this, it is still very unclear what the general situation will be. It may be possible to use the theory of Cartan connections to answer our question at least in a wide class of cases. It may also be possible to attack the question more directly with the aid of methods in hypoelliptic equations. At any rate, it is a meaningful geometrical question, and it is conceivable that it has an interesting answer.

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