Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 4(2005), pp. 115 - 119.

On projective embeddings

by Giuseppe Marino

Abstract[1]. Some embedding of semilinear and linear spaces in projective spaces is investigated.

1. Introduction and basics.

A *semilinear space* $\Gamma = (P_\Gamma, \mathcal{L}_\Gamma)$ is a point-line geometry such that any two distinct points lie on at most one line, every line contains at least two points and every point lies on at least one line. A line of a semilinear space is called *thick* (*thin*) if it contains at least three (exactly two) points. We say Γ *thick-lined* (*thin-lined*) if all of its lines are thick (thin). Two distinct points p and q are *collinear*, and we write $p \sim q$, if there exists a line containing p and q. We denote by $p \vee q$ such a line. The graph $(P_\Gamma, \sim)$ is the *collinearity graph* of Γ. Γ is *connected* if its collinearity graph is connected. For a point p, let $p^\sim$ denote the set of points of Γ either collinear with p or equal to p. If X is a subset of P_Γ, we put $X^\sim = \bigcap_{p \in X} p^\sim$. A point p such that $p^\sim = P_\Gamma$ is called *singular point* of Γ. Γ is *non-degenerate* if all of its points are non-singular, i.e. $P_\Gamma^\sim = \emptyset$. If $P_\Gamma = P_\Gamma^\sim$, then Γ is called a *linear space*.

A *subspace* of $\Gamma = (P_\Gamma, \mathcal{L}_\Gamma)$ is a subset $S \subseteq P_\Gamma$ such that for every two collinear points of S, the line joining them is contained in S. Clearly, a non-empty intersection of subspaces is a subspace, thus it is possible to define the span $\langle X \rangle_\Gamma$ of a subset $X \subseteq P_\Gamma$ as the intersection of all subspaces containing X. Moreover, a *singular* subspace of Γ is a subspace S such that any two points of S are collinear. We say that the *rank* of a singular subspace S of Γ is k if $k+1$ is the maximum lenght of all saturated chains of singular subspaces $S_0 \subset S_1 \subset \cdots \subset S_t$, such that S_0 is a point and $S_t = S$. It follows that points and lines of Γ are singular subspaces of rank 0 and 1, respectively. Finally, Γ has rank $rk\,(\Gamma) = n$ if n-1 is the maximum rank of its proper singular subspaces. We say that a subset X of P_Γ is *independent* if $\langle Y \rangle_\Gamma \subset \langle X \rangle_\Gamma$ for every proper subset $Y \subset X$. A *basis* of a subspace S is an independent spanning set of it.

Henceforth, $\mathbb{P}(V)$ denotes the projective space of linear subspaces of a K-vector space V, where K is a skewfield. The skewfield $K_\Sigma := K$ is said to be the *underlying* skewfield of $\Sigma := \mathbb{P}(V)$. If V has finite dimension d, we write $\mathbb{P}(d, K)$ to denote the corresponding projective space. Furthermore, if $K = GF(q)$, then we write $PG(d, q)$ instead of $\mathbb{P}(d, K)$.

Let $\Gamma = (P_\Gamma, \mathcal{L}_\Gamma)$ be a semilinear space of rank $rk(\Gamma) \geq 2$ (if $rk(\Gamma) = 2$, we assume that Γ is not a line) and $\Sigma = (P_\Sigma, \mathcal{L}_\Sigma)$ be a projective space of dimension $dim\,\Sigma \geq 2$. If dim $\Sigma = 2$, we suppose that Σ is desarguesian and K_Σ denotes its underlying skewfield.

[1]Author's address: G. Marino, Università degli Studi di Napoli "Federico II", Dipartimento di Matematica "Renato Caccioppoli", via Cintia, Complesso di Monte Sant'Angelo, edificio T, 80126 Napoli, Italy; e-mail: giuseppe.marino@unina2.it .

Keywords: Polar spaces, projective spaces, embeddings.
AMS Subject Classification: 51A45, 51A50.

According to [7], a *lax projective embedding* (or simply *embedding*) of Γ in Σ, and we write $e : \Gamma \to \Sigma$, is an injective map $e : P_\Gamma \to P_\Sigma$ such that:

(E1) $\langle e(P_\Gamma)\rangle_\Sigma = P_\Sigma$;
(E2) For every line $L \in \mathcal{L}_\Gamma$, $\langle e(L)\rangle_\Sigma \in \mathcal{L}_\Sigma$;
(E3) For any two distinct lines $L_1, L_2 \in \mathcal{L}_\Gamma$, $\langle e(L_1)\rangle_\Sigma \neq \langle e(L_2)\rangle_\Sigma$.

An embedding $e : \Gamma \to \Sigma$ is called *full* if $\langle e(L)\rangle_\Sigma = e(L)$ for every line $L \in \mathcal{L}_\Gamma$, and *flat* if $dim\,\Sigma = 2$.

Following [4], an embedding $e : \Gamma \to \Sigma$ is *sub-weak* if the following property holds:

(W1) For every point $p \in P_\Gamma$, $\langle e(p^\sim)\rangle_\Sigma \cap e(P_\Gamma) = e(p^\sim)$.

Moreover, a sub-weak embedding $e : \Gamma \to \Sigma$ is said to be *weak* if it satisfies the further condition:

(W2) For any two lines $L_1, L_2 \in \mathcal{L}_\Gamma$, if $x \in \langle e(L_1)\rangle_\Sigma \cap \langle e(L_2)\rangle_\Sigma$ then $x \in e(P_\Gamma)$.

Now, let $\Gamma = (P_\Gamma, \mathcal{L}_\Gamma)$ be a linear space, $e : \Gamma \to \Sigma$ be an embedding of Γ and $k \geq 2$ an integer. We say that e is k-$firm$ if for every independent subset $X \subseteq P_\Gamma$, with $|X| \leq k$, $e(X)$ is an independent subset of P_Σ. Obviously, an embedding $e : \Gamma \to \Sigma$ is 2-firm, being an injective map between points, and it is 3-firm, in virtue of $(E3)$. The embedding e is said to be *firm* if it is k-firm for all $k \geq 4$.
If Γ is a semilinear space and $e : \Gamma \to \Sigma$ is an embedding, we say that e is k-firm if the map $e_{|_S} : S \to \langle e(S)\rangle_\Sigma$ is k-firm, for any singular subspace S of Γ.

According to [1], a *polar space* is a semilinear space $\Gamma = (P_\Gamma, \mathcal{L}_\Gamma)$ satisfying the so-called *one-all axiom: for every point p and for every line L, $p^\sim$ contains either exactly one point of L or all points of L.* Following Tits [6], there exist three classes of non-degenerate thick-lined polar spaces of finite rank $n \geq 3$.
Classical polar spaces. All of its singular subspaces of rank 3 (which are projective planes) are desarguesian and defined on the same skewfield K_Γ, called the underlying skewfield of Γ. Moreover, either every line belongs to at least three planes or $n = 3$, K_Γ is a field and Γ is the Klein quadric $Q^+(5, K_\Gamma)$.
Non-classical grassmannians. Here $n = 3$ and Γ is the Grassmann space of indices $(3, 1)$, $Gr(\mathbb{P}(3, K))$, whose points are the lines of a 3-dimensional projective space $\mathbb{P}(3, K)$ over a non-commutative skewfield K and whose lines are the pencils of lines of $\mathbb{P}(3, K)$. A plane of Γ is coordinatized by either K or its opposite K^{op}, according to weather it arises from a set of lines of $\mathbb{P}(3, K)$ either containing a point of $\mathbb{P}(3, K)$, or contained in a plane of $\mathbb{P}(3, K)$.
Tits polar space. Here $n = 3$ and the planes of Γ are Moufang but non-desarguesian.

By [6], a non-degenerate thick-polar space $\Gamma = (P_\Gamma, \mathcal{L}_\Gamma)$ of rank $n \geq 3$ is classical if and only if it admits a full embedding $e : \Gamma \to \mathbb{P}(V)$, where V is defined on K_Γ. Furthermore, $e(\Gamma)$ arises from a non-degenerate trace-valued sesquilinear form or a non-singular pseudoquadratic form of V.

2. Embeddings of semilinear and linear spaces.

In this subsection let $\Gamma = (P_\Gamma, \mathcal{L}_\Gamma)$ be a semilinear space and $e : \Gamma \to \Sigma$ be an embedding of Γ.

If e is full, then (W2) easily holds, but (W1) may be not valid. In fact, if L_1, L_2 and L_3 are three non-concurrent lines of $\mathbb{P}(2, K)$, then the inclusion mapping of $(L_1 \cup L_2 \cup L_3, \{L_1, L_2, L_3\})$ in $\mathbb{P}(2, K)$ is a full embedding but it does not satisfy (W1).

It can be easily proved the following proposition.

Proposition 2.1. *If e is sub-weak, then $\langle e(L)\rangle_\Sigma \cap e(P_\Gamma) = e(L)$ for every line $L \in \mathcal{L}_\Gamma$.*

Proof. Clearly, $e(L)$ is contained in $\langle e(L)\rangle_\Sigma \cap e(P_\Gamma)$. Conversely, we suppose, by way of contradiction, that $e(w)$ is a point of $\Big(\langle e(L)\rangle_\Sigma \cap e(P_\Gamma)\Big) \setminus e(L)$, and consider a point x of L. Since $L \subseteq x^\sim$, we have $\langle e(L)\rangle_\Sigma \subseteq \langle e(x^\sim)\rangle_\Sigma$. Since $e(w)$ is a point of $\langle e(L)\rangle_\Sigma \cap e(P_\Gamma) \subseteq \langle e(x^\sim)\rangle_\Sigma \cap e(P_\Gamma)$, then $e(w) \in e(x^\sim)$, owing to condition (W1). Then, there exists a line $M \in \mathcal{L}_\Gamma$ containing w and x. The line M is different from L, since it contains $w \notin L$. Moreover, the lines $\langle e(M)\rangle_\Sigma$ and $\langle e(L)\rangle_\Sigma$ coincide, as they contain the two different points $e(x)$ and $e(w)$, contradicting (E3). □

In the next proposition (suggested by Eva Ferrara Dentice and Nicola Melone) we show that if a linear space admits a weak-embedding, then it is a projective space.

Proposition 2.2. *If e is weak, then every singular subspace S of Γ of rank at least 2 is an irriducible projective space over a sub-skewfield K_S of K_Γ.*

Proof. As S is a singular subspace of Γ, S is a linear space. Thus, S is an irreducible projective space if, and only if, it satisfies the so-called *Veblen-Young axiom: every line intersecting two sides of a triangle intersects the third side, as well.* Therefore, let L, M and N be three lines of S pairwise intersecting at distinct points, and let R be a line of S intersecting two of them, say L and M, at distinct points. By the Veblen-Young axiom in Σ, the lines $\langle e(R)\rangle_\Sigma$ and $\langle e(N)\rangle_\Sigma$ meets at a point x, and, by (W2), $x \in e(P_\Gamma)$, hence $x \in \Big(\langle e(N)\rangle_\Sigma \cap e(P_\Gamma)\Big) \cap \Big(\langle e(R)\rangle_\Sigma \cap e(P_\Gamma)\Big) = e(N) \cap e(R)$, by Proposition 2.1. Then, N and R meet at the point $e^{-1}(x)$ of P_Γ. In order to prove that S can be coordinatized over a skewfield, we have to verify the *Desargues axiom: the intersecting points of the corresponding sides of two perspective triangles are collinear.* Therefore, let $x\overset{\triangle}{y}z$ and $x'\overset{\triangle}{y'}z'$ be two perspective triangles of S. By the Veblen-Young axiom in S, the corresponding sides $y \vee z$ and $y' \vee z'$ meets at a point $x'' \in S$, $(x \vee z) \cap (x' \vee z') = y'' \in S$ and $(x \vee y) \cap (x' \vee y') = z'' \in S$. By holding the Desargues axiom in Σ, there exists a line ℓ of Σ containing $e(x'')$, $e(y'')$ and $e(z'')$. Denoted by R the line of S passing through x'' and y'', clearly $\langle e(R)\rangle_\Sigma = \ell$, hence $e(z'') \in \ell \cap e(P_\Gamma) = \langle e(R)\rangle_\Sigma \cap e(P_\Gamma) = e(R)$ (by Proposition 2.1) and the points x'', y'' and z'' are collinear in S. Then, there exists a skewfield K_S such that S is a projective space coordinatized on it. Finally, we show that K_S is a sub-skewfield of K_Σ. Let us recall that, if we fix a line of a projective space over a skewfield, then there is a bijection between the elements of the skewfield and the points of the line with one point deleted (see [3]). So, let x be a point of S, L be a line of S passing through x and, up to bijections, $K_S = L \setminus \{x\}$. Since $e(L) = \langle e(L)\rangle_\Sigma \cap e(P_\Gamma)$ and $K_\Sigma = \langle e(L)\rangle_\Sigma \setminus \{e(x)\}$, up to bijections, K_S is contained in K_Σ and, by condition (W2), the operations $+$ and $\cdot$ defined on K_S are $+$ and $\cdot$ of K_Σ restricted to K_S.

3. Embeddings of projective spaces

In this subsection let $\Gamma = (P_\Gamma, \mathcal{L}_\Gamma)$ be an irriducible projective space of dimension $dim\,(\Gamma) \geq 2$ and $e : \Gamma \to \Sigma$ be an embedding of Γ.

The embedding e is sub-weak, being condition (W1) trivially satisfied. Moreover, the following propositions can be proved (see [2]).

Proposition 3.1. (1) *For every plane S of Γ, $e(S)$ spans a plane of Σ.*
(2) *For every spanning set X of Γ, $e(X)$ is a spanning set of Σ.*
(3) *$dim\,(\Gamma) \geq dim\,(\Sigma)$.*
(4) *Γ is desarguesian and its underlying skewfield K_Γ is a sub-skewfield of K_Σ.*

Proposition 3.2. *Let e be firm. Then, (W2) is satisfied and e maps every basis of Γ onto a basis of Σ. Thus, $dim\,(\Gamma) = dim\,(\Sigma)$. Furthermore, if $dim\,(\Gamma)$ is finite, then e is firm if and only if $dim\,(\Gamma) = dim\,(\Sigma)$.*

Remark 3.3. In general, a projective space admits non-firm embeddings. For instance, given a field K_0 and integers d, h, with $h > d > 2$, let $K = K_0(\omega)$ be a simple extension of K_0 of degree h. Let $\Gamma = \mathbb{P}(d, K_0)$ and $\Sigma = \mathbb{P}(d-1, K)$. The function e which maps every point of P_Γ, whose homogeneous projective coordinates are $x_0, x_1, \ldots, x_d$, onto the point $[x_1 + x_0\omega, \ldots x_d + x_0\omega^d]$ of P_Σ is an embedding e of Γ in Σ. Since $dim\,(\Gamma) > dim\,(\Sigma)$, e is non-firm (but it is d-firm).

Remark 3.4. If $dim\,(\Gamma) > 2$ then e is weak if and only if it is 4-firm.

4. Embeddings of polar spaces

In [4] and [5] Thas and Van Maldeghem obtain the following results. Let us consider a classical polar space Γ of rank $n \geq 3$, defined over a commutative field K_Γ, and a sub-weak embedding s of Γ in $\Sigma = \mathbb{P}(d, K_\Sigma)$, d finite and K_Σ a field. Then, under suitable assumptions when Γ admits at least two full embeddings, K_Γ is a subfield of K_Σ, e is weak and firm and it induces a full embedding of Γ in $\mathbb{P}(d, K_\Gamma)$. Moreover, if Γ is a finite polar space and e is an embedding of Γ in $PG(d, q)$, then there exist $n \geq d$ and $s|q$ such that e induces a full embedding of Γ in $PG(n, s)$ (except the case $n = 3$ and Γ of symplectic type).

In [2], the authors deal with a more general situation, where K_Γ and K_Σ are arbitrary skewfields and d can also be infinite. In the paper [2] the following results are proved.

Theorem 4.1. *Let Γ be a non-degenerate thick-lined polar space of finite rank $n \geq 3$ and $e : \Gamma \to \Sigma$ be an embedding of Γ. Then either*

(1) *Γ is classical and its underlying skewfield K_Γ is a sub-skewfield of K_Σ, or*
(2) *$n = 3$, $dim\,(\Sigma) = 2$ (namely e is flat) and $\Gamma \cong Gr(\mathbb{P}(3, K))$ for a non-commutative sub-skewfield K of K_Σ such that $K \cong K^{op}$.*

Theorem 4.2. *Let Γ be a non-degenerate thick-lined polar space of finite rank $n \geq 2$ and $e : \Gamma \to \Sigma$ be a sub-weak embedding of Γ. Then*

(1) *e is weak,*
(2) *e is firm,*
(3) *$dim\,(\Sigma) \geq 2n - 1$.*

By combining the two theorems above we immediately obtain the following corollary.

Corollary 4.3. *Let Γ be a non-degenerate thick-lined polar space of finite rank $n \geq 3$ and $e : \Gamma \to \Sigma$ be a sub-weak embedding of Γ. Then Γ is classical, K_Γ is a sub-skewfield of K_Σ, the embedding e is weak and firm, and* $\dim(\Sigma) \geq 2n - 1$.

We conclude this section with the following theorem.

Theorem 4.4. *Given a classical polar space Γ of finite rank $n \geq 3$ and a full embedding $e_0 : \Gamma \to \Sigma_0$ of Γ, let $e : \Gamma \to \Sigma$ be a sub-weak embedding of Γ. Let us make the the following assumptions:*

(A) *For any choice of three maximal singular subspaces M_1, M_2, M_3 of Γ, if $rk\,(M_1 \cap M_2 \cap M_3) = n - 1$ and $e_0(M_3) \subseteq \langle e_0(M_1 \cup M_2)\rangle_{\Sigma_0}$, then $e(M_3) \subseteq \langle e(M_1 \cup M_2)\rangle_\Sigma$;*
(B) $P_{\Sigma_0}^{\perp} = \emptyset$ ([2]).

Then there exists a firm embedding $e_1 : \Sigma_0 \to \Sigma$ such that $e = e_1\, e_0$.

Remark 4.5. When Γ admits at least two full embeddings, both conditions (A) and (B) are necessary. For instance, let Γ be the non-degenerate symplectic variety $W(2n-1, q)$ of $\mathbb{P}(2n-1, q)$, with q a power of 2. Let e_0 and e be the full embeddings of Γ as $W(2n-1, q)$ in $\mathbb{P}(2n-1, q)$ and as $Q(2n, q)$ in $\mathbb{P}(2n, q)$, respectively. Thus, condition (A) is not

[2] $\perp$ is the orthogonal relation associated to the sesquilinear form from which $e_0(\Gamma)$ arises.

verified. If we permute the roles of e_0 and e, then condition (B) fails. In both cases we cannot construct an embedding between the two projective spaces.

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