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Geometry of contact Riemannian manifolds whose Reeb vector field is harmonic

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Abstract[1]. A contact riemannian manifold whose Reeb vector field is a harmonic vector field is called a H-contact manifold. In this paper, we review some results related to the geometry of a H-contact manifolds and to the harmonicity of Hopf vector fields.

1. Introduction

Let (M, g) be a Riemannian manifold and (T^1M, g_s) its unit tangent sphere bundle equipped with the Sasaki metric g_s. A unit vector field V on (M, g), if it exists, determines a map between (M, g) and (T^1M, g_s). When M is compact, the energy of V is the energy $E(V)$ of the corresponding map. V is said to be a *harmonic vector field* if it is a critical point for the energy functional E defined on the space $\chi^1(M)$ of all unit vector fields on (M, g). The corresponding critical point condition has been determined in [43] and [44]. A harmonic vector field determines a *harmonic map* if an additional condition is satisfied [14], [23]. In [14], similar notions were introduced when M is also non-compact. In these last years, the study on the harmonicity of unit vector fields is very active. Several results and many examples of such vector fields have already been discovered (see for example [8], [9], [17], [19], [32], [33] and the survey paper [16]).

An interesting geometrical situation, in which a distinguished vector field appears in a natural way, is given by a contact manifold (M, η) where we have the *Reeb vector field* (also called the *characteristic vector field*) ξ. It is well known that ξ plays a fundamental role in the study of the geometry of a contact manifold from the Riemannian point of view [4], and moreover the first examples of harmonic vector fields have been the Hopf vector fields on the unit odd-dimensional spheres, that is, the Reeb vector fields of the standard Sasakian structure of the unit odd-dimensional spheres. On the other hand, the Hopf unit vector fields on odd-dimensional spheres are a particular case of a general construction on any orientable real hypersurface of a Kähler manifold. So, it is natural to study the harmonicity of the Reeb vector field of a general contact Riemannian manifold or of a orientable real hypersurface in a Kähler manifold and to see how its harmonicity is related to the geometry of the manifold.

A contact metric (or Riemannian) manifold (M, η, g) whose Reeb vector field ξ is a harmonic vector field is called a *H-contact manifold* [33]. The condition that characterizes the H-contact manifolds is a very natural condition, it appears in many examples and problems of contact Riemannian geometry. The aim of this paper, based on the lecture

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given by the author at the University of Potenza in April 2004, is to review some results related to the geometry of a H-contact manifold and to the harmonicity of Hopf vector fields. The content is the following:
Sect.2 Harmonicity of unit vector fields.
Sect.3 Harmonicity of the Reeb vector field.
Sect.4 Three-dimensional H-contact manifolds.
Sect.5 The energy (2.1) and the Chern-Hamilton energy.
Sect.6 Harmonicity of Hopf vector fields of real hypersurfaces in Kähler manifolds.
Sect.7 Real hypersurfaces of contact type.
Sect.8 Some questions.

2. Harmonicity of unit vector fields

Let (M,g) and (M',g') be two Riemannian manifolds with M compact. The energy of a smooth map $f : M \to M'$ is defined by

$$E(f) := \int_M e(f)\, dv \ ,$$

where $e(f) = \frac{1}{2}\|f_*\|^2$ is the energy density and $\|f_*\|$ is the norm of the differential of f with respect to metrics g and g'. Hence, $\|f_*\|^2 = \mathrm{tr}_g\, f^*g'$ and locally, on the domain of a local orthonormal basis $\{E_i\}_{i=1,\ldots,n}$, $n = \dim M$, it can be expressed as $\sum_{i=1}^n g'(f_*E_i, f_*E_i)$. The critical points of this functional on $C^\infty(M,M')$ are known as harmonic maps and have been characterized by Eells and Sampson [12] as maps with vanishing tension field:

$$\tau(f) = \mathrm{tr}\,(\nabla df) = \sum_{i=1}^n \left(\nabla'_{E_i} f_*E_i - f_*\nabla_{E_i}E_i\right) \ .$$

More precisely,

$$\left(\frac{dE(t)}{dt}\right)_{t=0} = -\int_M \bar{g}(V, \tau(f))\, dv \ ,$$

where $E(t) = E(f_t)$, f_t is a smooth variation of f, $V = (\partial f_t/\partial t)(0)$ and $\bar{g}$ is the bundle metric on $f^{-1}TM'$ induced from the metric g'.

If we consider the tangent bundle $\pi : TM \to M$ on the Riemannian manifold (M,g), we can construct a natural metric on TM, the *Sasaki metric* g_s, induced from the metric g. For any point $z = (p,u) \in TM$ and any tangent vectors $X_z, Y_z \in T_z(TM)$, consider two curves in TM, $\bar{\alpha}(t) = (\alpha(t), V(t))$ and $\bar{\beta}(t) = (\beta(t), W(t))$ such that $\bar{\alpha}(0) = \bar{\beta}(0) = z$, $\dot{\bar{\alpha}}(0) = X_z$ and $\dot{\bar{\beta}}(0) = Y_z$. Then

$${g_s}_z(X_z, Y_z) = g_p(\dot{\alpha}(0), \dot{\beta}(0)) + g_p\left(\frac{DV}{dt}(0), \frac{DW}{dt}(0)\right) \ ,$$

where DV/dt is the covariant derivative of the vector field $V(t)$ along the curve $\alpha(t)$. $\pi_*X_z = \pi_*\dot{\bar{\alpha}}(0) = \dot{\alpha}(0)$, and the map $K_z : T_zTM \to T_pM$, $X_z = \dot{\bar{\alpha}}(0) \mapsto (DV/dt)(0)$, is the *connection map*. A vector $X_z \in T_zTM$ is called *horizontal vector* if it is tangent to a horizontal curve $\bar{\alpha}(t) = (\alpha(t), V(t))$, that is $V(t)$ is parallel along $\alpha(t)$, and is called *vertical vector* if it is tangent to the fiber $\pi^{-1}(p)$. Horizontal and vertical vectors generate two complementary (orthogonal with respect to g_s) distribution:

$$T_zTM = V_zTM \oplus H_zTM \ ,$$

where $V_zTM = \ker \pi_*$ denotes the vertical subspace and $H_zTM = \ker K_z$ the horizontal subspace. Our interest lies in the unit tangent sphere bundle T^1M which is a hypersurface of TM consisting of all unit tangent vectors to (M,g).

On T^1M we consider the metric induced from the Sasaki metric g_s. A unit normal vector field N to T^1M is given by the vertical vector field $N_z = u^v$ (vertical lift of u),

$z = (p, u)$. In general, the vertical lift of a vector (field) is not tangent to T_1M. For this reason, we define the *tangential lift* of $X \in T_xM$ by $X^t_{(p,u)} = (X - g(X, u)u)^v$.

Now, we consider the set $\chi^1(M)$ of all smooth unit vector fields on M which is supposed to be non-empty. This assumption implies the vanishing of the Euler characteristic. Any $U \in \chi^1(M)$ defines a map between (M, g) and (TM, g_s). Then $U : M \to TM$ is a *harmonic map* if it is critical point of the energy functional

$$E(U) = \int_M e(U)\, dv$$

defined on the set $C^\infty(M, TM)$. For any orthonormal basis $\{e_i\}_{i=1,\dots,n}$ of T_pM, the energy density $e(U)_p$ is given by

$$e(U)_p = \frac{1}{2} \|U_{*p}\|^2 = \frac{1}{2} \mathrm{tr}_g \ (U^* g_s)_p \ .$$

Since

$$(U^* g_s)_p (e_i, e_i) = g_p(e_i, e_i) + g_p(\nabla_{e_i} U, \nabla_{e_i} U) \ ,$$

we get

$$e(U)_p = \frac{1}{2} \left(n + \sum \|\nabla_{e_i} U\|^2 \right) = \frac{1}{2} \left(n + \|\nabla U \ \|^2 \right)$$

and hence

$$E(U) = \frac{1}{2}\, n \,\mathrm{vol}\,(M) + \frac{1}{2} \int_M \|\nabla U\|^2 \ dv \ . \tag{2.1}$$

The relevant part of this formula $B(U) = (1/2) \int_M \|\nabla U\|^2 \ dv$ is called the *total bending* of the vector field U (see [43]). In particular,

$$E(U) \geq \frac{1}{2}\, n \,\mathrm{vol}\,(M) \ ,$$

where the equality holds if and only if U is parallel. The unit vector field $U : M \to TM$ is a harmonic map if the tension field $\tau(U)$ vanishes. The horizontal and the vertical components of $\tau(U)$ can be computed by the formula (see Ishihara [22])

$$\tau(U) = \tau(U)^H + \tau(U)^V = \{\mathrm{tr}\, R(\nabla_. U, U)\cdot\}^H + \{-\bar{\Delta} U\}^V \ , \tag{2.2}$$

where R denotes the curvature tensor of (M, g) defined by

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \ ,$$

and $\bar{\Delta} U$ is the *rough Laplacian*, that is

$$\bar{\Delta} U = -\mathrm{tr}\, \nabla^2 U = \sum_{i=1}^{n} \left\{ \nabla_{\nabla_{E_i} E_i} U - \nabla_{E_i} \nabla_{E_i} U \right\} \ ,$$

where $\{E_i\}$ is a local orthonormal basis. The covariant derivative ∇ admits a formal adjoint ∇^*. In particular, if S is a $(1,1)$-tensor on M, we have

$$\nabla^* S = -\mathrm{tr}\, \nabla S = -\sum_{i=1}^{n} (\nabla_{E_i} S)\, E_i \ .$$

Using the operator ∇^*, we obtain

$$\bar{\Delta} U = \nabla^* \nabla U \quad \text{and} \quad g(\bar{\Delta} U, U) = \|\nabla U\|^2 \ .$$

Then, from (2.2), we get the following

Theorem 2.1 (Ishihara [22])**.** *Let U be a unit vector field on M.*
Then $U : (M, g) \to (TM, g_s)$ is a harmonic map if and only if $\nabla U = 0$.

Therefore, it is difficult to find a vector field which is a harmonic map between (M, g) and (TM, g_s). In this case, (M, g) must be locally a Riemannian product, because a unit parallel vector field determines two orthogonal complementary totally geodesic foliations. However, the situation is different if we consider U as a map between the Riemannian manifolds (M, g) and (T^1M, g_s). In this case, the variation of U is restricted to $C^\infty(M, T^1M)$ and $U : (M, g) \to (T^1M, g_s)$ is a harmonic map if its tension field $\tau_1(U)$ vanishes. Now, we recall that the tension field of the composition of two maps

$$(M, g) \xrightarrow{f} (M', g') \xrightarrow{f'} (M'', g'')$$

is (see [12])

$$\tau(f' \circ f) = df' \circ \tau(f) + \mathrm{tr}\,(\nabla df')(df, df) .$$

In particular, if f' is an isometric immersion, $\tau(f)$ is the orthogonal projection of $\tau(f' \circ f)$ on M'. The map $U : (M, g) \to (TM, g_s)$ can be considered as the composition

$$(M, g) \xrightarrow{U} (T^1M, g_s) \xhookrightarrow{i} (TM, g_s) ,$$

then the tension $\tau_1(U)$ is the orthogonal projection of $\tau(U)$ on T^1M. Moreover,

$$T_{U_p}TM = T_{U_p}T^1M \oplus < N_{U_p} >$$

and, by (2.2), $\tau_1(U)$ is given by

$$\{\mathrm{tr}\, R(\nabla_. U, U)\cdot\}^h + \{\mathrm{tr}\, \nabla^2 U - g(\mathrm{tr}\, \nabla^2 U, U)U\}^v .$$

Then, we get (see also Han-Yim [23] and Gil-Medrano [14])

Theorem 2.2. *Let U be a unit vector field on M. Then, $U : (M, g) \to (T^1M, g_s)$ is a harmonic map if and only if*

$$\textbf{a)}\ \mathrm{tr}\, R(\nabla_. U, U)\cdot = 0 \ \ \textit{and} \ \ \textbf{b)}\ \bar\Delta U = \|\nabla U\|^2\, U . \tag{2.3}$$

A unit vector field U is called *harmonic vector field* if it is critical for the functional energy $E_{|\chi^1(M)}$. In this case, the variation of U is restricted to unit vector fields and we have the following

Theorem 2.3 (Wood [44] and Wiegmink [43])**.** *Let U be a unit vector field on M. Then, U is a harmonic vector field if and only if*

$$\bar\Delta U = \|\nabla U\|^2\, U . \tag{2.4}$$

The critical point conditions (2.3) and (2.4) have a tensorial character and may also be considered on non-compact manifolds.

Remark 2.4. Examples of harmonic unit vector fields which do not define harmonic maps can be found in [9] and [17].

3. Harmonicity of the Reeb vector field

3.1. **Contact metric manifolds.** We start this section by recalling some basic facts about contact metric manifolds. A $(2n + 1)$-dimensional manifold M is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. Given η, there exists a unique vector field ξ, called the *characteristic vector field* or the *Reeb vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. Furthermore, a Riemannian metric g is said to be an *associated metric* if there exists a tensor ϕ of type $(1, 1)$ such that

$$\eta = g(\xi, \cdot) \ \ , \ \ d\eta(\cdot, \cdot) = g(\cdot, \phi\cdot) \ \ , \ \ \phi^2 = -I + \eta \otimes \xi .$$

(η, g, ξ, ϕ), or (η, g), is called a *contact metric structure* and (M, η, g, ξ, ϕ) is called a *contact metric manifold* or *contact Riemannian manifold*. The scalar torsion $\|\tau\|$, $\tau = L_\xi g$, introduced in [11], and the tensor $h = (1/2)L_\xi\phi$ are fundamental in contact metric geometry. They are related by

$$\tau = 2g(h\phi\cdot, \cdot) \ \ , \ \ \|\tau\|^2 = 4\mathrm{tr}\, h^2 = 8n - 4g(Q\xi, \xi) .$$

Moreover, we have

$$\nabla \xi = -\phi - \phi h \ . \tag{3.1}$$

It is well-known that the unit tangent sphere bundle T^1M of a general Riemannian manifold has a natural contact metric structure $(\eta, \bar{g}, \xi, \phi)$. The metric $\bar{g}$ is homothetic to the metric g_s induced by the Sasaki metric: $\bar{g} = (1/4)g_s$. The Reeb vector field ξ is proportional to the geodesic flow vector field:

$$\xi_z = 2\dot{\tilde{\gamma}}(0) = 2u_z^h \ ,$$

where $\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ with $\gamma(t)$ geodesic and $\tilde{\gamma}(0) = z = (p, u) \in T^1M$. The tensor ϕ is given by: $\phi X_z^h = X_z^t$ and $\phi X_z^t = -X_z^h + g(X_p, u)u_z^h$.

Now, we recall the definitions of some classic subclasses of contact metric manifolds. A contact metric manifold is said to be *η-Einstein* if the Ricci operator Q is of the form $Q = aI + b\eta \otimes \xi$, where a and b are functions. A contact metric manifold is said to be a *K-contact manifold* if ξ is a Killing vector field, or equivalently, $h = 0$. In dimension $n \geq 5$, it is known that for any η-Einstein K-contact manifold, a and b are constant. A contact metric structure (ξ, η, ϕ, g) on M is called a *Sasakian* (or *normal*) structure if the almost complex structure J on $M \times \mathbb{R}$ defined by $J(X, f(d/dt)) = (\phi X - f\xi, \eta(X)(d/dt))$, where f is a smooth function on $\mathbb{R}$, is integrable. Moreover, a contact metric structure is Sasakian if and only if

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \ .$$

Any Sasakian manifold is K-contact and the converse also holds when $n = 1$, that is, for three-dimensional spaces. On a K-contact manifold ξ is an eigenvector of the Ricci operator Q, more precisely we have

$$Q\xi = 2n\xi \ .$$

A locally symmetric K-contact (or Sasakian) manifold necessarily has constant curvature 1 (Tanno [36]). For this reason T. Takahashi (see for example [4], p.115) introduced the *locally ϕ-symmetric Sasakian spaces*. Such spaces are defined as Sasakian manifolds satisfying:

$$\phi^2 \left(\nabla_V R\right)(X, Y, Z) = 0 \quad \forall X, Y, Z, V \in \ker \eta \ . \tag{3.2}$$

Geometrically, these spaces can be characterized as Sasakian manifolds such that: *the local reflections with respect to the integral curves of the Reeb vector field are local isometries.* Boeckx and Vanhecke [6] proposed this property as definition of local ϕ-symmetry for a general contact metric manifold, moreover in [7] the same authors and Bueken formalized the following two notions. A contact metric manifold is called *strongly locally ϕ-symmetric* if the characteristic reflections are local isometries while it is called *weakly locally ϕ-symmetric* if the curvature condition (3.2) is satisfied. The notion of strongly locally ϕ-symmetric space implies an infinite number of curvature conditions ([6]), in particular implies the curvature condition (3.2). Moreover, the Reeb vector field of a strongly locally ϕ-symmetric space is an eigenvector of the Ricci operator. Another interesting subclass of contact metric manifolds, which extends the subclass of Sasakian manifolds, is that of so-called (k, μ)-spaces introduced by D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [5]. Such spaces are defined as contact metric manifolds satisfying

$$R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\} \tag{3.3}$$

where k and μ are constant. Moreover, we have

$$Q\xi = 2nk\xi \ \ , \ \ k \leq 1 \ ,$$

and if $k = 1$ or $h = 0$, then M is Sasakian (see [5]). In the sequel, a contact metric manifold whose characteristic vector field ξ is a harmonic vector field is called an *H-contact manifold*. We refer to [4] for more information about contact metric geometry.

3.2. **Harmonicity of ξ.** C. M. Wood [44] proved that the Hopf vector field on the unit $(2n+1)$-sphere S^{2n+1}, that is, the Reeb vector field of the standard Sasakian structure on S^{2n+1}, is a harmonic vector field. G. Wiegmink [43] proved that the Reeb vector field of any Sasakian manifold is harmonic, moreover it determines a harmonic map (see [23]). S. D. Han and J. W. Yim [23] proved that the Hopf vector field on the unit 3-sphere S^3 is the only unit vector field which defines a harmonic map into its unit tangent bundle. Harmonicity of the Reeb vector field of K-contact manifolds and (k,μ)-spaces was proved in [20], [8]. Now, let M be a general contact metric three-manifold. In [20], Gonzalez-Davila and Vanhecke, using a convenient expression of the Levi-Civita connection given in [10] and a criterium of harmonicity given in [14], proved that the Reeb vector field ξ of M is harmonic if and only if ξ is an eigenvector of the Ricci operator. In arbitrary dimension we have the following results (see [33]).

Theorem 3.1. *Let (M,η,ξ,g,ϕ) be a $(2n+1)$-dimensional contact metric manifold. Then*

$$\bar{\Delta}\xi = 4n\xi - Q\xi = \|\nabla\xi\|^2\xi + \phi^2 Q\xi \ ,$$

where $\|\nabla\xi\|^2 = 2n + \operatorname{tr} h^2$. In particular: $\Delta\eta = 4n\eta$, where Δ is the Hodge-de Rham Laplacian, and M is K-contact *if and only if $\bar{\Delta}\eta = 2n\eta$.*

Then, as a consequence of this theorem, we get

Theorem 3.2. *A contact metric manifold is an H-contact manifold, that is ξ is a harmonic vector field, if and only if ξ is an eigenvector of the Ricci operator.*

We will see that the condition which characterizes the class of H-contact manifolds, naturally appears in many problems and examples. Einstein contact metric manifolds, and more in general η-Einstein manifolds, are the first examples of H-contact manifolds.

Remark 3.3. For a general Riemannian manifold (M,g), harmonic vector fields are substantially different from harmonic vector fields in the conventional sense viz. the dual 1-form is harmonic in Hodge theory. The following Weitzenböck's formula for differential 1-forms ω is well known:

$$\Delta\omega = \bar{\Delta}\omega + g(QX_0,\cdot) \tag{3.4}$$

where X_0 is the vector field defined by $\omega = g(X_0,\cdot)$. Then, when a vector field is harmonic, the associated 1-form, in general is not Hodge Harmonic. If ω is a harmonic 1-form in Hodge theory and $QX_0 = \lambda X_o$, then X_o is a harmonic vector field.

The geometry of T^1M has been deeply studied and involves the geometry of the base manifold in a way that, although very natural, produces complicated formulas if the curvature has no particular properties. Let us recall that the curvature tensor appears in the covariant derivative of Sasaki metric. So, one can not expect to get results for a general manifold, without curvature assumptions. For example we have:

Theorem 3.4 ([39], [5], [6]). *Let (M,g) be a general Riemannian manifold. Then*
a) *T^1M is K-contact (or Sasakian) if and only if (M,g) is of constant curvature $+1$;*
b) *T^1M is a (k,μ)-space (or a strongly locally ϕ-symmetric space) if and only if (M,g) is of constant curvature c.*

About the harmonicity of the Reeb vector field of T^1M, we have the following result due to Boeckx-Vanhecke:

Theorem 3.5 ([8]). *Let (M,g) be a general Riemannian manifold. If (M,g) is a two-point homogeneous space, then the natural contact metric structure of T^1M is H-contact and $\xi : T^1M \to T^1T^1M$ is a harmonic map. If $\dim M = 2,3$, then T^1M is H-contact if and only if (M,g) is of constant sectional curvature.*

Now, we investigate how the class $\mathcal{M}_H$ of all H-contact manifolds is related to classical classes of contact metric manifolds with respect to inclusion relations. We put $\mathcal{M}_S$:= the set of all Sasakian manifolds, $\mathcal{M}_K$:= the set of all K-contact manifolds, $\mathcal{M}_{k,\mu}$:= the set of all (k,μ)-spaces, and $\mathcal{M}_{\phi s}$:= the set of all strongly locally ϕ-symmetric spaces. The Reeb vector field of a contact metric manifold $M \in \mathcal{M}_S \cup \mathcal{M}_{k,\mu} \cup \mathcal{M}_K \cup \mathcal{M}_{\phi s}$ is an eigenvector of the Ricci operator. Moreover, Theorems 3.4 and 3.5 imply that the unit tangent sphere bundle of a two-point homogeneous space of non-constant sectional curvature gives an example of H-contact manifold which does not belong to $\mathcal{M}_S \cup \mathcal{M}_{k,\mu} \cup \mathcal{M}_K \cup \mathcal{M}_{\phi s}$. Then we have

Proposition 3.6 ([33])**.** *The class $\mathcal{M}_H$ extends the classes $\mathcal{M}_S$, $\mathcal{M}_{k,\mu}$, $\mathcal{M}_K$ and $\mathcal{M}_{\phi s}$*

We end this section with the problem of conformally flat contact metric spaces. Tanno [36] proved that a conformally flat $(2n+1)$-dimensional K-contact manifold is of constant curvature $+1$. In [10], the authors proved that a conformally flat three-dimensional H-contact manifold is of constant sectional curvature 0 or $+1$. Recently, this result has been extend in arbitrary dimension. In fact, we have

Theorem 3.7 (Bang-Blair [1])**.** *A $(2n+1)$-dimensional conformally flat H-contact manifold is of constant sectional curvature c.*

We note that the constant $c = 0$ or $+1$ if $n = 1$, and $c = 1$ if $n > 1$.

4. Three-dimensional H-contact manifolds

In this section we briefly review some results on the geometry of a contact metric three-manifold related to the notion of H-contact. For a contact metric three-manifold, the Webster scalar curvature W is defined by (see Chern-Hamilton [11], Tanno [37])

$$W = \frac{r - \rho(\xi,\xi) + 4}{8} \ ,$$

where r denotes the scalar curvature and ρ the Ricci tensor. Moreover, when the scalar torsion $\|\tau\| \neq 0$, we put

$$p := \frac{4\sqrt{2}\, W}{\|\tau\|} \ .$$

A contact metric manifold is called *homogeneous* if there is a connected Lie group of isometries acting transitively which leave the contact form invariant. We recall the following result about three-dimensional homogeneous contact metric manifolds obtained by the author in [31].

Theorem 4.1. *Let (M,η,g) be a simply connected homogeneous contact metric three-manifold. Then M is a Lie group G and both η and g are left invariant. More precisely, we have the following classification.*

Sasakian case*:*

(1) *If G is unimodular, then it is one of the following Lie groups :*
the Heisenberg group H when $W = 0$;
the 3-sphere group $SU(2)$ when $W > 0$;
the group $\tilde{SL}(2,\mathbb{R})$ when $W < 0$;

(2) *If G is non-unimodular, its Lie algebra is given by*

$$[e_1, e_2] = \alpha\, e_2 + 2\xi \ \ , \ \ [e_1,\xi] = [e_2,\xi] = 0 \ ,$$

where $e_1, e_2 = \phi e_1 \in \ker\eta$ and $\alpha \neq 0$; in this case $W = -\alpha^2/4 < 0$.

Non Sasakian case*:*

(1) *If G is unimodular, then it is one of the following Lie groups :*
the 3-sphere group $SU(2)$ when $p > 1$;
the group $\tilde{SL}(2,\mathbb{R})$ when $-1 \neq p < 1$;

the group $\tilde{E}(2)$, universal covering of the group of rigid motions of Euclidean 2-space, when $p = 1$;
the group $E(1,1)$ of rigid motions of Minkowski 2-space when $p = -1$;
(2) *If G is non-unimodular, its Lie algebra is given by*

$$[e_1, e_2] = \alpha e_2 + 2\xi \ , \ [e_1, \xi] = \gamma e_2 \ , \ [e_2, \xi] = 0 \ ,$$

where $e_1, e_2 = \phi e_1 \in \ker \eta$ and $\alpha, \gamma \neq 0$; in this case $p < 1$.

A Riemannian manifold such that the volume of all sufficiently small geodesic balls only depends on the radius is called a *ball-homogeneous space* (Kowalski-Vanhecke [27]). Based on the above theorem, in [10] Calvaruso, the author and Vanhecke derived the classification of ball-homogeneous H-contact metric three-manifolds. More precisely, we have

Theorem 4.2. *A H-contact three-manifold M is ball-homogeneous if and only if it is locally isometric to a unimodular Lie group equipped with a left-invariant contact metric structure.*

T. Koufogiorgos and C. Tsichlias [25] introduced a new class of contact metric three-manifolds : *the generalized (k,μ)-spaces.* These spaces are defined by the equation (3.3) where k nd μ are functions. They proved that, if $\dim M > 3$, then k and μ are necessarily constant and if $\dim M = 3$, there exist examples of generalized (k,μ)-spaces which are not (k,μ)-spaces.
Now, we have a characterization of such spaces in terms of harmonic maps. Let (M, η, g, ξ, ϕ) be a contact metric three-manifold and m a point of M. Then there exists a smooth local orthonormal basis of the form $\{\xi, e_1, e_2 = \phi e_1\}$ in a neighborhood of m. Let U_1 be the open subset of M where $h \neq 0$ and let U_2 be the open subset of points $m \in M$ such that $h = 0$ in a neighborhood of m. $U_1 \cup U_2$ is an open dense subset of M. On U_1 we put $he_1 = \lambda e_1$ and hence, since $h\phi = -\phi h$, we have $he_2 = -\lambda e_2$ where λ is a non-vanishing smooth function. Then, we have

Theorem 4.3 ([32]). *Let M be a contact metric three-manifold. Then the Reeb vector field $\xi : (M, g) \to (T^1M, \bar{g})$ defines a harmonic map if and only if M is a generalized (k,μ)-space on the dense open subset $U_1 \cup U_2$.*

In [26], the authors classify the 3-dimensional generalized (k,μ)-spaces which satisfy the condition $\|\mathrm{grad}\, k\| = \mathrm{const.}\ (\neq 0)$. This class of manifolds is determined by two arbitrary functions of one variable. Of course, this gives a classification of H-contact three-manifolds with $\|(\mathrm{grad}\ \|\tau\|)\|$=const. $(\neq 0)$. We note that these manifolds are non-compact (and parallelizable).

H-contact three-manifolds are also related to the study of strongly (and weakly) locally ϕ-symmetric spaces. In fact, we have:

Theorem 4.4 ([10], [35]). *Let M be a contact metric three-manifold. Then:*
a) *M is strongly locally ϕ-symmetric if and only if it is H-contact and locally homogeneous;*
b) *if M is H-contact, then M is weakly locally ϕ-symmetric if and only if it has constant ϕ-sectional curvature.*

Next, we recall that the *volume of an immersion* $f : M \to (M', g')$ is the volume of the Riemannian submanifold (M, f^*g'), that is

$$\mathrm{vol}\,(f) = \int_M dv_{f^*g'} \ ,$$

where M is a compact manifold. If we choose a metric g on M, then

$$\mathrm{vol}\,(f) = \int_M \sqrt{\det f^*g'}\, dv_g = \int_M \sqrt{\det L_f}\, dv_g \ ,$$

where L_f is the endomorphism defined by $g(L_f X, Y) = f^* g'(X, Y)$.

Now, a unit vector field V on a Riemannian manifold (M, g) determines a map $(M, g) \to (T^1M, g_s)$ which is a immersion and hence it also determines a submanifold of (T^1M, g_s). When M is compact, the volume of V, that is, the volume of the corresponding submanifold $(M, V^* g_s)$ of (T^1M, g_s), is given by

$$\mathrm{vol}\,(V) = \int_M \sqrt{\det L_V}\, dv_g \ .$$

In this case, L_V is defined by

$$g\,(L_V(X), Y) = g(X, Y) + g\,(\nabla_X V, \nabla_Y V)\ ,$$

i.e. $L_V = I + (\nabla V)^t \circ \nabla V$. This yields the volume functional on the set $\chi^1(M)$. In [15], it is proved that V is a critical point for the volume functional defined on $\chi^1(M)$ if and only if the 1-form

$$\omega_V(X) = \mathrm{tr}\,(Z \mapsto (\nabla_Z S_V)X)$$

vanishes on $V^\perp$, where $S_V = (\sqrt{\det L_V} L_V^{-1}(\nabla V)^t))$. V is called **minimal vector field** if it is critical for this volume functional. This is equivalent with the condition that the submanifold of (T^1M, g_s), determined by V, is minimal. Such a unit vector field is called minimal unit vector field even when M is possibly non-compact.
In [20] the authors proved the following

Proposition 4.5. *The Reeb vector field of a contact metric three-manifold is minimal if and only if on U_1, we have*

$$A\lambda\left\{(\lambda-1)^2+1\right\} = 4(\phi e)(\lambda) \ \text{ and } \ B\lambda\left\{(\lambda+1)^2+1\right\} = -4e(\lambda)\ .$$

Then, using the Proposition 4.5, we get the following result (see [32]).

Theorem 4.6. *Let (M, η, g, ξ, ϕ) be a contact metric three-manifold. Then ξ is harmonic and minimal if and only if M is Sasakian or is locally isometric to a unimodular Lie group G equipped with a non-Sasakian left-invariant contact metric structure (η, g), more precisely (using the invariant p):*
if $p > 1$, $\tilde{G}$ is the 3-sphere group $SU(2)$;
if $p = 1$, $\tilde{G}$ is the group $\tilde{E}(2)$, i.e. the universal covering of the group of rigid motions of Euclidean 2-space;
if $-1 \neq p < 1$, $\tilde{G}$ is the group $\tilde{SL}(2, \mathbb{R})$;
if $p = -1$, $\tilde{G}$ is the group $E(1, 1)$ of rigid motions of the Minkowski 2-space; where $\tilde{G}$ denotes the universal covering of G.

Corollary 4.7 ([32])**.** *The only compact three-manifold which admit a H-contact structure whose Reeb vector field is minimal are compact left quotient, under a discrete subgroup, of the Lie groups $SU(2)$, H^3, $\tilde{SL}(2, \mathbb{R})$, $\tilde{E}(2)$ or $E(1, 1)$.*

Moreover, using (3.1), we obtain the following.

Proposition 4.8. *Let M be a compact H-contact three-manifold whose Reeb vector field ξ is minimal. Then, the energy and the volume of ξ satisfy:*

$$E(\xi) = \left(\frac{5}{2} + \frac{1}{2}\,\left\|\frac{\tau}{2}\right\|^2\right)\mathrm{vol}\,(M) \geq \frac{5}{2}\,\mathrm{vol}\,(M)\ ,$$

where the equality holds if and only if M is Sasakian; and

$$\mathrm{vol}\,(\xi) = \left(4 + \frac{1}{4}\,\left\|\frac{\tau}{2}\right\|^4\right)^{\frac{1}{2}}\mathrm{vol}\,(M) \geq 2\mathrm{vol}\,(M)\ ,$$

where the equality holds if and only if M is Sasakian.

In [19], it is proved that on a compact Sasakian three-manifold with ϕ-sectional curvature $c > 1$, $\pm\xi$ minimize the energy and the volume and moreover they are the unique minimizers.

Remark 4.9. About the minimality of the Reeb vector field ξ of a contact metric manifold of arbitrary dimension, we know that ξ is minimal when M is K-contact, a (k,μ)-space or a *Kenmotsu manifold* (see [21]).

5. The energy (2.1) and the Chern-Hamilton energy

Let (M,η) be a $(2n+1)$-dimensional compact contact manifold. The energy functional

$$F(g) = \frac{1}{2}\int_M \|\tau\|^2\, dv \ , \tag{5.1}$$

defined on the set $\mathcal{A}(\eta)$ of all metrics associated to η, was studied by Chern and Hamilton [11] in the three-dimensional case, by Blair (see [4], p.167) when the contact form is regular, and by Tanno [37] for a general contact manifold. The critical point condition for the functional (5.1) is

$$\nabla_\xi \tau = 2\tau\phi \tag{5.2}$$

and was determined by Tanno [37]. More recently, the equation (5.2), called Tanno's equation, has been studied by Barletta and Dragomir [2]. In particular, they proved that Tanno's equation is the critical point condition of a large class of functionals. We note that the Chern-Hamilton functional is related to the energy functional considered in Section 2. Note that a contact metric structure (η,g,ξ,ϕ) on M is completely determined by the tensors g and ξ. In fact, the contact form is given by $\eta = g(\xi,\cdot)$ and the tensor ϕ is defined by $d\eta = g(\cdot,\phi\cdot)$. Now, consider the energy of the map $V \in \chi^1(M)$ with respect to the contact metric g:

$$L(V,g) = \frac{1}{2}\int_M \|V_*\|^2\, dv_g = \frac{2n+1}{2}\operatorname{vol}(M,g) + \frac{1}{2}\int_M \|\nabla V\|^2\, dv_g \ .$$

The energy L depends on $(V,g) \in \chi^1(M)\times\mathcal{A}(\eta)$. If we restrict this functional L to the set $\chi^1(M)\times\{g\}$, where g is a fixed metric associated to η, then we get the energy functional E considered in Section 2 and the characteristic vector field ξ is critical for L (that is, M is an H-contact manifold) if and only if ξ is an eigenvector of the Ricci operator (Theorem 3.2). If we consider the same functional L defined on the set $\{\xi\}\times\mathcal{A}(\eta)$, then a metric $g \in \mathcal{A}(\eta)$ is critical for such functional if and only if it is critical for the Chern-Hamilton energy functional $F(g)$. In fact, all metrics associated to η have the same volume element (see [4], p.38), moreover from (3.1) we get

$$\|\nabla\xi\|^2 = (2n + \frac{\|\tau\|^2}{4}) \ .$$

So, we say that a *contact metric structure* (ξ,g) on M is *critical* if ξ is critical for the functional L defined on $\chi^1(M)\times\{g\}$ and g is critical for the same functional defined on the set $\{\xi\}\times\mathcal{A}(\eta)$. The critical point condition (5.2) has a tensorial character and may also considered on non-compact manifolds.

A K-contact structure is critical in the above sense (in fact, $\tau = 0$ and $Q\xi = 2n\xi$), but in general the converse does not hold. In fact, the standard contact metric structure on $T^1M(-1)$, the unit tangent sphere bundle of a Riemannian manifold of constant curvature -1, which is not K-contact, has ξ critical for the energy E (see [8]) and g critical for the energy F (Blair [4], p.168). If M is a compact K-contact manifold (that is, $\tau = 0$) and the Ricci tensor satisfies $\rho > -2g$, then $b_1(M) = 0$ (Tanno [38]). Generalizing Tanno's result, we have the following theorem (which is Theorem 4 of Goldberg-Perrone-Toth [18] reformulated).

Theorem 5.1. *Let (M,g,ξ) be a $(2n+1)$-dimensional compact contact metric manifold where the contact metric structure is critical. If the Ricci tensor satisfies $\rho > -cg > (-2 + (\|\tau\|/\sqrt{2n}))g$, then the first Betti number $b_1(M)$ vanishes.*

In the 3-dimensional case, we have a complete classification of such manifolds.

Theorem 5.2 ([33]). *Let (M, ξ, g) be a contact metric three-manifold. Then the contact metric structure (ξ, g) is critical if and only if M is K-contact or locally isometric to the Lie group $\tilde{SL}(2, \mathbb{R})$ equipped with a left-invariant contact metric structure which is not K-contact.*

H. Geiges [13] proved that a compact three-manifold admits a normal contact form (i.e. a K-contact structure) if and only if it is diffeomorphic to a left quotient, under a discrete subgroup, of one of the following Lie groups: $SU(2)$, H^3 and $\tilde{SL}(2, \mathbb{R})$. Theorem 5.2 and the mentioned theorem of Geiges imply the following result.

Corollary 5.3 ([33]). *The only compact three-manifolds which admit a critical contact metric structure are compact left quotient, under a discrete subgroup, of the Lie groups $SU(2)$, H^3 and $\tilde{SL}(2, \mathbb{R})$.*

6. Harmonicity of Hopf vector fields of real hypersurfaces in Kähler manifolds

The Hopf unit vector fields on odd-dimensional spheres are a particular case of a general construction on any orientable real hypersurface (M, g) of a Kähler manifold $(\bar{M}, \bar{g}, J)$. Let N be an unit vector field normal to the hypersurface M. Then $\xi = JN$, called *Hopf* (or *Reeb*) *vector field*, is the characteristic vector field of an almost contact metric structure (η, g, ξ, ϕ) on M. The tensor ϕ is defined by $JX = \phi X - \bar{g}(X, JN)N$ and $\eta = g(\xi, \cdot)$. The hypersurface M is called a *Hopf hypersurface* if the Hopf vector field ξ determines a principal direction, that is $A\xi = \alpha\xi$, where A denotes the shape operator. In this case some results on the harmonicity of ξ were obtained by Tsukada and Vanhecke ([40], [41]) mainly when $\bar{M}$ is a complex space form or the complex 2-plane Grassmannian. When $\bar{M}$ is a general Kähler manifold we get the following results (see [34]).

Theorem 6.1. *Let M be an orientable real hypersurface in a general Kähler manifold $\bar{M}$ of complex dimension $n+1$. Then*

$$\bar{\Delta}\xi = \phi \operatorname{tr} \nabla A + \phi^2 A^2 \xi + \|\nabla\xi\|^2 \xi = -\phi \nabla H + \operatorname{pr}_{|\ker \eta} \bar{Q}\xi + \phi^2 A^2 \xi + \|\nabla\xi\|^2 \xi \ ,$$

where $\bar{Q}$ is the Ricci operator of $\bar{M}$.

Corollary 6.2. *Let M be an orientable real hypersurface of a Kähler manifold $\bar{M}$. Then the Hopf vector field ξ is harmonic if and only if*

$$\phi \operatorname{tr} \nabla A + \phi^2 A^2 \xi = 0 \ .$$

In particular:
a) *If $\operatorname{div} A = 0$, then ξ is harmonic if and only if ξ is eigenvector of A^2.*
b) *If M is a Hopf hypersurface, then ξ is harmonic if and only if* $\operatorname{tr} \nabla A$ *is parallel to the Hopf vector field ξ.*

Theorem 6.3. *Let M be an orientable Hopf hypersurface of a Kähler manifold $\bar{M}$ with ξ eigenvector of $\bar{Q}$. Then:*
(i) *The Hopf vector field ξ is harmonic if and only if $\nabla H = \xi(H)\xi$.*
(j) *If α is constant along the integral curves of ξ, then the Hopf vector field ξ is harmonic if and only if the mean curvature H is constant.*

The principal curvature α of a Hopf hypersurface M, in a complex space form $\bar{M}(c)$ of constant holomorphic sectional curvature $c \neq 0$, is constant and the other principal curvatures are constant along the integral curves of ξ (see [41] Proposition 2.2), therefore $\xi(H) = 0$. Moreover for such spaces $\operatorname{tr} R(\nabla_{\cdot}\xi, \xi)\cdot = 0$, so we get the following (see also [41]).

Corollary 6.4. *Let M be an orientable Hopf hypersurface of a complex space form $\bar{M}(c)$, $c \neq 0$. Then ξ determines a harmonic map from (M, g) to (T^1M, g_s) if and only if the mean curvature H is constant.*

Another application of Theorem 6.1 is related to the Hodge-de Rham Laplacian Δ of ξ ($\Delta\xi$ is the vector field dual of the one form $\Delta\eta$). We consider the Weitzenböck formula (3.4) with $\omega = \eta$, that is:

$$\Delta\eta = \bar{\Delta}\eta + g(Q\xi, \cdot) \ \ \text{, i.e. } \ \Delta\xi = \bar{\Delta}\xi + Q\xi \ .$$

On the other hand, from Gauss equation (see for example [34]) we get

$$Q\xi = \bar{Q}\xi - g(\bar{Q}\xi, N)N + \bar{R}(N, \xi)N - A^2\xi + HA\xi$$

where $\bar{R}$ denotes the curvature tensor of $\bar{M}$. Therefore, applying Theorem 6.1, we have

$$\Delta\xi = \phi\nabla H + \left\{HA\xi - 2A^2\xi + \bar{R}(N, \xi)N\right\}_{|\ker\eta} +$$

$$+ \left\{g(\bar{Q}\xi, \xi) + \bar{R}(N, \xi, N, \xi) + Hg(A\xi, \xi) - \|A\xi\|^2 + \|\nabla\xi\|^2\right\}\xi \ .$$

Then, this formula and (i) of Theorem 6.3 imply the following

Theorem 6.5. *Let M be an orientable Hopf hypersurface in a Kähler manifold $\bar{M}$ with ξ eigenvector of the Jacobi operator $\bar{R}_N := \bar{R}(\cdot, N)N$, then*

$$\Delta\xi = \phi\nabla H + \left\{g(\bar{Q}\xi, \xi) - \beta + H\alpha - 2\alpha^2 + \|A\|^2\right\}\xi \ ,$$

where $A\xi = \alpha\xi$ and $\bar{R}_N\xi = \beta\xi$. If, in addition, $\bar{M}$ is Kähler-Einstein, then ξ is a harmonic vector field if and only if η is an eigenform of the Hodge-de Rham Laplacian.

If M is a real hypersurface in a complex space form $\bar{M}$, then the Hopf vector field ξ of M is an eigenvector of the Jacobi operator $\bar{R}_N$. In [3], Berndt and Suh provided examples of Hopf hypersurfaces in the complex Grassmannian manifold $G_2(\mathbb{C}^{m+2})$ and in its non-compact dual which are Kähler-Einstein manifolds. Such hypersurfaces are also discussed in [40] and the condition "ξ is an eigenvector of the Jacobi operator $\bar{R}_N$" is satisfied.

The torsion $\tau = L_\xi g$ plays a fundamental role in the study of the geometry of a general contact metric manifold (see [4]), and in particular for the harmonicity of the characteristic vector field ξ (see [32], [33]). Now, we consider the same torsion τ for our study of the harmonicity of Hopf vector fields. So, let M be a real hypersurface of a Kähler manifold $\bar{M}$, equipped with the induced almost contact metric structure. Denote by T the tensor of type $(1, 1)$ corresponding to the torsion τ:

$$\tau(X, Y) = g(TX, Y) \ .$$

Then we get the following theorem.

Theorem 6.6 ([34])**.** *Let M be an orientable real hypersurface of a Kähler manifold $\bar{M}$. Then*

$$\bar{\Delta}\xi = Q\xi + \nabla^*T = Q\xi - \operatorname{tr}\nabla T \ , \tag{6.1}$$

where ∇^* denotes the formal adjoint of ∇.

If the almost contact metric structure of M is contact metric, then $T = 2h\phi$ and hence (see [33]) we get

$$\operatorname{tr}\nabla T = -2\nabla^*h\phi = 2Q\xi - 4n\xi \ ,$$

and the equation (6.1) becomes

$$\bar{\Delta}\xi = -Q\xi + 4n\xi \ ,$$

which is the equation obtained in Section 3 in the context of the contact metric geometry. Therefore, Theorem 6.6 is a natural generalization of Theorem 3.1.

If the Ricci tensor ρ of a real hypersurface M is of the form $\rho = ag + b\eta \otimes \eta$ for some $a, b \in \mathbb{R}$, then M is called a *pseudo-Einstein* real hypersurface of the Kähler manifold

$\bar{M}$ (see Kon [24]). In the same paper, Kon determined all complete pseudo-Einstein real hypersurfaces in a complex projective space $\mathbb{CP}^{n+1}$ and in a complex Euclidean space $\mathbb{C}^{n+1}$ for $n > 1$. For such hypersurfaces, using Theorem 6.6, we have the following criterium of harmonicity.

Corollary 6.7. *Let M be an orientable pseudo-Einstein real hypersurface in a Kähler manifold $\bar{M}$. Then, ξ is harmonic if and only if $\phi\nabla^* T = 0$.*

Hypersurfaces satisfying the commutative condition $T = 0$, that is, $A\phi = \phi A$, have been classified by Okumura [30] when $\bar{M}$ is the complex projective space $\mathbb{CP}^{n+1}$, by Montiel-Romero [28] when $\bar{M}$ is the complex hyperbolic space $\mathbb{CH}^{n+1}$ and by Berndt-Shu [3] when $\bar{M}$ is the complex Grassmannian manifold $G_2(\mathbb{C}^{n+2})$. For a general Kähler manifold $\bar{M}$, *quasi-umbilical* hypersurfaces (i.e., the shape operator has the form $A = aI + b\eta \otimes \xi$ where a and b are smooth functions) satisfy the commutative condition $T = 0$. On the other hand, to the light of Proposition 4.1 of [34], the condition $T = 0$ can be replaced by the condition $T = 0$ on $\ker \eta$. For such hypersurfaces we find the same harmonicity criterium for ξ as found in the context of contact metric geometry.

Corollary 6.8. *Let M be an orientable real hypersurface of a Kähler manifold $\bar{M}$ satisfying the commutative condition $T = 0$ on $\ker \eta$. Then, ξ is harmonic if and only if ξ is eigenvector of the Ricci operator.*

7. Real hypersurfaces of contact type

Let M be a real hypersurface of a Kähler manifold $(\bar{M}, \bar{g}, J)$ and (η, g, ξ, ϕ) the induced almost contact metric structure. Following Okumura [29] M is said to be of *contact type* if on M there exists a function r which nowhere vanishes and satisfies the condition

$$d\eta = rg(\cdot, \phi\cdot) \ . \tag{7.1}$$

In this case, we have

$$d\eta = ri^*\Omega \ ,$$

where Ω is the Kähler form of $\bar{M}$ and $i : M \hookrightarrow \bar{M}$. Then, since $\Phi = g(\cdot, \varphi\cdot)$ has rank $2n$, we obtain that $\eta \wedge (d\eta)^n$ is a volume form, that is, η is a contact form on M. The function r in the condition (7.1) can be positive or negative depending on the orientation. So, we can suppose that r is positive. When $r = 1$, the induced almost contact metric structure is a contact metric structure. A hypersurface M of contact type in a Kähler manifold is a Hopf hypersurface. Moreover, when $\dim M > 3$, r is a constant and if $\dim M = 3$, then r is constant along the integral curves of ξ (see [34]).

Theorem 7.1 ([34])**.** *Let M be a hypersurface of contact type in a Kähler manifold $\bar{M}$ with ξ eigenvector of the Ricci operator $\bar{Q}$. If $\dim M > 3$, then the following properties are equivalent:*
a) *ξ is harmonic;*
b) *the mean curvature H is constant;*
c) *ξ is an eigenvector of the Ricci operator Q;*
d) *ξ is an eigenvector of the Jacobi operator $\bar{R}_N := \bar{R}(\cdot, N)N$.*
If $\dim M = 3$, we have that the properties a) *and* b) *are equivalent.*

For hypersurfaces of contact type in a complex space forms $\bar{M}(c)$, the function r is constant (see Okumura [29]) for $\dim M \geq 3$. Such hypersurfaces have been completely classified by Okumura [29] in a complex Euclidean space, by Kon [24] in a complex projective space and by Vernon [42] in a complex hyperbolic space. The examples of Hopf hypersurfaces in a complex two-plane Grassmannian (which is Kähler-Einstein), studied by Berndt and Shu [3], are of contact type.

From Theorem 7.1 we get

Corollary 7.2. *The Hopf vector field of an orientable real hypersurface of contact type M in a complex space form $\bar{M}^{n+1}(c)$, is harmonic and determines a harmonic map.*

Corollary 7.3. *Let M be a hypersurface of contact type in a Kähler-Einstein manifold $\bar{M}^{n+1}$ and (η, g, ξ, φ) the induced almost contact metric structure.*
(i) *If $n \geq 1$, then (η, g, ξ, φ) is a H-contact structure if and only if M has constant mean curvature.*
(j) *If $n > 1$, then (η, g, ξ, φ) is a H-contact structure if and only if ξ is an eigenvector of the Ricci operator.*

8. Some questions

We conclude with some questions related to the exposed results:
1. Classify the 3-dimensional compact H-contact manifolds or the 3-dimensional compact contact metric manifolds whose Reeb vector field defines a harmonic map.

2. Are the two-point homogeneous space the only Riemannian manifolds whose the corresponding unit tangent sphere bundles are H-contact?

3. When is the characteristic vector field of an almost contact metric manifold harmonic?

4. To study the minimality of the Reeb vector field of a contact metric manifold.

5. To study the stability (for the energy) of the Reeb vector field of a H-contact manifold.

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