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Riesz product type measures on the Cantor group

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Abstract[1]. Let $\Omega = \{-1, 1\}^{\mathbb{N}}$ and $\{\omega_j\}$ be independent random variables taking values in $\{-1, 1\}$ with equal probability. Ω is a compact abelian group, called Cantor group, under the product topology and the group operation of pointwise product. We give a brief discussion[2] on the study of the weak limit of the Riesz type products on Ω as follows

$$P_n = \prod_{j=1}^{n} (1 + a_j \omega_j + b_j \omega_{j+1}) \ ,$$

where a_j, b_j be real numbers with $|a_j|+|b_j| < 1$. We state some results when the coefficients are constants and present several cases which are possible to deal with.

1. Classical Riesz product measures

The classical Riesz product measure is first introduced in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by Riesz, and later by Zygmund [1], as the weak limit of finite products as follows

$$\prod_{1}^{N} (1 + a_n \cos(2\pi\lambda_n t))$$

as N tends to infinity, where $a_n's$ are bounded by 1 and the integers $\lambda_n's$ are lacunary in the sense $\lambda_{n+1}/\lambda_n \geq 3$. In other words, there is a Radon measure μ such that

$$\lim_{N\to\infty} \int_{\mathbb{T}} f(t) \prod_{1}^{N} (1 + a_n \cos(2\pi\lambda_n t)) \, dt = \int_{\mathbb{T}} f(t) \, d\mu(t) \ \ , \ \ \forall f \in C(\mathbb{T}) \ .$$

Moreover, this measure is continuous, that is, $\mu(\{t\}) = 0$ for $\forall t \in \mathbb{T}$. It is absolutely continuous or singular with respect to the Lebesgue-Haar measure on $\mathbb{T}$ if and only if $\{a_n\}_n$ is square summable or not.

Later, Hewitt and Zuckerman [2] defined Riesz products on a general non-discrete compact abelian group. A short description of their approach is as follows.

Let G be a nondiscrete compact abelian group with discrete dual group Γ, Λ be a subset of Γ and denote by $W(\Lambda)$ the set of all elements $\gamma \in \Gamma$ of the form

$$\gamma = \lambda_1^{\epsilon_1} \lambda_2^{\epsilon_2} \cdots \lambda_n^{\epsilon_n} \ , \tag{1.1}$$

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where $\epsilon_k \in \{-1,1\}$ and λ_k are distinct elements of Λ. Suppose Λ satisfies the requirement that each element of $W(\Lambda)$ has a unique representation of the form (1.1) up to the order of the factors, and let α be any complex function on Λ bounded by 1, we define

$$P(\Phi,\alpha)=\prod\{1+\mathrm{Re}\,[\alpha(\lambda)\lambda]\ :\ \lambda\in\Phi\}$$

for any finite set $\Phi\subset\Lambda$. Hewitt and Zuckerman showed that there exists a unique continuous probability measure $\mu_{\alpha,\lambda}$ on G which is the weak limit of $P(\Phi,\alpha)dm$ in the topology of $M(G)$, where $M(G)$ is the convolution algebra of all Radon measures on G and m is the normalized Haar measure on G. A famous theorem of Kakutani says that $\mu_{\alpha,\lambda}$ is either absolutely continuous or singular with respect to the Lebesgue-Haar measure on G, according to whether $\alpha\in l^2(\Lambda)$ or not.

Riesz products are proved to be a source of powerful ideas that can be used to produce concrete examples of measures with desired properties, such as singularity. The topics to determine Hausdorff dimensions and multifratal analysis of Riesz product measures were incepted by Peyriere [3] and extensively studied by Fan [4], et al.

2. The Cantor Group Ω

Throughout this paper, let

$$\Omega=\prod_{1}^{\infty}\Omega_j=\{-1,1\}^{\mathbb{N}}$$

be the cartesian product with all factors equal to $\Omega_j=\{-1,1\}(\forall j\geq 1)$, and write its elements

$$\varepsilon=(\varepsilon_n)_{n\in\mathbb{N}}\ \text{ or }\ \varepsilon=\varepsilon_1\varepsilon_2\cdots .$$

Ω is well known as an abelian group under the operation of pointwise product. With the discrete topology on each factor, the product topology on Ω makes it a compact abelian group, the so-called Cantor group. This topology can also be induced by a metric: the distance between two elements $\varepsilon=(\varepsilon_n)_{n\in\mathbb{N}},\delta=(\delta_n)_{n\in\mathbb{N}}$ in Ω is defined by

$$d(\varepsilon,\delta)=2^{-\inf\{n\geq 0:\varepsilon_{n+1}\neq\delta_{n+1}\}} .$$

Denote the projection $\omega_n:\Omega\to\{-1,1\}$ by $\omega_n(\varepsilon)=\varepsilon_n$. Elements in the dual group Γ of Ω, which are continuous group homomorphisms from Ω into the multiplicative group of complex numbers of modulus 1, are provided by the projection functions. Precisely, let $\mathcal{R}=\{\omega_n : n\in\mathbb{N}\}\subset\Gamma$, each nontrivial element of Γ can be uniquely written as $\omega_{j_1}\omega_{j_2}\cdots\omega_{j_k}$, where $1\leq j_1<j_2<\cdots<j_k<\infty$. Note that for the normalized Haar measure m on Ω, $\{\omega_j\}$ may be viewed as independent random variables taking values in $\{-1,1\}$ with equal probability. By abuse of language, we will write $\mathrm{d}m$ as $\mathrm{d}\omega$, and the Haar measure on $\Omega_j=\{-1,1\}$ by $\mathrm{d}\omega_j$ in the sequel.

The classical Riesz product measure on Ω is of the form

$$\prod_{j\geq 1}(1+a_j\omega_j)d\omega\ \ (a_j\in\mathbb{R},|a_j|<1)\ . \tag{2.1}$$

As we have known in Section 1, it is a continuous probability measure, and is either absolutely continuous or singular with respect to the normalized Haar measure m on Ω according to whether $\{a_j\}$ is square summable or not. If a_j are constants for all j, i.e. $a_j=a(\forall j\geq 1)$, where $|a|<1$, the dimension and multifractal structure of μ are completely known. We sketch these results in the below.

We recall some useful notations ([5], [6]). If (X,d) is a metric space and ν is a Borel measure on X. The upper and lower logarithmic densities of ν at $x\in X$ are defined by

$$\overline{D}(\nu,x)=\overline{\lim_{r\to 0}}\ \frac{\log\,\nu(B_r(x))}{\log\,r}\ ,$$

$$\underline{D}(\nu, x) = \varliminf_{r \to 0} \frac{\log \nu(B_r(x))}{\log r} ,$$

where $B_r(x)$ is the closed ball with centre x and radius r. If they are agree, their common value

$$D(\nu, x) = \overline{D}(\nu, x) = \underline{D}(\nu, x)$$

is called the logarithmic density of ν at x. The dimension of ν is defined by

$$\dim \nu = \sup\{\alpha \ : \ \underline{D}(\nu, x) \geq \alpha \ , \ \nu\text{−a.e. } x \in X\} .$$

It is also characterized by

$$\dim \nu = \inf\{\dim E \ : \ E \text{ is a Borel set such that } \nu(E) > 0\} .$$

If $D(\nu, x)$ is constant for $x \in X, \nu$-a.e., ν is called unidimensional. For $\beta > 0$, put

$$E(\beta) = \{x \in X \ : \ \underline{D}(\nu, x) = \beta\} .$$

If exist infinite many of β such that $\dim E(\beta) > 0$, one says that ν has multifractal structure.

To consider slightly different context from the above for convenient. Let $\Omega_{(n)}$ be the cartesian product $\{-1, 1\}^n$, and by convention $\Omega_{(0)}$ has a single element denoted by ϵ. The set $\Omega^* = \bigcup_{n \geq 0} \Omega_{(n)}$ is the collection of all finite sequences of elements in $\{-1, 1\}$ which are called words on $\{-1, 1\}$. For arbitrary $u \in \Omega^*$ and $v \in \Omega^* \cup \Omega$, we naturally define the concatenation of u and v to be the sequence uv obtained by putting u in front of v. If $u \in \Omega^*$, its length $|u|$ is the integer n such that $u \in \Omega_{(n)}$ and denote by Ω_u the set $\{u\varepsilon : \varepsilon \in \Omega\}$. The balls of radii (or diameter) 2^{-n} in Ω are precisely the sets Ω_u for $u \in \Omega_{(n)}$. Clearly, a topology basis of Ω is $\{\Omega_u : u \in \Omega^*\}$. If $u \in \Omega^*$ and $0 \leq n \leq |u|$, or $u \in \Omega$, we denote by $u|_n$ the element $u_1 u_2 \dots u_n \in \Omega_{(n)}$.

Denote T as the left shift operator on Ω,

$$T(\varepsilon_1 \varepsilon_2 \cdots) = \varepsilon_2 \varepsilon_3 \cdots .$$

It is easy to check that the measure μ defined by $\prod_{j \geq 1}(1 + a\omega_j)$ is invariant and ergodic with respect to T.

For $\varepsilon \in \Omega$, the diameter of $\Omega_{\varepsilon|_n}$ is $|\Omega_{\varepsilon|_n}| = 2^{-n}$, and

$$\mu(\Omega_{\varepsilon|_n}) = \lim_{m \to \infty} \int_{\Omega_{\varepsilon|_n}} \prod_{j=1}^{m} (1 + a\omega_j) \ d\omega = \frac{1}{2^n} \prod_{j=1}^{n} (1 + a\varepsilon_j) .$$

So

$$\frac{\log \mu(\Omega_{\varepsilon|_n})}{\log |\Omega_{\varepsilon|_n}|} = 1 - \frac{1}{\log 2} \left[\frac{1}{n} \sum_{j=1}^{n} \log (1 + a\varepsilon_j) \right] .$$

Let $f(\omega) = \log (1 + a\omega_1)$. Since μ is ergodic, by Birkhoff ergodic theorem, we have

$$\frac{1}{n} \sum_{j=1}^{n} \log (1 + a\varepsilon_j) = \frac{1}{n} \sum_{j=0}^{n-1} f(T^j \varepsilon) \to \int f \ d\mu \ , \ \ \mu\text{−a.e.} .$$

Thus

$$\lim_{n \to \infty} \frac{\log \mu(\Omega_{\varepsilon|_n})}{\log |\Omega_{\varepsilon|_n}|} = 1 - \frac{1}{\log 2} \int \log (1 + a\omega_1) \ d\mu =$$

$$= 1 - \frac{1}{2 \log 2} \left[(1 + a) \log (1 + a) + (1 - a) \log (1 - a) \right] \ , \ \ \mu\text{−a.e.} .$$

So μ is unidimensional, and $\dim \mu = 1 - (1/2 \log 2)[(1+a) \log (1+a) + (1-a) \log (1-a)]$.

If let $\mathbf{p} = (p_1, p_{-1})$ be a probability vector, i.e. its entries are nonnegative and add up to 1. For $i \in \{-1, 1\}$, let

$$\varphi_i(\varepsilon, n) = \frac{1}{n} \text{card}\{k \ : \ 1 \leq k \leq n \ , \ \varepsilon_k = i\} ,$$

and set

$$\Omega_{\mathbf{p}} = \{\varepsilon \in \Omega \ : \ \lim_{n\to\infty} \varphi_i(\varepsilon, n) = p_i \ , \ \text{for } i \in \{1, -1\}\} \ .$$

We have, for any $\varepsilon \in \Omega_{\mathbf{p}}$,

$$D(\mu, \varepsilon) = 1 - \frac{1}{\log 2} \left[p_1 \log (1+a) + p_{-1} \log (1-a)\right] \, ,$$

and $\dim \Omega_{\mathbf{p}} = -(1/\log 2)\,[p_1 \log p_1 + p_{-1} \log p_{-1}]$ (a special case of Billingsley theorem).

It is easy to see that $\Omega(\beta) = \Omega_{\mathbf{p}}$ with $\beta = 1 - (1/\log 2)[p_1 \log (1+a) + p_{-1} \log (1-a)]$. Thus, if $\beta \in [1 - (\log (1+|a|)/\log 2) \ , \ 1 - (\log (1-|a|)/\log 2)]$, then

$$\dim \Omega(\beta) = -\frac{1}{\log (1+a) - \log (1-a)} \left\{ (1-\beta) \log \frac{(1-\beta)\log 2 - \log (1-a)}{-(1-\beta)\log 2 + \log (1+a)} + \right.$$
$$\left. + \frac{1}{\log 2} \left[\log (1+a) \log \left| \log \frac{2^{1-\beta}}{1+a} \right| - \log (1-a) \log \left| \log \frac{2^{1-\beta}}{1-a} \right| \right] \right\} +$$
$$+ \frac{1}{\log 2} \ \log\log \frac{1+|a|}{1-|a|} \ .$$

Thus, the multifractal analysis of the measure given by $\prod_{j\geq 1}(1 + a\omega_j)$ are completely clear.

3. Riesz product type measures on Ω

Now it is natural to consider following products

$$\prod_{j=1}^{n} (1 + a_j\omega_j + b_j\omega_{j+1}) \ , \ \ n \geq 1 \ , \tag{3.1}$$

where a_j, b_j are real numbers and $|a_j| + |b_j| < 1 (\forall j \geq 1)$. These are generalization of (2.1), and give birth to essentially different properties compared with the classical ones.

Firstly, we expect to determine if $\{(P_n \, d\omega)/\mathbf{E}(P_n)\}$ converges to a certain probability measure in the weak topology of $M(\Omega)$, here we denote $\mathbf{E}(P) = \int_\Omega P d\omega$. And then study the singularity of this measure if it exists. We will also study the Hausdorff dimension and multifractal analysis of this measure.

In the following we will briefly state some results in case that a_j, b_j are constants and present several cases which are possible to deal with.

3.1. Constant coefficients case.

In [7] we consider a Riesz product type measure associated to the following products

$$P_n = \prod_{j=1}^{n} (1 + a\omega_j + b\omega_{j+1}) \ , \ \ n \geq 1 \ , \tag{3.2}$$

where a, b are two real numbers and $|a| + |b| < 1$. We have the following results

Proposition 3.1. *$\{\mu_n = (P_n \, d\omega)/\mathbf{E}(P_n)\}$ converges to a probability measure μ in the weak topology of $M(\Omega)$.*

Proposition 3.2. *μ is a continuous measure.*

Proposition 3.3. *If $a + b \neq 0$, the measures μ and the normalized Haar measure m on Ω are mutually singular.*

Proposition 3.4. *μ is a Gibbs measure, thus has multifractal structure and multifractal formalism holds for it.*

3.2. **Discussions of other cases.** It is possible to deal with the case that a_j, b_j are periodic real numbers. There are also opportunities in the scenarios where a_j, b_j are random variables uniformly distributed in $[-(1/2), (1/2)]$; a_j, b_j are automatic sequences; and so on.

More generally, we may consider the products

$$\prod_{j=1}^{n}(1 + a_j\omega_j + b_j\omega_{j+1} + c_j\omega_j\omega_{j+1}) \ , \quad n \geq 1 \ ; \tag{3.3}$$

and

$$\prod_{j=1}^{n}(1 + a_j\omega_j + b_j\omega_{j+1} + c_j\omega_{j+2}) \ , \quad n \geq 1 \ , \tag{3.4}$$

or, more items in the above brackets. We have considered a special case of (3.3) in [8].

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