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Analytic hypoellipticity for a class of sums of squares of vector fields with non-symplectic characteristic variety

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Abstract[1]. Recently, N. Hanges proved that the operator

$$P = \partial_t^2 + t^2\Delta_x + \partial_{\theta(x)}^2$$

in $\mathbb{R}^3$ is analytic hypoelliptic in the sense of germs at the origin and yet fails to be analytic hypoelliptic 'in the strong sense' in any neighborhood of the origin (there is no neighborhood U of the origin such that for every open subset V of U and distribution u in U, Pu analytic in V implies that u is analytic in V). Here $\partial_t = \partial/\partial t, \partial_{\theta(x)} = iD_{\theta(x)} = x_1\partial/\partial x_2 - x_2\partial/\partial x_1, x = (x_1, x_2)$, and $\Delta_x = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. We give a very short L^2 proof of this result[2], obtained jointly with A. Bove and M. Derridj, which generalizes easily and suggestively to other operators. It is striking that these operators have non-symplectic characteristic varieties. Finally we point out that these results are consistent with Treves' conjecture.

1. Introduction and Generalizations

In his recent paper [3], Hanges considered the operator

$$P_H = \partial_t^2 + t^2\Delta_x + \partial_{\theta(x)}^2 = \sum_1^4 X_j^2 \tag{1.1}$$

in $\mathbb{R}^3$ where $\partial_{\theta(x)} = x_1\partial/\partial x_2 - x_2\partial/\partial x_1$ and made the interesting distinction between analytic hypoellipticity in the germ sense and a.h.e. in the *strict* sense. Hanges proved that the operator P was not analytic hypoelliptic *in the strict sense* in any open set U containing the origin, i.e., did not have the property that for any open subset V of U, if Pu is analytic in V then so is the solution u, yet had the property that if Pu was analytic in some neighborhood of the origin then the so was u in a (possibly smaller) neighborhood of the origin. He ties this result to the conjecture of Treves concerning the Poisson strata of the operator P, namely that if one writes $P = \sum_1^4 X_j^2$, and considers the successive strata where 1) all X_j vanish, 2) all X_j and their first brackets vanish, 3) all X_j and their first and second brackets vanish, etc., then the operator should be analytic hypoelliptic in the strict sense if and only if all these strata are symplectic. In the case of the particular operator being considered here, not even the characteristic variety is symplectic, being given by $t = \tau = x_1\xi_2 - x_2\xi_1 = 0$.

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Here we give a very elementary, and flexible, proof of the affirmative part of his result and argue that the negative part is entirely reasonable as well, though we avoid entirely the mention of so-called Treves curves, which foliate the characteristic variety of P, in our proof. We hope to pursue related, but more degenerate, examples in a future paper.

The generalizations we consider may be motivated by observing that while the "added" term $\partial^2_{\theta(x)}$ in P, which suggests the celebrated non-analytic hypoelliptic example of Baouendi and Goulaouic,

$$P_{BG} = \partial_t^2 + t^2\partial_x^2 + \partial_y^2 = \sum_1^3 Z_j^2 \ , \tag{1.2}$$

differs from this example in one essential factor—the integral curves of ∂_y are non-compact yet those of ∂_θ which start close to the origin remain close. Hence a propagation of singularities result may be rephrased in terms of a germ result on analyticity.

To put the matter differently, the L^2 proof of propagation of singularities for P_{BG} hinges (writing $iD = \partial$) on the fact that in estimating localized high derivatives of a solution u in the x-direction, $\varphi(x,y)D_x^p u$, via the L^2 *a priori* estimate (we take φ independent of t since for $t \neq 0$, the operator is elliptic), one encounters and cannot avoid the derivation (and bracket)

$$\sum_j \|Z_j\varphi D_x^p u\|^2_{L^2} \lesssim |(P_{BG}\varphi D_x^p u, \varphi D_x^p u)_{L^2}| \lesssim$$

$$\lesssim \sum_j \|[Z_j, \varphi D_x^p]u\|^2_{L^2} + \cdots \lesssim$$

$$\lesssim \|\varphi' D_x^p u\|^2_{L^2} + \cdots$$

When the value of j is 3, i.e, we are trying to estimate D_x derivatives of u and encounter a y-derivative of φ with no gain in the number of x-derivatives, we cannot proceed, even with Ehrenpreis type localizing functions, to obtain analytic growth. Unless, of course, the y-derivative of the localizing function is supported in a region where the solution is known to be analytic already.

However, if the localizing function φ could be written as a function independent of the y-variable as well, this situation would not arise and analyticity would follow (after some calculation, admittedly, but elementary calculations with no sophisticated ingredients.)

This is what occurs when the open set under consideration is global in the "y-direction", as in proofs of analyticity which are local in some variables and global in others, as on a tube or torus, or when the vector field D_y is replaced by a vector field whose integral curves remain in any neighborhood of the point under consideration, as in Hanges' example, where D_y is replaced by D_θ.

Thus the following generalization of Hanges' example suggest themselves rapidly: in $(t,x) \in \mathbb{R}^\ell \times \mathbb{R}^k$, and with $\partial_j = \partial/\partial x_j$,

$$P_1 = \Delta_t + |t|^2\Delta_x + \sum_{i,j=1}^k a_{ij}(x,t)(x_i\partial_j - x_j\partial_i)^2 \tag{1.3}$$

for positive definite and analytic matrix valued function a_{jk}. Note the critical feature of this operator that the Laplacian in x commutes with each of the angular operators $x_i\partial_j - x_j\partial_i$. Actually, in terms of estimates what is crucial is that there be a C^ω basis, $\{X_j\}$, of vector fields in the x variables in in $\mathbb{R}^k \setminus \{0\}$ such that any bracket, $[X_j, x_i\partial_\ell - x_\ell\partial_i]$ be a linear combination of the angular vector fields $x_i\partial_j - x_j\partial_i$ (over which we have coercive control).

We also remark that when $a_{ij}(x,t) \equiv 1$, then it is not hard to see that the last sum is a constant multiple of the Laplace Beltrami operator on the unit sphere.

Still more generally, let us consider k vector fields $X_1, \ldots, X_k$ in the $x-$ variables with analytic coefficients (of x,t) and s vector fields $Y_1, \ldots, Y_s$ in the $x-$ variables which may be singular but have analytic coefficients . Let $x_0 \in \mathbb{R}^k$ be a fixed point and denote by

$U \subset \mathbb{R}^k$ an open neighborhood of x_0. Without loss of generality we may suppose that $x_0 = 0$. We assume that

1 - The $\{\partial/\partial t_m, X_j\}_{\substack{j=1,\dots,k\\ m=1,\dots,\ell}}$ span the tangent space on every point $(t,x) \in \mathbb{R}^\ell \times U$, with $x \neq 0$.

2 - $Y_1, \dots, Y_s$ have a compact closed family of integral manifolds which foliate U.

3 - We assume that the following commutation relations hold:

$$[\frac{\partial}{\partial t_m}, X_\ell] = C^\omega \text{ linear combination of the } Y, \frac{\partial}{\partial t}\ , \text{ and } tX\ \ ,\ \ \forall\, m, \ell\ , \tag{1.4}$$

and for some C^ω positive definite quadratic form Λ in the X_j's, with coefficients independent of the t variables,

$$[\Lambda, Y_\ell] = C^\omega \text{ quadratic expression in the } Y, \frac{\partial}{\partial t}\ , \text{ and } tX\ \ ,\ \ \forall\, \ell\ . \tag{1.5}$$

Consider then the operator

$$P_2 = \Delta_t + |t|^2 \sum_{i,j=1}^{k} a_{ij} X_i X_j + \sum_{i,j=1}^{s} b_{ij} Y_i Y_j\ , \tag{1.6}$$

where $a_{ij}(t,x)$ and $b_{ij}(t,x)$ are C^ω positive definite matrices. We will show that we may argue as in the particular case to obtain the result that P_2 is analytic hypoelliptic in the sense of germs at the origin.

Note that assumption 2 implies that we may choose a localizing function constant on the integral curves of $Y_1, \dots Y_s$, of Ehrenpreis type, identically equal to one on any given compact subset of U but vanishing outside of U.

We state our theorem:

Theorem 1.1. *Let us consider the operator P_2 as in* (1.6)*, where the coefficients a_{ij}, b_{ij} are real analytic in a neighborhood of the origin. Let U be a neighborhood of the origin with the properties in Assumptions* 1-3 *above. Let $P_2 u = f$ hold on the same open set U, with $f \in C^\omega(U)$. Then u is also in $C^\omega(U)$.*

2. Proof in the case of Hanges' operator (1.1)

As remarked above, we may take localizing functions to be independent of t, since were a derivative in t to land on such a localizer, one would be in the region where the operator was clearly elliptic and the analyticity of the solution u is well known. We denote such an Ehrenpreis type localizing function by $\varphi(x) = \varphi_N(x)$ subject to the usual growth of its derivatives: $|D^\alpha \varphi| \leq C^{|\alpha|+1} N^{|\alpha|}$ for $|\alpha| \leq N$, where the constant C is (universally) inversely proportional to the width of the band separating the regions where $\varphi \equiv 0$ and $\varphi \equiv 1$.

Next, since P is C^∞ hypoelliptic we may assume that u is smooth and proceed to obtain estimates for $D_t^p u$ and $D_{x_j}^p u$ near 0.

The *a priori* estimate for P, while subelliptic, is more importantly maximal: for $v \in C_0^\infty$,

$$\begin{aligned} &\|D_t v\|_{L^2}^2 + \sum_1^2 \|t D_{x_j} v\|_{L^2}^2 + \|D_{\theta(x)} v\|_{L^2}^2\ (+\ \|v\|_{1/2}^2) \leq \\ &\leq C|\langle Pv, v\rangle| + C\|v\|_{L^2}^2\ . \end{aligned} \tag{2.1}$$

Setting $v = \varphi D_t^p u$, to begin with, we obtain

$$\begin{aligned} &\|D_t \varphi D_t^p u\|_{L^2}^2 + \sum_1^2 \|t D_{x_j} \varphi D_t^p u\|_{L^2}^2 + \|D_{\theta(x)} \varphi D_t^p u\|_{L^2}^2\ (+\|\varphi D_t^p u\|_{1/2}^2) \leq \\ &\leq C|\langle P\varphi D_t^p u, \varphi D_t^p u\rangle| + C\|\varphi D_t^p u\|_{L^2}^2 \leq \end{aligned}$$

$$\leq C|\langle \varphi D_t^p Pu, \varphi D_t^p u\rangle_{L^2}| + C\sum_1^4 |([X_j^2, \varphi D_t^p]u, \varphi D_t^p u)| + C\|\varphi D_t^p u\|_{L^2}^2 \ .$$

Now crucial in the brackets are the quantities (recall that we may take φ independent of t, and clearly to localize in x we may take it to be purely *radial* in (x_1, x_2) i.e. we choose φ to be constant on the integral curves of X_4), so that $X_4\varphi = 0$,

$$[X_1, \varphi D_t^p] = [X_4, \varphi D_t^p] = 0 \ ,$$

and

$$[X_j, \varphi D_t^p] = t\varphi' D_t^p - \underline{p}\ \varphi D_x D_t^{p-1} \quad , \quad j = 2, 3 \ .$$

In the first case, we may ignore the factor t and recognize the passage from one power of D_t to a derivative on φ as an acceptable swing, which, upon iteration, will lead to $C^{p+1}N^p \sim C^{p+1}p!$ when $p \sim N$. The second term takes two powers of D_t (e.g., X_1 from the estimate and one power of D_t and produces a factor of p and a 'bad' vector field D_x. Iterating this will yield $p!!\, D_x^{p/2}u \sim p!^{1/2} D_x^{p/2}u$ on the support of φ.

On the other hand, setting $v = \varphi D_{x_j}^q u$, with perhaps $q = p/2$, or, better, $v = \varphi\Delta_x^{q/2}u$, where we write $\Delta_x = \sum_j D_{x_j}^2$,

$$\|D_t\varphi\Delta_x^{q/2}u\|_{L^2}^2 + \sum_1^2 \|tD_{x_j}\varphi\Delta_x^{q/2}u\|_{L^2}^2 + \|D_{\theta(x)}\varphi\Delta_x^{q/2}u\|_{L^2}^2 + (+\|\varphi\Delta_x^{q/2}u\|_{1/2}^2) \leq$$

$$\leq C|\langle P\varphi\Delta_x^{q/2}u, \varphi\Delta_x^{q/2}u\rangle| + C\|\varphi\Delta_x^{q/2}u\|_{L^2}^2 \leq$$

$$\leq C|\langle \varphi\Delta_x^{q/2}Pu, \varphi\Delta_x^{q/2}u\rangle_{L^2}| + C\sum_1^4 |\langle [X_j^2, \varphi\Delta_x^{q/2}]u, \varphi\Delta_x^{q/2}u\rangle| + C\|\varphi\Delta_x^{q/2}u\|_{L^2}^2 \ ,$$

and now the crucial brackets are

$$[X_1^2, \varphi\Delta_x^{q/2}] = 0 \quad , \quad [X_4^2, \varphi\Delta_x^{q/2}] = 0$$

and

$$[X_j^2, \varphi\Delta_x^{q/2}] = 2X_j t\varphi'\Delta_x^{q/2} - t^2\varphi^{(2)}\Delta_x^{q/2} \quad , \quad j = 2, 3$$

(where we have used rather heavily the fact that $X_4\varphi = 0$ since φ depends only on x, and radially so, and that in fact $[D_\theta, \Delta_x] = 0$.)

This last line leads to two kinds of terms, namely, for $j = 2, 3$,

$$\langle 2X_j t\varphi'\Delta_x^{q/2}u, \varphi\Delta_x^{q/2}u\rangle$$

and

$$\langle t^2\varphi^{(2)}\Delta_x^{q/2}u, \varphi\Delta_x^{q/2}u\rangle \ .$$

Morally, these terms show the correct gain to lead to analytic growth of derivatives, namely one must think of $t\Delta^{1/2}$ as an X_j with $j = 2$ or $j = 3$, and so in the first term above one merely integrates by part noting that $X_j^* = -X_j$ and obtains, after a weighted Schwarz inequality, a small multiple of the left hand side of the *a priori* inequality and the square of a term with one derivative on φ and $t\Delta^{1/2}$ and q reduced by one, though one more commutator is required to make the order correct, and this will introduce another derivative on φ and q again decreased by one unit, etc. The second term is of a different character, though the same observation reduces us essentially to

$$\langle X\varphi^{(2)}\Delta_x^{(q-1)/2}u, X\varphi\Delta_x^{(q-1)/2}u\rangle$$

in which instead of each copy of φ receiving one derivative, we have two derivatives on one copy and none on the other. Fortunately, the Ehrenpreis-type cut-off functions may be differentiated not merely N times with the usual growth but $2N$ or $3N$ with no change—so in the above inner product we include a factor CN with the copy of φ which remains undifferentiated and a factor of $(CN)^{-1}$ with the other. The estimates work out just as before.

3. Proof in the general case (1.6)

The general case is not more complicated than the first, simplest case, with $\Delta_t = \sum_{m=1}^{\ell} D_{t_m}^2$ requiring us to consider each t-variable separately; Δ_x is replaced by $\sum_{i,j=1}^{k} a_{ij}(t,x)X_iX_j$ and the square of the angular derivative by the sum $\sum_{i,j=1}^{s} b_{ij}(t,x)Y_iY_j$. Thus we merely give a brief sketch of the proof.

We have an *a priori* estimate of the form

$$\sum_{j=1}^{\ell} \|D_{t_j}u\|^2 + \sum_{j=1}^{\ell}\sum_{h=1}^{k} \|t_jX_hu\|^2 + \sum_{j=1}^{s} \|Y_su\|^2 + \|u\|_{1/2}^2 \leq \tag{3.1}$$

$$\leq C\left(|\langle P_2u, u\rangle| + \|u\|^2\right) .$$

Let us write, as introduced above,

$$\Lambda = \text{a positive definite quadratic expression in the } X_i$$

and

$$A = \sum_{i,j=1}^{k} a_{ij}(t,x)X_iX_j \ .$$

Analogously to what has been done before we denote by φ a cut-off function of Ehrenpreis type, constant on the integral manifold of the fields $Y_1, \dots, Y_s$ and independent of t.

The problem of estimating the growth rate of the derivatives of u then reduces to estimating

$$\|\varphi\Lambda^{q/2}u\| \ ,$$

for every natural number $q \leq N$, where $|\partial^\alpha \varphi| \leq C^{1+|\alpha|}N^{|\alpha|}$, for $0 \leq |\alpha| \leq 3N$.

We have thus to examine the structure of the commutator

$$[P_2, \varphi\Lambda^{q/2}] = [\Delta_t + |t|^2A + B, \varphi\Lambda^{q/2}] \ ,$$

where we wrote

$$B = \sum_{i,j=1}^{s} b_{ij}(t,x)Y_iY_j \ .$$

The above quantity becomes:

$$[\Delta_t + |t|^2A + B, \varphi\Lambda^{q/2}] = \frac{q}{2}\,\varphi[\Delta_t, \Lambda]\Lambda^{q/2-1} + |t|^2[A, \varphi]\Lambda^{q/2} +$$

$$+\frac{q}{2}\,|t|^2\varphi[A, \Lambda]\Lambda^{q/2-1} + \varphi[B, \Lambda^{q/2}] = T_1 + T_2 + T_3 + T_4 \ ,$$

modulo lower order terms whose treatment is easier. Let us look at each term in the above formula, denoting by 'elliptic' any term which contains, in the inner product, two factors of the form maximally estimated by the operator, namely two factors each of the form Y, tX, or $\partial/\partial t$. Such terms will be subject to the a priori inequality (after an integration by parts) in a recursive manner and will cause little trouble.

3.1. T_1. Since the commutator appears in a scalar product, taking one t-derivative to the other side, we have to estimate, for some coefficient $a(x)$,

$$q|\langle \varphi[D_{t_s}, \Lambda]\Lambda^{q/2-1}u, D_{t_s}\varphi\Lambda^{q/2}u\rangle| \sim q|\langle \varphi[D_{t_s}, aX]\Lambda^{(q-1)/2}u, D_{t_s}\varphi\Lambda^{q/2}u\rangle| \ .$$

But by (1.4), this bracket is elliptic, hence the factor of q balances the decrease in the exponent of Λ and will iterate analytically.

3.2. T_2. We have to estimate

$$|\langle |t|^2[\Lambda,\varphi]\Lambda^{q/2}u, \varphi\Lambda^{q/2}u\rangle| \ .$$

The commutator in the left hand side factor of the above scalar product leads to an expression of the type

$$2|\langle |t|^2 X_i\varphi'\Lambda^{q/2}u, \varphi\Lambda^{q/2}u\rangle| \sim 2|\langle Z\varphi'\Lambda^{(q-1)/2}u, Z\varphi\Lambda^{q/2}u\rangle|$$

with elliptic Z modulo (easier) lower order terms. Here we used just the form of Λ. A weighted Schwarz inequality shows that this term iterates analytically, since with the Ehrenpreis-type localizing functions, a derivative on φ balances a decrease in q.

3.3. T_3. We have to estimate the scalar product

$$|\langle \frac{q}{2}\ |t|^2\varphi[A,\Lambda]\Lambda^{q/2-1}u, \varphi\Lambda^{q/2}u\rangle| \ .$$

Since

$$[A,\Lambda] = \sum_{i,j=1}^{k}\sum_{\alpha,\beta=1}^{k}[a_{ij}X_iX_j, aX_\alpha X_\beta] = \tag{3.2}$$
$$= \sum \tilde{a}X^2[X,X] = \sum \tilde{a}X^2\,\{X \text{ or } \frac{\partial}{\partial t}\}$$

again modulo lower order terms, using assumption 3, part 1.

Now one of the X factors raises $q/2-1$ to $q/2-1/2=(q-1)/2$, with the factor of q balancing the decrease from q to $q-1$, and the other two X's (or one X and one $\partial/\partial t$ which is better, combine with t^2 to produce Z^2, with Z elliptic. Thus this inner product, as well, iterates analytically.

3.4. T_4. Since

$$\varphi[B,\Lambda^{q/2}] \sim \frac{q}{2}\varphi[B,\Lambda]\Lambda^{q/2-1} \ ,$$

the estimate of T_4 boils down to computing the commutator $[B,\Lambda]$ and estimating the resulting terms. But this goes as before in view of the second part of assumption 3, since the bracket contains a product of three vector fields, two of which are elliptic and one serves to convert $\Lambda^{q/2-1}$ to $\Lambda^{(q-1)/2}$.

This ends the proof of the Theorem in the general case.

4. Remarks vis-à-vis the conjecture of Treves

The conjecture of Treves states, in this context, that the operator P should be analytic hypoelliptic at the origin if and only if all layers of the Poisson stratification are symplectic. The first of these layers is the characteristic manifold, which is patently non-symplectic in all of these cases. And the operators are analytic hypoelliptic *in the sense of germs* at the origin.

The distinction is crucial. For to contradict Treves' conjecture, there would have to exist an open neighborhood V of the origin, in which one could have analytic data with a non-analytic solution. And that may still well be the case. What we have shown is that for neighborhoods of the origin *of a certain geometry relative to the operator*, analytic data forces analyticity of the solution.

It is well known however that in the case of the operator P_2 there is in general propagation of the analytic wave front set (or rather of the analytic regularity) along the Hamilton leaves of the characteristic manifold (which are non trivial in this case), see e.g. [1].

From the above proof we may see that the analyticity of the solution is forced, in some "adapted" open set, by a "global" phenomenon, that might be described by saying that the analytic singularities of the solution in the open set under consideration comes from points outside the open set lying on some Hamilton leaf of the characteristic manifold. This is actually prevented by the "large scale" geometry of the open set.

It can thus be asserted that, far from being in contradiction with Treves' conjecture, the present result is in complete agreement with it and points out the importance of the geometry of the non-symplectic strata of the Poisson-Treves stratification.

References

[1] A. Bove & D. S. Tartakoff, *Propagation of Gevrey regularity for a class of hypoelliptic equations*, Trans. Amer. Math. Soc., 348(1996), 2533-2575.

[2] A. Bove & D. S. Tartakoff, *Optimal non-isotropic Gevrey exponents for sums of squares of vector fields*, Comm. Partial Differential Equations, (7-8)22(1997), 1263-1282.

[3] N. Hanges, *Analytic regularity for an operator with Treves curves*, J. Functional Analysis, 210(2004), 295-320.

[4] D. S. Tartakoff, *Local analytic hypoellipticity for $\Box_b$ on non-degenerate Cauchy Riemann manifolds*, Proc. Nat. Acad. Sci. U.S.A., 75(1978), 3027-3028.

[5] D. S. Tartakoff, *On the local real analyticity of solutions to $\Box_b$ and the $\overline{\partial}$-Neumann problem*, Acta Math., 145(1980), 117-204.

[6] F. Treves, *Analytic hypo-ellipticity of a class of pseudo-differential operators with double characteristics and application to the $\overline{\partial}$-Neumann problem*, Comm. Partial Differential Equations, (6-7)3(1978), 475-642.

[7] F. Treves, *Symplectic geometry and analytic hypo-ellipticity*, in *Differential equations: La Pietra 1996 (Florence)*, Proc. Sympos. Pure Math., 65(1999), 201-219.