

**Locally conformally Kähler manifolds.
A selection of results**

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Abstract¹. I present a selection of results on locally conformally Kähler geometry published after 1997. The proofs are mainly sketched, some of them are even omitted. Several open problems are indicated in the end.

1. FOREWORD

Although known since the 50's, from P. Libermann's work, locally conformally Kähler (l.c.K.) structures have been systematically studied only starting with 1976, after the impulse given by I. Vaisman's work. Twenty-two years after, the monograph [23] gathered almost all known results in the subject. But already during the printing of the book new results were announced that could not be reported. Somehow, I had the feeling that the book came a little bit too early. On the other hand, I cherish the hope that it gave an impetus for new research.

Soon after the book was published, P. Gauduchon and the present author proved the existence of l.c.K. structures on all Hopf surfaces, a construction later generalized in the paper [39] to produce Hopf manifolds in any dimension. Almost in the same time, in his thesis, F.A. Belgun classified the compact complex surfaces with l.c.K. metrics with parallel Lee form (Vaisman metrics). In the same time, Y. Kamishima studied the l.c.K. structures from the uniformization viewpoint in a series of papers. He also initiated the study of the automorphism group of l.c.K. manifolds. The properties of the Ricci-like curvature were investigated by authors like B. Alexandrov, S. Ivanov; using Bochner type formulas, they found some new topological properties of l.c.K. manifolds with positive Weyl-Ricci tensor. With a completely different approach, M. Verbitsky recently produced similar restrictions and a structure theorem for compact Einstein-Weyl Vaisman manifolds, relating this geometry to the Sasakian one, very much in the spirit of the characterization given in [39] and [28]. The latter paper also discusses Hamiltonian fields and extends the reduction procedure from symplectic and Kähler geometry to l.c.K. structures. This study was further developed in [29], where a new equivalent definition was proposed,

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in terms of presentations; it makes more clear the relation between Kähler and l.c.K. geometries and provides a new invariant for l.c.K. structures. The structure of compact Vaisman manifolds is now completely understood in [53] and, on the other hand, algebro-geometric techniques were applied in order to derive vanishing theorems and a Kodaira type immersion theorem for Vaisman manifolds, see [67], [54]. Moreover, Riemannian properties of the l.c.K. metrics, in particular harmonicity of transversal holomorphic maps and harmonicity and minimality of vector fields and distributions were also discussed in [9] and [52]. Very recently, also the Inoue and Inoue-Hirzebruch surfaces were generalized to higher dimension, in [50] (where, as a side result, the authors also disproved the 30 years lasting Vaisman conjecture stating that any compact l.c.K. manifold has an odd odd Betti number) and [58]. Further on, considering in [55] l.c.K. manifolds with potential, a class including the Vaisman one, the Hopf surfaces of Kähler rank 0 were also generalized to higher dimension; any compact l.c.K. manifold with potential, of complex dimension at least 3, can be embedded in the resulting linear Hopf manifold. Finally, using the Gauduchon metric on compact Vaisman manifolds, a stability theory was developed in [68], with particular striking results for stable bundles over diagonal Hopf manifolds.

All this development is intimately connected with the late achievements in Sasakian and 3-Sasakian geometry. One may argue that the structure theorem in [53] reduces Vaisman geometry, in the compact case, to Sasakian one. But on the other hand, also results in l.c.K. geometry served to obtain significant progress in Sasakian geometry: Belgun used his classification of compact complex surfaces admitting metrics with parallel Lee form to completely classify Sasakian structures on compact 3-manifolds, cf. [11], [13], and the embedding theorem for compact Vaisman manifolds produced, cf. [54] and [55], a CR-embedding of any compact Sasakian manifold into a Sasakian weighted sphere. However, it is worth stressing that all these results hold only in the compact case.

Let me finally mention that much work was done in the late years on understanding the locally conformally Kähler structures in quaternionic geometry, but I shall not report here on this topic. The interested reader may consult my former survey [51] and the more recent paper [67].

Needless to say, the present survey does not aim to exhaustive comprehension. For example, I have left the recent results on special submanifolds of l.c.K. manifolds (cf. [8], [16] *et. al.*) for a future paper. This report only reflects my knowledge, information, taste and power of understanding.

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2. REVIEW OF OLD RESULTS AND A NEW DEFINITION

2.1. Definitions. For this subsection, I mainly refer to [23] and the references therein.

In the sequel (M, J, g) will be a connected Hermitian manifold of complex dimension $n \geq 2$. I shall denote by ω its fundamental two-form given by $\omega(X, Y) = g(X, JY)$.

Definition 2.1. (M, J, g) is called *locally conformally Kähler*, *l.c.K.* for short, if there exists an open cover $\mathcal{U} = \{U_\alpha\}$ such that each locally defined metric $g_\alpha = e^{-f_\alpha} g|_{U_\alpha}$ is Kähler for some smooth function f_α on U_α .

Equivalently, (M, J, g) is l.c.K. if and only if there exists a *closed* one-form θ such that

$$(2.1) \quad d\omega = \theta \wedge \omega .$$

Of course, locally, $\omega|_U = df_U$. Note also that, except on complex surfaces, the equation (2.1) implies $d\theta = 0$.

The globally defined one-form θ is called *the Lee form*² and its metrically equivalent (with respect to g) vector field $B = \theta^\sharp$ is called *the Lee vector field*; the vector field JB is called the *anti-Lee vector field*.

It can be easily seen that (M, J, g) is l.c.K. if and only if the following equation is satisfied for any $X, Y \in \mathcal{X}(M)$:

$$(2.2) \quad (\nabla_X J)Y = \frac{1}{2} \{ \theta(JY)X - \theta(Y)JX + g(X, Y)JB - \omega(X, Y)B \},$$

where ∇ is the Levi-Civita connection of g . Note that the above equation shows that l.c.K. manifolds belong to the class W_4 of the Gray-Hervella classification.

Another very useful characterization of l.c.K. manifolds is obtained *via* the universal Riemannian cover. Let $p : \tilde{M} \rightarrow M$ be the covering map and denote also by J the lifted complex structure. The lifted metric \tilde{g} is globally conformal with a Kähler metric \tilde{h} , because $p^*\theta$ is exact. The fundamental group of M will then act by holomorphic conformal transformations with respect to (J, \tilde{h}) . But conformal transformations of a symplectic form (in real dimension at least 4) are in fact homotheties. The converse is also true. But, in fact, one can see that it is not important to work with the universal cover, but with any cover on which the pull-back of the Lee form is exact. Summing up, the following equivalent definition holds:

Theorem 2.1. *A complex manifold (M, J) is l.c.K. if and only if admits a covering endowed with a global Kähler metric with respect to which the deck group of the covering acts by holomorphic homotheties.*

I shall denote by $\mathcal{H}(\tilde{M}, \tilde{h}, J)$ the above group of holomorphic homotheties and by $\rho : \mathcal{H}(\tilde{M}, \tilde{h}, J) \rightarrow \mathbb{R}_+$ the homomorphism associating to each homothety its scale factor.

2.2. Presentations. For this subsection I refer to [29] where all the definitions and results are to be found.

The above theorem suggests the following definition (where by homothetic Kähler one understands that the Kähler metric is fixed only up to homotheties):

Definition 2.2. A pair (K, Γ) is a *presentation* if K is a homothetic Kähler manifold and Γ a discrete Lie group of biholomorphic homotheties acting freely and properly discontinuously on K . If M is l.c.K. and $M = K/\Gamma$ as l.c.K. manifolds, then (K, Γ) is a *presentation of M* .

There can be more than one presentation for a given l.c.K. manifold. One cannot give a hierarchy, but can distinguish the extremes and, consequently, give a new definition of l.c.K. manifolds:

Definition 2.3. Let (K, Γ) be a presentation. The associated *maximal* presentation is $(\tilde{K}, \tilde{\Gamma})$, where \tilde{K} is the homothetic Kähler universal covering of K and $\tilde{\Gamma}$ is the lifting of Γ to \tilde{K} ³. Then

$$(K_{\min}, \Gamma_{\min}) = \left(\frac{\tilde{K}}{\text{Isom}(\tilde{K}) \cap \tilde{\Gamma}}, \frac{\tilde{\Gamma}}{\text{Isom}(\tilde{K}) \cap \tilde{\Gamma}} \right)$$

is called the associated *minimal* presentation.

The presentations (K, Γ) and (K', Γ') are equivalent if $(\tilde{K}, \tilde{\Gamma}) = (\tilde{K}', \tilde{\Gamma}')$. The = sign means that there exists an equivariant map, namely, there exists a pair of maps (f, h) , where $f : \tilde{K} \rightarrow \tilde{K}'$ is a biholomorphic homothety, $h : \tilde{\Gamma} \rightarrow \tilde{\Gamma}'$ is an isomorphism and for any $\gamma \in \tilde{\Gamma}$, $x \in \tilde{K}$ we have $f(\gamma x) = h(\gamma)f(x)$. Equivalently, we say that (K, Γ) is equivalent to (K', Γ') if $(K_{\min}, \Gamma_{\min}) = (K'_{\min}, \Gamma'_{\min})$.

²After the name of Hwa-Cwung Lee, cf. [45].

³See [29] for a thorough discussion about the properties of lifted actions.

A l.c.K. manifold is an equivalence class $[(K, \Gamma)]$ of presentations.

Of course, the second part of the definition is motivated by the fact that the maximal and minimal presentations are uniquely associated (as $\tilde{\Gamma}$, Γ_{\min} -spaces respectively) to a given l.c.K. structure.

Note that any minimal presentation is necessarily (K, \mathbb{Z}) .

In this context, I shall denote with ρ_K the scale homomorphism (the character) associated to the presentation (K, Γ) , $\rho_K : \Gamma \rightarrow \mathbb{R}_+$. Its image is a finitely generated subgroup (see also [55]) and its rank provides a nice invariant of the l.c.K. structure, measuring the "truly conformal" part of Γ :

Proposition 2.1. *For any presentation (K, Γ) , the rank of the free abelian group $\rho_\Gamma(\Gamma)$ depends only on the equivalence class $[(K, \Gamma)]$. Hence, one can speak about the rank R_M of l.c.K. manifold.*

The rank is bounded:

Proposition 2.2.

$$0 \leq r_M \leq b_1(M)$$

and $r_M = 0$ if and only if M is g.c.K. In particular, if $b_1(M) = 1$ then $r_M = 1$.

For example, the manifolds $M_s = (H^s \times \mathbb{C})\Gamma$ constructed in [50] (cf. Section 4) all have rank s .

2.3. Vaisman manifolds. A strictly smaller class of l.c.K. manifolds is the one formed by those with parallel (with respect to the Levi Civita connection) Lee form. I call them *Vaisman manifolds*, as I. Vaisman was the first to study them systematically (under the name of generalized Hopf manifolds, [64], a name which later proved to be inappropriate). On such a manifold, the length of the Lee vector field is constant and I shall always assume it is nonzero. Hence, in what follows, I shall normalize and consider that on a Vaisman manifold $\|B\| = \|JB\| = 1$.

Note that on a compact l.c.K. manifold, the metric with parallel Lee form, if it exists, is unique up to homothety in its conformal class and coincide with the Gauduchon (standard) metric, [47].

Proposition 2.3. *Let (M, J, g) be a Vaisman manifold. Then the Lee and anti-Lee vector fields commute ($[B, JB] = 0$), are Killing ($\mathcal{L}_B g = \mathcal{L}_{JB} g = 0$) and holomorphic ($\mathcal{L}_B J = \mathcal{L}_{JB} J = 0$). Consequently, the distribution generated by B and JB is a holomorphic Riemannian foliation.*

I denote by \mathcal{F} the foliation generated by B and JB . Note also that the leaves of the foliation generated by the nullity of the Lee form carry an induced α -Sasakian structure (see [15] as concerns metric contact manifolds) with JB as characteristic (Reeb) vector field. If these foliations are quasi-regular, then the leaf spaces are, respectively, a Kähler and a Sasakian orbifold.

The first known examples of l.c.K. manifolds which are not Kähler (because they have $b_1 = 1$) were the (diagonal) Hopf manifolds: $H_\lambda = (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$, $g_{flat}/|z|^2$, J_{can} , where \mathbb{Z} is generated by the transformation $z_j \mapsto \lambda z_j$, with $\lambda \in \mathbb{C}$, $|\lambda| \neq 0, 1$. The Lee form here is (when read on $\mathbb{C}^n \setminus \{0\}$): $\theta = -d \log |z|^2$ and is seen to be parallel. These manifolds are diffeomorphic with the product $S^1 \times S^{2n-1}$. What matters here is the Sasakian structure of S^{2n-1} . Indeed it was then proved that the total space of a flat principal circle bundle over a compact Sasakian manifold carries a Vaisman metric whose Lee form is identified with the connection form of the bundle.

A Vaisman structure obtained as above is regular. Non-regular examples can be obtained as suspensions over S^1 with fibre a Sasakian manifold, [26] and, for a general structure theorem, [67], [53] (see below).

Remark 2.1. Particular examples of Vaisman structures are the locally conformally hyperkähler ones. If compact, they have $b_1 = 1$, hence also rank $r_M = 1$. We shall see later that, in fact, all compact Vaisman manifolds have rank 1. On the other hand, examples of compact Vaisman manifolds with arbitrarily large b_1 can be obtained as induced Hopf bundles over curves of large genus in $\mathbb{C}P^2$.

L.c.K. metrics without parallel Lee form were constructed by F. Tricerri on some of the Inoue surfaces and by I. Vaisman on the diagonal Hopf manifold.

3. LOCALLY CONFORMALLY KÄHLER METRICS ON NON-KÄHLER COMPACT COMPLEX SURFACES

3.1. **Vaisman structures.** Locally, non-global, conformal Kähler metrics may exist only on the surfaces in classes VI and VII of the Kodaira classification, [43].

A generic Hopf surface is defined by Kodaira as a complex surface whose universal cover is $\mathbb{C}^2 \setminus \{0\}$. Hopf surfaces with fundamental group isomorphic with \mathbb{Z} are called primary. Kodaira also proved that the fundamental group of any primary Hopf surface can be realized as the cyclic group generated by the transformation:

$$(z_1, z_2) \mapsto (\alpha z_1 + \lambda z_2^m, \beta z_2),$$

where $m \in \mathbb{N}$ and α, β, λ are complex numbers subject to the conditions:

$$(\alpha - \beta^m)\lambda = 0, \quad |\alpha| \geq |\beta| > 1.$$

A Vaisman metric on Hopf surfaces $H_{\alpha, \beta}$ with $\lambda = 0$ was constructed in [26]. It can be described as follows:

Step 1. One starts with the canonical Sasakian structure (g_0, R_0) on S^3 , where g_0 is the metric induced by the flat one on \mathbb{C}^2 and $R_0(z) = (iz_1, iz_2)$. With $k_1 = \log |\alpha|$, $k_2 = \log |\beta|$, this one is deformed using the function

$$\Delta(z) = \frac{2k_1|z_1|^2 + 2k_2|z_2|^2}{k_1 + k_2},$$

namely one puts:

$$(3.1) \quad R_\Delta = R_0 + \frac{2(k_1 - k_2)}{k_1 + 1 + k_2}(iz_1, -iz_2)$$

and define a new metric g_Δ by the conditions:

- (1) R_Δ has g_Δ -norm 1.
- (2) R_Δ is orthogonal to the canonical contact distribution \mathcal{D} of S^3 .
- (3) On \mathcal{D} , $g_\Delta = \Delta^{-1}g_0$.

Note that this is not a D-homothetic transformation in the sense of [60]. Still, the new structure (g_Δ, R_Δ) is Sasakian on S^3 (cf. *loc. cit.* for a general criterion for such deformations to be Sasakian).

Step 2. One uses the flow of g_Δ -isometries

$$\sigma_{\alpha, \beta}((z, t)) = \left(e^{-i \text{Arg } \alpha} \cdot z_1, e^{-i \text{Arg } \beta} \cdot z_2, t \right)$$

to obtain a suspension over S^1 with fibre S^3 . The total space will then be a Vaisman manifold diffeomorphic with $S^3 \times S^1$.

To describe the underlying complex structure $J_{\alpha, \beta}$, let T be the vector field tangent to S^1 , viewed as a vector field on $S^1 \times S^3$, and let $E = (\bar{z}_2, -\bar{z}_1)$ (so that E, iE generate \mathcal{D}). Also let

$$F(z) = \log |\alpha| |z_1|^2 + \log |\beta| |z_2|^2.$$

Now $J_{\alpha,\beta}$ is described by the following table:

$$(3.2) \quad \begin{aligned} J_{\alpha,\beta}T &= \frac{1}{\operatorname{Re} F} (-\operatorname{Im} F T + |F|^2 R_0 + i\bar{F}(\log \alpha - \log \beta)z_1 z_2 E), \\ J_{\alpha,\beta}R_0 &= \frac{1}{\operatorname{Re} F} (-T + \operatorname{Im} F R_0 + (\log \alpha - \log \beta)z_1 z_2 E), \\ J_{\alpha,\beta}E &= iE. \end{aligned}$$

Translated on $H_{\alpha,\beta}$, by means of the diffeomorphism

$$F_{\alpha,\beta}(z_1, z_2, t \bmod \mathbb{Z}) = [\alpha^t z_1, \beta^t z_2],$$

this Vaisman structure corresponds to the potential

$$\Phi_{\alpha,\beta}(z_1, z_2) = e^{(\log |\alpha| + \log |\beta|)\tau/(2\pi)},$$

with τ given as (unique) solution as the equation:

$$\frac{|z_1|^2}{e^{\tau \log |\alpha| \pi}} + \frac{|z_2|^2}{e^{\tau \log |\beta| \pi}} = 1.$$

Namely:

Theorem 3.1 (Gauduchon-Ornea [26]). *The metric:*

$$g_{\alpha,\beta} = \frac{dd^c \Phi_{\alpha,\beta}}{\Phi_{\alpha,\beta}}(JX, Y)$$

is globally conformally Kähler on $\mathbb{C}^2 \setminus \{0\}$, has parallel Lee form and descends to a Vaisman metric on $H_{\alpha,\beta}$.

In fact, this way one gets a 1-parameter family of Vaisman metrics on $H_{\alpha,\beta}$: indeed, for any $l \in \mathbb{R}_+$, the potential $\Phi_{\alpha,\beta}^l$ produces again a Vaisman metric (this amounts to taking lk_1, lk_2 instead of k_1, k_2 above).

This family of structures has been generalized in [56] by including it in a larger family of l.c.K. structures (preserving the complex structure and modifying only the metrics). Let $h : \mathbb{R} \rightarrow \mathbb{R}_+$ be any function with period 2π . Parton's metrics are given by the following Hermitian matrix (written in the complex basis $\{R_0, -E\}$):

$$(3.3) \quad g_{\alpha,\beta}^h = \begin{pmatrix} \frac{\pi h}{(\operatorname{Re} F)^2} + \frac{|z_1|^2 |z_2|^2 \log^2(|\alpha|/|\beta|)}{(\operatorname{Re} F)^3} & \frac{iz_1 z_2 \log(|\alpha|/|\beta|)}{(\operatorname{Re} F)^2} \\ \frac{\overline{iz_1 z_2} \log(|\alpha|/|\beta|)}{(\operatorname{Re} F)^2} & \frac{1}{\operatorname{Re} F} \end{pmatrix}$$

For a constant h , the family of Vaisman metrics in [26] is recovered for $l = 2\pi h/(\log |\alpha| + \log |\beta|)$. This is not by chance; indeed, by a direct computation one proves:

Proposition 3.1 (Parton [56]). *The metric $g_{\alpha,\beta}^h$ is Vaisman if and only if $h = \text{const}$.*

The full list of compact complex surfaces which admit l.c.K. metrics with parallel Lee form was given by Belgun:

Theorem 3.2 (Belgun [10]). *A non-Kähler compact complex surface admits a Vaisman metric if and only if it is: a properly elliptic surface, or a Kodaira surface (primary or secondary), or a Hopf surface $H_{\alpha,\beta}$.*

Belgun's proof relies on the following criterion for the existence of Vaisman metrics which encodes also the construction in [26]:

Theorem 3.3 (Belgun [10]). *Let (M, J) be a compact complex surface with universal covering space (\tilde{M}, J) . A Vaisman metric on (M, J) is equivalent with the following data:*

i) *A real holomorphic vector field V without zeros on M . Denote with \tilde{V} its lift to \tilde{M} and with Φ its complex flow.*

ii) *A group homomorphism $\tau : \pi_1(M) \cdot \Phi \rightarrow \mathbb{R}$ such that*

(1) $\tau|_{\pi_1(M)} \neq 0$;

(2) $\tau(\varphi_t^{\tilde{V}}) = \varepsilon \neq 0$ and $\tau(\varphi_t^{J\tilde{V}}) = 0$, where φ_t^X denotes the flow of the vector field X .

iii) *A 3-dimensional foliation of \tilde{M} given by the level hypersurfaces of a τ -equivariant function $f : \tilde{M} \rightarrow \mathbb{R}$ whose leaves carry a contact structure induced by the distribution Ω of complex lines contained in $\text{Ker } df$.*

In this case, a Vaisman metric on M is given by the Kähler form $\omega = k_1 H^{-1} d(dH \circ J)$, where $H = e^{k_2 f}$ and $k_1, k_2 \in \mathbb{R}^$.*

Here is the outline of the proof.

Let first g be a Vaisman metric with (parallel) Lee form θ on M . By the de Rham decomposition theorem, $(\tilde{M}, \tilde{g}) = (\tilde{S}, g) \times (\mathbb{R}, \text{can})$, where \tilde{S} is the universal cover of a leaf S of the foliation $\text{Null } \theta$ (note that S carries an induced Sasakian structure which underlying contact structure the statement refers to). Now \tilde{V} is identified with $\|V\|\partial_t$ and f can be defined as the projection of \tilde{M} on the \mathbb{R} factor. To define τ , one first checks that for any $a \in \pi_1(M)$, the scalar $f((a \circ \varphi_z^{\tilde{V}})(x, 0))$ does not depend on $x \in \tilde{S}$. Then one sets:

$$\tau(a \circ \varphi_z^{\tilde{V}}) = f((a \circ \varphi_z^{\tilde{V}})(x, 0)) .$$

To show that τ is a group homomorphism, it is enough to see that its restriction to $\pi_1(M)$ is such. This follows from the following computation:

$$\begin{aligned} \tau(a_1 a_2) &= f(a_1(a_2(x, 0))) = f(a_1(\varphi_{\tau(a_2)}^{\tilde{V}}(x, 0))) = f(\varphi_{\tau(a_2)}^{\tilde{V}}(a_1(x, 0))) \\ &= f(\varphi_{\tau(a_2)}^{\tilde{V}}(\varphi_{\tau(a_1)}^{\tilde{V}}(x, 0))) = \tau(a_1) + \tau(a_2) . \end{aligned}$$

The properties of τ are now straightforward ($\varepsilon = \|\tilde{V}\|$).

The Kähler form is now defined for $k_1 = -1/4$ and $k_2 = 2\varepsilon$.

For the converse, let $k_1 = -1/4$ and $k_2 = \pm 2df(\tilde{V})$ with the sign given by the one of $d(dH \circ J)(JY, Y)$, for $Y \in \text{Null } \theta$. Then $\omega' = k_1 H^{-1} d(dH \circ J)$ is clearly l.c.K. with Lee form $\theta' = (k_2/2)df$ and Lee field $(k_2/2)\tilde{V}$. This a unit Killing field (because $\mathcal{L}_{\tilde{V}} f = 0$ and $\mathcal{L}_{\tilde{V}} J = 0$). Hence the symmetric part of $\nabla\theta'$ is 0. As the antisymmetric part of $\nabla\theta'$ is identified with $d\theta' = 0$, the proof is complete.

Remark 3.1. For the first part of the proof one may also note the general fact that the cone $S \times \mathbb{R}$ over a Sasakian manifold (S, g) has the Vaisman metric $2e^{-t}g_c$, with Lee form $-dt$, where $g_c = e^t(dt^2 + g)$ is the Riemannian cone metric.

Now the proof of Theorem 3.2 proceeds by a case by case analysis, using the above criterion and the following consequence:

Proposition 3.2. *Let W be the lift of the parallel Lee field of a Vaisman metric to the universal cover. Then the orbits of JW and of the cyclic group generated by $a \circ \varphi_{-r}^W$ are relatively compact in $\mathbb{C}^2 \setminus \{0\}$, where $a \in \pi_1(M)$ and $r = \tau(a)$.*

3.2. Locally conformally Kähler metrics with non-parallel Lee form.

Other surfaces in the classes VI and VII - but not all of them - may admit l.c.K. metrics with non-parallel Lee form. The first examples (on surfaces which were not known to admit also Vaisman structures) were constructed in [61] on the Inoue surfaces S_M , $S_{n;p,q,r}^-$ and $S_{n;p,q,u}^+$ with $u \in \mathbb{R}$. These metrics have harmonic (but non-parallel) Lee form and are locally homogeneous.

A l.c.K. metric with non-parallel Lee form was constructed in [26] on the Hopf surface of Kähler rank 0 (with $\lambda \neq 0$). The construction uses a small deformation of the structure on $H_{\alpha,\beta}$, but the argument is specific to this case: in general, the l.c.K. class is not closed under small deformations [10] (even if one can always deform a Vaisman structure on a compact manifold to a quasi-regular one, see below the proof of the embedding theorem in [54]); see also §7 for a subclass of l.c.K. manifolds which is stable to small deformations. Moreover, Belgun proved:

Proposition 3.3 (Belgun [10]). *The Hopf surfaces of Kähler rank 0 do not admit any Vaisman structure.*

The proof is again an application of the criterion in Theorem 3.3. The general form of a holomorphic vector field on $\mathbb{C}^2 \setminus \{0\}$ which descends on this Hopf surface can be found as:

$$W(z_1, z_2) = (mbz_1 + cz_2^m)\partial_{z_1} + bz_2\partial_{z_2} \quad , \quad b, c \in \mathbb{C} .$$

Now one may compute the flows of W and JW and, supposing the existence of a parallel Lee vector field (necessarily holomorphic), apply Proposition 3.2: if the orbits of JW are relatively compact, then $b \in \mathbb{R}$ and $c = 0$. Then the second condition in the Proposition 3.2 assures $\lambda = 0$.

Besides, Belgun proved that the Inoue surfaces cannot admit l.c.K. metrics with parallel Lee form. And, most important, showing that compact non-Kähler surfaces are not necessarily of l.c.K. type, he proved:

Theorem 3.4 (Belgun [10]). *Inoue surfaces $S_{n;p,q,u}^+$ with $u \in \mathbb{C} \setminus \mathbb{R}$ cannot admit any l.c.K. metric.*

This is in fact the only case where the method in [61] of constructing l.c.K. metrics on Inoue surfaces didn't work. The considered surface is a quotient of Sol_1^4 by an integer lattice. Belgun's proof goes like this: he first shows that Sol_1^4 admits a bi-invariant volume form; this is done by writing down explicitly the generators of the Lie algebra of Sol_1^4 and checking that the multivector they define is ad-invariant. He then proves that for $\text{Im } u \neq 0$, there is no left invariant l.c.K. metric on Sol_1^4 (with another proof, this was also observed in [65]). This follows from a very nice computation on the above determined generators of \mathfrak{sol}_1^4 read as vector fields. Finally, he shows that if $S_{n;p,q,u}^+$ with $u \in \mathbb{C} \setminus \mathbb{R}$ admits an l.c.K. metric, then, by averaging with respect to the bi-invariant volume form found at the first step, it also admits a Sol_1^4 -invariant l.c.K. metric, which is the desired contradiction.

As $S_{n;p,q,u}^+$ with $u \in \mathbb{C} \setminus \mathbb{R}$ can be obtained by a small deformation of an Inoue surface which admits l.c.K. metric, this proves

Corollary 3.1 (Belgun [10]). *Unless the Kähler class, the l.c.K. class is not stable under small deformations.*

On the other hand, as already mentioned, l.c.K. metrics with non-parallel Lee form on some of the Inoue surfaces were constructed in [61]. I shall briefly recall this construction for S_M . Let $H = \{w = w_1 + iw_2 \in \mathbb{C} \mid w_2 > 0\}$ be the open half-plane and let $M = (a_{ij}) \in \text{SL}(3, \mathbb{Z})$ be a uni-modular matrix with one real eigenvalue $\alpha > 1$ with eigenvector (a_1, a_2, a_3) , and a non-real complex eigenvalue β , with eigenvector (b_1, b_2, b_3) . Consider the following transformations:

$$(w, z) \mapsto (\alpha w, \beta z) ,$$

$$(w, z) \mapsto (w + a_j, z + b_j)$$

They generate a group Γ_M which acts on $H \times \mathbb{C}$, the quotient being a compact complex surface, the Inoue surface S_M . The metric $g = w_2^{-2} dw \otimes d\bar{w} + w_2 dz \otimes d\bar{z}$ on $H \times \mathbb{C}$ is

globally conformal Kähler with Lee form $\omega = d \log w_2$. Being compatible with the action of Γ_M , it induces a l.c.K. metric on S_M .

This metric, a warped-product of the flat metric of \mathbb{C} and the -1 -constant curvature metric of the Poincaré half-plane, has also interesting Riemannian properties. In fact, it was shown that its Lee and anti-Lee vector fields provide examples of harmonic vector fields (for definitions, see e.g. [19], [27]):

Proposition 3.4 (Ornea-Vanhecke [52]). *The Lee and anti-Lee vector fields of the Tricerri metric on an Inoue surface S_M satisfy the following properties:*

- i) *they are harmonic and minimal;*
- ii) *the distribution locally generated by them (which is not a foliation in this case) is harmonic and determines a minimal immersion of (S_M, g) into $(G_2^{or}(S_M), g^S)$.*

Remark 3.2. A l.c.K. structure on a principal S^1 -bundle over a compact 3-dimensional solvable Lie group was constructed in [4]. We reported it as such in [23]. In fact, as it was shown later on in [38], this structure is homothetically holomorphic with the l.c.K. structure on the Inoue surface of type Sol_1^4 constructed by Tricerri.

Recently, in [24], a new construction was announced of anti-self-dual Hermitian metrics on some compact surfaces in class VII with $b_2 > 0$. Such metrics are automatically l.c.K. by a result in [18].

3.3. A characterization in terms of Dolbeault operator. L.c.K. manifolds do not usually appear among the limiting cases of inequalities involving the eigenvalues of Dirac-type operators. One motivation could be that l.c.K. geometry has not a holonomy formulation. However, such a characterization is available for some l.c.K. surfaces in terms of the eigenvalues of the Dolbeault operator (which, on Kähler manifolds, acts essentially as the Dirac one). The proof is too technical to be reported here, I only state the result:

Theorem 3.5 (Alexandrov-Grantcharov-Ivanov [1]). *Let M be a compact Hermitian spin surface of positive conformal scalar curvature k . Then the first eigenvalue of the Dolbeault operator satisfies the inequality:*

$$\lambda^2 \geq \frac{1}{2} \inf k .$$

In the limiting case $k = \text{const.}$ and M is l.c.K.

4. A GENERALIZATION OF THE INOUE SURFACE S_M

I shall outline here the construction in [50]. As far as I know, it is for the first time that (algebraic) number theory is used to construct examples of l.c.K. structures. It is remarkable that their examples also lead to disproving an almost thirty years old conjecture by Vaisman.

Let K be an algebraic number field of degree $n := (K : \mathbb{Q})$. Let then $\sigma_1, \dots, \sigma_s$ (resp. $\sigma_{s+1}, \dots, \sigma_n$) be the real (resp. complex) embeddings of K into \mathbb{C} , with $\sigma_{s+i} = \bar{\sigma}_{s+i+t}$, for $1 \leq i \leq t$. Let \mathcal{O}_K be the ring of algebraic integers of K . Note that for any $s, t \in \mathbb{N}$, there exist algebraic number fields with precisely s real and $2t$ complex embeddings.

Using the embeddings σ_i , K can be embedded in \mathbb{C}^m , $m = s + t$, by

$$\sigma : K \rightarrow \mathbb{C}^m \quad , \quad \sigma(a) = (\sigma_1(a), \dots, \sigma_m(a)) .$$

This embedding extends to \mathcal{O}_K and $\sigma(\mathcal{O}_K)$ is a lattice of rank n in \mathbb{C}^m , see, for example, [17, p. 95 ff.]. This gives rise to a properly discontinuous action of \mathcal{O}_K on \mathbb{C}^m . On the other hand, K itself acts on \mathbb{C}^m by

$$(a, z) \mapsto (\sigma_1(a)z_1, \dots, \sigma_m(a)z_m) .$$

Note that if $a \in \mathcal{O}_K$, $a\sigma(\mathcal{O}_K) \subseteq \sigma(\mathcal{O}_K)$. Let now \mathcal{O}_K^* be the group of units in \mathcal{O}_K and set

$$\mathcal{O}_K^{*,+} = \{a \in \mathcal{O}_K^* \mid \sigma_i(a) > 0, 1 \leq i \leq s\} .$$

The only torsion elements in the ring \mathcal{O}_K^* are ± 1 , hence the Dirichlet units theorem asserts the existence of a free abelian group G of rank $m - 1$ such that $\mathcal{O}_K^* = G \cup (-G)$. Choose G in such a way that it contains $\mathcal{O}_K^{*,+}$ (with finite index). Now $\mathcal{O}_K^{*,+}$ acts multiplicatively on \mathbb{C}^m and, taking into account also the above additive action, one obtains a free action of the semi-direct product $\mathcal{O}_K^{*,+} \ltimes \mathcal{O}_K^*$ on \mathbb{C}^m which leaves invariant $H^s \times \mathbb{C}^t$ (as above, H is the open upper half-plane in \mathbb{C}). Again by Dirichlet units theorem the authors show that it is possible to choose a subgroup U of $\mathcal{O}_K^{*,+}$ such that the action of $U \ltimes \mathcal{O}_K$ on $H^s \times \mathbb{C}^t$ be properly discontinuous and co-compact. Such a subgroup U is called *admissible* for K . The quotient

$$X(K, U) := (H^s \times \mathbb{C}^t) / (U \ltimes \mathcal{O}_K)$$

is then shown to be a m -dimensional compact complex (affine) manifold, differentially a fiber bundle over $(S^1)^s$ with fiber $(S^1)^n$.

Observe that for $s = t = 1$ and $U = \mathcal{O}_K^{*,+}$, $X(K, U)$ reduces to an Inoue surface S_M as described in the preceding paragraph.

Theorem 4.1 (Oeljeklaus-Toma [50]).

- i) For $t = 1$, $X(K, U)$ admits locally conformally Kähler metrics.
- ii) For $t > 1$ and $s = 1$, there is no locally conformally Kähler metric on $X(K, U)$.

Indeed,

$$\varphi : H^s \times \mathbb{C} \rightarrow \mathbb{R}, \quad \varphi = \frac{1}{\prod_{j=1}^s (i(z_j - \bar{z}_j))} + |z_m|^2$$

is a Kähler potential on whose associated 2-form $i\partial\bar{\partial}\varphi$ the deck group acts by linear holomorphic homotheties.

A particular class of manifolds $X(K, U)$ is that of *simple type*, when U is not contained in \mathbb{Z} and its action on \mathcal{O}_K does not admit a proper non-trivial invariant submodule of lower rank (which, as the authors show, is equivalent to the assumption that there is no proper intermediate field extension $\mathbb{Q} \subset K' \subset K$ with $U \subset \mathcal{O}_{K'}$). The information about the topology of $X(K, U)$ is gathered (I report only what is relevant to l.c.K. geometry, but the whole computation is beautiful) in the following

Theorem 4.2 (Oeljeklaus-Toma [50]).

- i) $b_1(X(K, U)) = s$, hence for $s = 2p$ $X(K, U)$ cannot be Vaisman.
- ii) If $X(K, U)$ is of simple type, then $b_2(X(K, U)) = \binom{s}{2}$.
- iii) The tangent bundle $TX(K, U)$ is flat and $\dim H^1(X(K, U), \mathcal{O}_{X(K, U)}) \geq s$. In particular, $X(K, U)$ are non-Kähler.

Now, for $s = 2$ and $t = 1$, the six-dimensional $X(K, U)$ is of simple type, hence has the following Betti numbers: $b_0 = b_6 = 1$, $b_1 = b_5 = 2$, $b_2 = b_4 = 1$, by i) and ii), $b_3 = 0$ by iii). This proves:

Theorem 4.3 (Oeljeklaus-Toma [50]). *Vaisman's conjecture claiming that a compact locally conformally Kähler, non-Kähler manifold must have an odd odd Betti number is false.*

This ends the story of a conjecture that most of the interested people believed true, especially because it holds for Vaisman manifolds (b_1 is odd).

Remark 4.1. No general classification of l.c.K. structures is available for the moment for 3-dimensional complex manifolds. Only recently, in [63], l.c.K. structure were classified on *nilmanifolds*. Using representation theory, the author first determines the nilpotent Lie algebras that can admit l.c.K. structures: these are \mathfrak{h}_1 and \mathfrak{h}_3 . He then proves that *any l.c.K. structure on the product of the circle with a compact quotient of the Heisenberg group is Vaisman* and, on the other hand, *compact complex parallelizable nilmanifolds which are not tori do not admit l.c.K. metrics.*

5. THE AUTOMORPHISM GROUP OF A LOCALLY CONFORMALLY KÄHLER MANIFOLD

By definition, a diffeomorphism of a l.c.K. manifold is a (l.c.K.)-automorphism if it is biholomorphic and preserves the conformal class:

$$\text{Aut}(M) = \{f \in \text{Diff}(M) \mid f_*J = Jf_*, f^*g \in [g]\} .$$

Proposition 5.1 (Kamishima [36]). *The automorphism group of a compact l.c.K. manifold is a compact Lie group.*

In fact, it is a Lie group as it is closed in the group of all conformal transformations of $(M, [g])$ and it is compact as a consequence of the Obata-Ferrand theorem.

This follows also from a more general argument. As I already noted, on a compact Vaisman manifold the metric with parallel Lee form coincides (up to homothety) with the Gauduchon metric. But the Gauduchon metric exists on any compact l.c.K. manifold. Hence one may thus think about the relation between $\text{Aut}(M)$ and the isometry group of the latter. And indeed:

Theorem 5.1 (Madsen-Pedersen-Poon-Swann [47]). *The automorphism group of a compact l.c.K. manifold coincides with the isometry group of the Gauduchon metric.*

This implies that, when working on compact Vaisman manifolds, considering isometric actions with respect to the Vaisman metric instead of conformal actions does not represent a loss of generality (see Section 8).

Vaisman structures can be characterized among the l.c.K. ones (but only on compact spaces) by means of group actions:

Theorem 5.2 (Kamishima-Ornea [39]). *Let (M, g, J) be a compact l.c.K. manifold. Then $[g]$ contains a metric with parallel Lee form if and only if $\text{Aut}(M)$ contains a complex 1-dimensional Lie subgroup.*

Clearly, from Proposition 2.3, the condition is necessary. The proof of the sufficiency is based on exhibiting the universal cover \tilde{M} of M as a Riemannian cone (endowed with the associated Kähler structure) over a Sasakian manifold, then showing that $\pi_1(M)$ acts by holomorphic homotheties with respect to the Kähler metric. The needed Sasakian manifold is obtained as follows. Let T be the complex 1-dimensional Lie subgroup in the statement. One first shows that a lift \tilde{T} of T cannot contain only isometries, hence $\rho(T) = \mathbb{R}_+$. It is then possible to pick generators $\xi, J\xi$ such that the flow φ_t^ξ act by conformal maps (and is isomorphic to \mathbb{R}) and the flow $\varphi_t^{J\xi}$ act by isometries. Next, using the compactness of M , one sees that this action of \mathbb{R} on \tilde{M} is free and proper, in particular, ξ does not vanish. Finally, the 1-level set W of the squared norm of ξ (with respect to the Kähler metric of \tilde{M}) is shown to have an induced Sasakian structure.

On the other hand, it is known that the orbit map ev of any action of a torus T^2 , $ev_x(t) = t \cdot x$, induces a map $ev_* : \mathbb{Z}^2 \rightarrow H_1(M, \mathbb{Z})$ in homology. If M is compact Kähler and the action is holomorphic, then ev_* is injective. By contrast, Kamishima proves:

Theorem 5.3 (Kamishima [37]). *Let M be a compact, non-Kähler l.c.K. manifold of complex dimension at least 2. If $\text{Aut}(M)$ contains a complex torus $T_{\mathbb{C}}^1$, then the induced action in homology has rank 1.*

Note that some elliptic surfaces, in particular the diagonal Hopf surfaces, more generally, the regular compact Vaisman manifolds, do admit such actions, generated by the Lee and anti-Lee vector fields.

6. GEOMETRY AND TOPOLOGY OF VAISMAN MANIFOLDS

6.1. A general example. Examples of (compact), non-Kähler, l.c.K. manifolds are now abundant. More precisely, there are many examples of compact Vaisman manifolds. It

was known that the total space of a flat principal circle bundle over a compact Sasakian manifold carries a Vaisman metric whose Lee form is identified with the connection form of the bundle. I here present such a concrete example (see [39] for the general case and [26] for the surface case) which plays in Vaisman geometry the role of the projective space in Kähler geometry.

Let a_j be real numbers such that $0 < a_1 \leq \dots \leq a_n$. Deform the standard contact form of the sphere S^{2n-1} , $\eta = \sum (x_i dy_i - y_i dx_i)$, to $\eta_A = (1/\sum a_i |z_i|^2) \eta$ which underlies the same contact structure, but whose Reeb field is $R_A = \sum a_i (x_i \partial y_i - y_i \partial x_i)$. Accordingly, define a metric g_A on S^{2n-1} by letting it be equal to the round one on $\text{Ker } \eta_A$ and by requesting that R_A be unitary and orthogonal to $\text{Ker } \eta_A$. Now (S^{2n-1}, η_A, g_A) is Sasakian and will be denoted S_A^{2n-1} . It is called *the weighted sphere*⁴. The cone $\mathbb{R} \times S_A^{2n-1}$ is then Kähler, *e.g.* according to [20], the Kähler form being identified with $\omega_A = d(e^t \eta_A)$. The underlying complex manifold can be shown to be biholomorphic with $\mathbb{C}^n \setminus \{0\}$ (with the standard complex structure) by means of the map $(t, (z_j)) \mapsto (e^{-a_j t} z_j)$. Now pick a group $\Gamma \subset \mathbb{R} \times \text{PSH}(S_A^{2n-1})$ which acts properly discontinuously by holomorphic homotheties with respect to ω_A . The quotient space will be locally conformally Kähler with Lee form identified with $-dt$, hence parallel. Equivalently, we may take the corresponding action of Γ on $\mathbb{C}^n \setminus \{0\}$ and “read” this Vaisman structure on a quotient of $\mathbb{C}^n \setminus \{0\}$. In particular, let $(c_1, \dots, c_n) \in (S^1)^n$, $s \neq 0$ and consider an action of \mathbb{Z} on $\mathbb{C}^n \setminus \{0\}$ generated by: $(z_1, \dots, z_n) \mapsto (e^{a_1 s} c_1 z_1, \dots, e^{a_n s} c_n z_n)$. This group satisfies the above construction, hence the quotient $S^1 \times S_A^{2n-1}$, isometric with $H_\Lambda : \mathbb{C}^n \setminus \{0\}/\mathbb{Z}$, with $\lambda_i = e^{a_i s} c_i$, is a Vaisman manifold that was called a (general, or non-standard, or diagonal) Hopf manifold. Reversing the construction, we see that for every complex numbers λ_j satisfying $1 < |\lambda_1| \leq \dots \leq |\lambda_n|$, there exists a corresponding Hopf manifold. In the simplest case, when $\lambda_i = 1/2$, one recovers the standard Hopf manifold with l.c.K. metric (read on $\mathbb{C}^n \setminus \{0\}$) $g_0 = (\sum |z_i|^2)^{-1} \sum dz_i \otimes d\bar{z}_i$ and Lee form $\omega_0 = -(\sum |z_i|^2)^{-1} \sum (\bar{z}_i dz_i + z_i d\bar{z}_i)$; here the Lee field is the one tangent to the S^1 factor.

The construction of the above example obeys a general pattern: one starts with a Sasakian manifold, constructs the Riemannian cone over it, this has a Kähler structure, then selects a group which acts by holomorphic homotheties with respect to this Kähler structure. The quotient is then a Vaisman manifold. As will be clear later, this pattern in fact exhausts all compact Vaisman manifolds.

Still another way of looking at the above diagonal Hopf manifold, which will play an important role in §7, is from the viewpoint of the potential existing on the universal cover $\mathbb{C}^n \setminus \{0\}$, cf. [68]. In fact, if, for simplicity, we let Γ be generated by the diagonal operator A with eigenvalues α_i , then the Kähler potential

$$\varphi(z_1, \dots, z_n) = \sum |z_i|^{\beta_i} \quad , \quad \beta_i = \log_{|\alpha_i|^{-1}} C \quad , \quad C = \text{const.} > 1 \quad ,$$

is acted on by A as follows: $A^* \varphi = C^{-1} \varphi$. Hence the associated Kähler form ω_K will satisfy the same relation: $A^* \omega_K = C^{-1} \omega_K$. This proves that the quotient $H_A = (\mathbb{C}^n \setminus \{0\})/\langle A \rangle$ is l.c.K. Now Theorem 5.2 assures that it is Vaisman with respect to the metric ω_K/φ (remember also the construction in [26]). It can be shown (cf. [68]) that this is the Gauduchon metric of its conformal class and that its Lee field is given by the formula

$$\theta^\sharp = - \sum z_i \log |\alpha_i| \partial z_i \quad .$$

The diagonal Hopf manifold may be identified among the compact Vaisman manifolds by means of the existence of a non-compact flow \mathbb{C}^* consisting of a kind of transformations that I now describe. Let $\{\theta^1, \dots, \theta^{n-1}, \bar{\theta}^1, \dots, \bar{\theta}^{n-1}\}$ be complex 1-forms that together with θ and $J\theta$ determine a coframe field adapted to the l.c.K. structure of M .

⁴This deformed Sasakian structure of the sphere has been recently encountered in a different context, related to the weighted projective spaces, cf. [22]

A diffeomorphism f of M is called a *Lie-Cauchy-Riemann transformation* (cf. [39]) if it satisfies:

$$\begin{aligned} f^*\theta &= \theta, \\ f^*(J\theta) &= \lambda J\theta, \\ f^*\theta^\alpha &= \sqrt{\lambda} \cdot \theta^\beta U_\beta^\alpha + J\theta \cdot v^\alpha, \\ f^*\bar{\theta}^\alpha &= \sqrt{\lambda} \cdot \bar{\theta}^\beta \bar{U}_\beta^\alpha + J\theta \cdot \bar{v}^\alpha, \end{aligned}$$

for some smooth functions $\lambda \in \mathbb{R}_+$, $v^\alpha \in \mathbb{C}$, $U_\beta^\alpha \in U(n-1)$. The totality of these transformations form a group which contain the holomorphic isometries and preserve the specific G -structure of l.c.K. manifolds. The announced result is:

Theorem 6.1 (Kamishima-Ornea [39]). *Let (M, g, J) be a compact, connected, l.c.K. non-Kähler manifold with parallel Lee form θ . Suppose that M admits a closed subgroup $\mathbb{C}^* = S^1 \times \mathbb{R}_+$ of Lie-Cauchy-Riemann transformations whose S^1 subgroup induces the Lee field θ^\sharp . Then M is holomorphically isometric, up to scalar multiple of the metric, to a Hopf manifold H_Λ .*

6.2. The weight bundle. Let now $L \rightarrow M$ be the weight bundle canonically associated to a l.c.K. manifold (see [25], [21]). It is a real line bundle associated to the representation $GL(2n, \mathbb{R}) \ni A \mapsto |\det A|^{(1/2n)}$. The Lee form can be interpreted as a connection one-form in L , associated to the Weyl connection determined on M by the l.c.K. metric and the Lee form. L is thus flat and one may speak about its monodromy. Nothing can be said in general about the monodromy of L . However, for Vaisman manifolds, the following result is true and it gives the key for the structure of such compact manifolds:

Theorem 6.2 (Ornea-Verbitsky [53]). *Let M be a compact Vaisman manifold. Then the monodromy of L is isomorphic to \mathbb{Z} .*

For the proof, let Γ be the monodromy (it is a subgroup of \mathbb{R}_+) and let $\widehat{M} \rightarrow M$ be the associated covering. Let $\text{Aut}(\widehat{M}, M)$ be the group of conformal automorphisms of \widehat{M} which are lifts of a l.c.K. automorphism of M . One may compose the scale homomorphism $\rho : \mathcal{H}(\widehat{M}) \rightarrow \mathbb{R}_+$ with the natural forgetful homomorphism $\Phi : \text{Aut}(\widehat{M}, M) \rightarrow \text{Aut}(M)$ and obtain a homomorphism

$$\text{Aut}(\widehat{M}, M) \xrightarrow{\Phi \times \rho} \text{Aut}(M) \times \mathbb{R}_+$$

which proves to be injective. One looks now to the Lie group G generated by the Lee flow, which is isomorphic to a torus $(S^1)^k$ (see above), and to its counter-image in $\text{Aut}(\widehat{M}, M)$, $\widehat{G} = \Phi^{-1}(G)$. Observing that $\Gamma = \ker \Phi \subset \widehat{G}$, one is left to prove that \widehat{G} is isomorphic to $\mathbb{R} \times (S^1)^{k-1}$. This is done in two steps. One first shows that the connected component \widehat{G}_0 of \widehat{G} is non-compact, hence isomorphic to $\mathbb{R} \times (S^1)^{k-1}$. The second step here is to show that \widehat{G} is connected by showing that the quotient $H = \widehat{G}/G_0$ is trivial. To this end, one first shows that H is finite, by proving that each connected component of \widehat{G} meets the compact ρ^{-1} . Second, one notes that from the structure theorem for abelian groups, $\widehat{G} \cong \mathbb{R} \times (S^1)^{k-1} \times H$, hence H is embedded in \widehat{G} . As such, it can be seen that H is in fact embedded in Γ which, as monodromy of a real line bundle, is included in \mathbb{R} , thus cannot have non-trivial compact subgroups.

The breakthrough towards obtaining new significant results in l.c.K. geometry was the use of algebro-geometric techniques by M. Verbitsky. The key tool of his approach is the use of the 2-form

$$\omega_0 := d^c\theta = \frac{\partial - \bar{\partial}}{\sqrt{-1}} \theta = -dJ\theta.$$

Its main property is the following:

Proposition 6.1 (Verbitsky [67]). *Let M be a Vaisman manifold. Then all eigenvalues of ω_0 are positive, except the one corresponding to the Lee field which is zero.*

On the other hand, it can easily be seen that on the orthogonal transverse of the canonical foliation \mathcal{F} , ω_0 is closed, hence plays the role of a Kähler form. This implies that, if the Vaisman manifold is quasi-regular, ω_0 projects on the Kähler form of the base orbifold:

Proposition 6.2 (Verbitsky [67]). *Let M be a compact, quasi-regular Vaisman manifold. Let p denote the canonical projection over the Kähler orbifold (Q, ω_Q) . Then $p^*\omega_Q = \omega_0$.*

The form ω_0 can be interpreted as the curvature of the Chern connection in the complexified $L_{\mathbb{C}}$ of the weight bundle. Indeed, being L flat, its complexified has a canonical holomorphic structure given by the $(0, 1)$ part of the flat covariant derivative. Moreover, an adapted Hermitian metric can also be considered, by decreeing the canonical section of L to be unitary. This way one can look at the Chern connection of $L_{\mathbb{C}}$. A direct computation proves:

Theorem 6.3 (Verbitsky [67]). *The curvature of the Chern connection in $L_{\mathbb{C}}$ equals $-2\sqrt{-1}\omega_0$.*

This is the result that opens the way for the use of the algebraic-geometric tools in l.c.K. geometry.

6.3. Topological properties. On a Vaisman manifold, one can consider differential operators adapted to the form ω_0 instead of to the fundamental form ω . Precisely, let L_0 be the wedging operator with ω_0 : $L_0(\eta) = \eta \wedge \omega_0$, let Λ_0 be its Hermitian formal adjoint and let $H_0 = [L_0, \Lambda_0]$ be their commutator. If ξ_1, \dots, ξ_{n-1} , $\sqrt{2}\theta^{1,0}$ is an orthonormal frame of $T^{1,0}M$, then (cf. [67]):

$$(6.1) \quad \begin{aligned} \omega_0 &= \sqrt{-1} (\xi_1 \wedge \bar{\xi}_1 + \dots + \xi_{n-1} \wedge \bar{\xi}_{n-1}), \\ H_0(\eta) &= (p - n + 1)\eta, \end{aligned}$$

for any $\eta = \xi_{i_1} \wedge \dots \wedge \xi_{i_k} \wedge \bar{\xi}_{i_{k+1}} \wedge \dots \wedge \bar{\xi}_p \wedge R$, where R is a monomial in $\theta^{1,0}$ and $\theta^{0,1}$ only and p is the number of ξ -s in η . Verbitsky then shows that, as in the case of Kähler geometry, the triple L_0, Λ_0, H_0 is a $SL(2)$ -triple. He then associates to H_0 the weight decomposition $\Lambda^*(M) = \oplus \Lambda_i^*(M)$ such that the monomial η above has weight p . This way one has, at the level p , the decomposition: $\Lambda^p(M) = \Lambda_{p-2}^p(M) \oplus \Lambda_{p-1}^p(M) \oplus \Lambda_p^p(M)$. Let $d_0 : \Lambda_i^p(M) \rightarrow \Lambda_{i+1}^{p+1}(M)$ be the weight 1 component of the usual de Rham differential and denote with ∂_0 and $\bar{\partial}_0$ its $(1, 0)$ and $(0, 1)$ part respectively. One immediately checks that $d_0\omega_0 = 0$, $d_0\theta = 0$, $d_0J\theta = 0$. Using algebraic techniques introduced by Grothendieck (algebraic differential operators), Verbitsky shows that

$$d_0^2 = \partial_0^2 = \bar{\partial}_0^2 = 0$$

and he computes the Kodaira identities for the new introduced operators:

Theorem 6.4 (Verbitsky [67]). *On a Vaisman manifolds, the above operators satisfy the following commutation relations:*

$$(6.2) \quad \begin{aligned} [\Lambda_0, \partial_0] &= \sqrt{-1} \bar{\partial}_0^* & , & \quad [L_0, \bar{\partial}_0] = -\sqrt{-1} \partial_0^* \\ [\Lambda_0, \bar{\partial}_0] &= -\sqrt{-1} \partial_0 & , & \quad [L_0, \bar{\partial}_0^*] = \sqrt{-1} \bar{\partial}_0. \end{aligned}$$

Here $*$ denotes the Hermitian adjoint.

The same commutation formulas can then be obtained for the similar operators acting on sections of a holomorphic bundle $V \rightarrow M$ endowed with a Chern connection with respect to which d_0 is now the weight 1 component. Preserving the same notations as above, but understanding now the operators with coefficients in V , one proves again the formulas (6.2). This implies the following vanishing result:

Theorem 6.5 (Verbitsky [67]). *Let M be a compact Vaisman manifold and let V be any positive tensor power of its weight bundle L equipped with its Hermitian structure. Let $\bar{\partial}_0 : \Lambda^{0,p} \otimes V \rightarrow \Lambda^{0,p+1} \otimes V$ be the ω_0 -differential associated to the Chern connection in V as above. Then all $\bar{\partial}_0$ -harmonic forms are trivial in dimension $p \leq \dim_{\mathbb{C}} M - 2$. Moreover, $\eta(\theta^\sharp) = 0$ for each $(0, p)$ $\bar{\partial}_0$ -harmonic form η with $p = \dim_{\mathbb{C}} M - 1$.*

For the proof, one first shows, as in the case of the operators on Kähler manifolds, using the commutation formulas, the following connecting formula between the Laplacians of ∂_0 and $\bar{\partial}_0$:

$$\Delta_{\partial_0} - \Delta_{\bar{\partial}_0} = \sqrt{-1} [\Theta_V, \Lambda_0],$$

where Θ_V is the curvature operator of the Chern connection in V . As by Theorem 6.3 $\sqrt{-1} \Theta_V = c\omega_0$, $c > 0$, one has

$$\Delta_{\bar{\partial}_0} = \Delta_{\partial_0} + cH_0.$$

But from the second formula in (6.1), H_0 is positive definite on $(r, 0)$ -forms for $r < n - 1$. Hence $\Delta_{\bar{\partial}_0}$ is the sum between a positive definite and a positive semi-definite operator and thus all his eigenvalues are strictly positive. This implies that no harmonic $(r, 0)$ -forms exist for $r < n - 1$. The last statement follows similarly.

On the other hand, if one refers to the notion of basic cohomology with respect to the canonical foliation \mathcal{F} , it is clear that on basic forms, the operators $\bar{\partial}$ and $\bar{\partial}_0$ coincide. This way, a Tsukada's result about basic cohomology can be expressed as follows:

Theorem 6.6 (Tsukada [62]). *Let η be a (p, q) -form, $p + q \leq n - 1$, on a compact n -dimensional Vaisman manifold. The following conditions are equivalent:*

- (i) η is $\bar{\partial}$ -harmonic.
- (ii) $\eta = \theta^{0,1} \wedge \alpha + \beta$, where α, β are $\bar{\partial}_0$ -harmonic basic forms and satisfy $\Lambda_0 \alpha = \Lambda_0 \beta = 0$.

For a first application of these results, suppose that the canonical bundle K of the Vaisman manifold is a negative power of L . This happens, for example, if M is Einstein-Weyl⁵ (cf. [67]). Now look at basic $\bar{\partial}$ -harmonic forms. By the above, these are precisely the $\bar{\partial}_0$ -harmonic forms. By Serre duality $\mathcal{H}_{\bar{\partial}_0}^p(\mathcal{O}_M) \cong \overline{\mathcal{H}_{\bar{\partial}_0}^{n-p}(K)}$. But from Theorem 6.5, $\overline{\mathcal{H}_{\bar{\partial}_0}^{n-p}(K)} = 0$ for $p > 1$ and for $p = 1$ the non-trivial $\eta \in \mathcal{H}_{\bar{\partial}_0}^{n-p}(K)$ satisfy $\eta(\eta^{0,1\sharp}) = 0$. Again by Serre duality one derives that the $\bar{\partial}_0$ -harmonic $(0, p)$ -forms in $\mathcal{H}_{\bar{\partial}_0}^p(\mathcal{O}_M)$ vanish for $p > 1$ and are proportional to $\theta^{0,1}$ if $p = 1$. Hence in this latter case the form cannot be basic, thus there are no non-trivial basic $\bar{\partial}$ -harmonic $(0, p)$ -forms. Summing up, one obtains the following information about the cohomology of the structural sheaf:

Theorem 6.7 (Verbitsky [67]). *Let M be a compact Vaisman manifold whose canonical bundle is a negative power of the weight bundle. Then $\mathcal{H}^i(\mathcal{O}_M) = 0$ for $i > 1$ and $\mathcal{H}^1(\mathcal{O}_M) = 1$.*

In fact, this result was independently proved by B. Alexandrov and S. Ivanov in a more general setting (here I quote it adapted to the present situation):

Theorem 6.8 (Alexandrov-Ivanov [2]). *Let M be a compact Vaisman manifold. Suppose that the Ricci tensor of its Weyl connection is non-negative. Then $b_1(M) = 1$ and the Hodge numbers satisfy:*

$$\begin{aligned} h^{p,q}(M) &= 0 \quad , \quad p = 1, 2, \dots, n \\ h^{0,q}(M) &= 0 \quad , \quad q = 2, 3, \dots, n \\ h^{0,1}(M) &= 1 \quad . \end{aligned}$$

⁵It is in fact redundant to say Einstein-Weyl Vaisman manifold, as any Hermitian-Einstein-Weyl manifold is Vaisman, cf. [57].

The proof of Alexandrov and Ivanov is different in spirit from Verbitsky's: it relies on a Bochner type formula in Weyl geometry. If D denotes the Weyl covariant derivative on M and Ric^D the symmetric part of its curvature tensor (but note that on Vaisman manifolds the Ricci tensor of the Weyl connection is symmetric), they prove the following formula:

$$(6.3) \quad Ric^D(X, X) \geq \frac{(n-2)(n-4)}{8} \{|X|^2 - \theta(X)^2\}.$$

It immediately follows that $b_1(M) \leq 1$ and, as on compact Vaisman manifolds the first Betti number is odd, it must be 1. On the other hand, the Chern covariant derivative ∇^C on M , is related to the Weyl covariant derivative by

$$\nabla^C = D + \frac{1}{2} \theta \otimes Id + \frac{1}{2} J\theta \otimes J.$$

This gives the relation between the respective curvature tensors:

$$R^C = R^D + \frac{1}{2} dJ\theta.$$

Let

$$R^C(\omega) = \frac{1}{2} \sum g(R^C(e_i, Je_i)X, Y),$$

$\{e_i\}$ being an orthonormal frame. Let $k^C(X, Y) = R^C(\omega)(JX, Y)$ be the associated symmetric, J -invariant bilinear form. One has

$$k^C = Ric^D + \frac{1}{2} \langle dJ\theta, \omega \rangle g.$$

But it is not hard to prove that

$$dJ\theta = \omega + \theta \wedge J\theta,$$

so that

$$k^C = Ric^D + (n-1)g.$$

As $Ric^D \geq 0$, $k^C > 0$ follows. Now one applies the classical vanishing theorems for holomorphic forms and derive $h^{p,0} = 0$ for $p = 1, 2, \dots, n$. For the rest, one uses the following relations proved in [62]:

$$h^{m,0} = h^{m-1,0}, \quad h^{0,p} = h^{p,0} + h^{p-1,0}, \quad p \leq n-1, \quad 2h^{1,0} - b_1 - 1, \quad 2h^{0,1} = b_1 + 1$$

and Serre duality.

Another vanishing result I want to mention is the following:

Theorem 6.9 (Verbitsky [67]). *Let M be a compact Hermitian-Einstein-Weyl manifold. Then all holomorphic p -forms on M vanish, for $p > 0$ and $h^1(M) = 1$.*

Note that in general h^1 can be arbitrarily big (still odd) on a Vaisman manifold: it is enough to think at the total space of the induced Hopf bundle over a projective curve of big genus.

The second statement of the theorem follows from the first one by looking at the Dolbeault spectral sequence $E_r^{p,q}$. Since $E_2^{p,q} = H^q(\Omega^p(M))$, one gets $H^0(\Omega^1(M)) = 0$ and, from Theorem 6.7, $H^1(\Omega_0(M)) = \mathbb{C}$. Thus

$$h^1(M) \leq \dim H^0(\Omega^1(M)) + \dim H^1(\Omega_0(M)) = 1.$$

Now, if $h^1 = 0$, the weight bundle would have trivial monodromy, contradiction with it being isomorphic to \mathbb{Z} .

As for the first statement of the theorem, it follows by first showing that TM , as a holomorphic Hermitian bundle, satisfies $\Lambda_0(\Theta_{TM}) = -c\sqrt{-1} Id_{TM}$, $c > 0$ (in other words, it is ω_0 -Yang-Mills with negative constant), then showing that such a bundle cannot have global holomorphic sections.

On a l.c.K. manifold one may also consider the twisted differential operator d^θ acting as

$$d^\theta \eta = d\eta - \theta \wedge \eta .$$

It is immediate that $d^\theta \circ d^\theta = 0$, hence d^θ produces a cohomology which groups are denoted with $H_\theta^*(M, \mathbb{R})$. It was first introduced and studied in [31] for locally conformally symplectic manifolds. For compact manifolds, the top d^θ cohomology was known (*loc. cit.*) to vanish. However, it was recently proved that on compact Vaisman manifolds (in fact, the original proof refers, more generally, to Riemannian manifolds endowed with a parallel one-form) all these cohomology groups vanish (see also Remark 6.4 for a proof which works only on Vaisman manifolds):

Proposition 6.3 (León-López-Marrero-Adrón [46]). *On a compact, non-Kähler Vaisman manifold all the cohomology groups $H_\theta^*(M, \mathbb{R})$ are trivial.*

The proof is technical and is based on the following relations (here one takes into account that $B = \theta^\sharp$ is Killing etc.):

$$\mathcal{L}_B = -d^* \circ e(\theta) - e(\theta) \circ d^* , \quad d^* \circ \mathcal{L}_B = \mathcal{L}_B \circ d^* ,$$

where $e(\theta)$ denotes the exterior product operator with θ and d^* the Riemannian adjoint of d . If $i(\theta)$ is the interior product with θ and $d^{\theta*}$ the formal adjoint of d^θ , then one may derive the formulas:

$$d^\theta \circ i(\theta) = -i(\theta) \circ d^\theta + \mathcal{L}_B + Id \quad , \quad d^{\theta*} \circ i(\theta) = -i(\theta) \circ d^{\theta*} ,$$

$$d^\theta \circ \mathcal{L}_B = \mathcal{L}_B \circ d^\theta \quad , \quad d^{\theta*} \circ \mathcal{L}_B = \mathcal{L}_B \circ d^{\theta*} .$$

With this, one may show that $\langle \mathcal{L}_B \eta, \eta \rangle = 0$ for any k -form η . Hence, if $d^\theta \eta = 0$, then $\mathcal{L}_B \eta = -\eta$ and $\eta = 0$ follows.

Remark 6.1. Recently, A. Banyaga computed the d^θ cohomology of a 4-dimensional compact locally conformally symplectic manifold which is diffeomorphic with the Inoue surface of the first kind and showed that this d^θ -cohomology is non-trivial, cf. [7]. In view of the previous result, the reason might be that, as Belgun proved (see above), as a complex surface, this manifold does not admit any metric with parallel Lee form.

6.4. The structure of compact Vaisman manifolds. From the proof of Theorem 5.2 (see also [28]) it follows that the Riemannian cone of a Sasakian manifold admits a globally conformally Kähler metric with parallel Lee form. Conversely, the universal cover of any Vaisman manifold admits such a structure, the Sasakian manifold being the universal cover of leaves of the foliation $\text{Ker } \theta$.

So, to obtain Vaisman manifolds one has the following receipt: pick a Sasakian manifold W , construct the Riemannian cone, then identify a group of holomorphic homotheties of the Kähler metric on the cone. One natural way of finding such a group is to select a Sasakian automorphism φ of W and to consider the group generated on the cone by the transformation $(w, t) \mapsto (\varphi(w), qt)$, $q \in \mathbb{R}$, $q > 1$. This is a homothety with scale factor q^2 . The quotient $M_{\varphi, q}$ is the Riemannian suspension of φ over the circle of length $2\pi q$ and is a Vaisman manifold (cf. [26], [39], [28], [53]).

This construction can be reversed, thus providing the general structure of compact Vaisman manifolds:

Theorem 6.10 (Ornea-Verbitsky [53]). *A compact Vaisman manifold M admits a canonical Riemannian submersion $p : M \rightarrow S^1$ with fibres isometric to a Sasakian manifold W . Moreover, there exists a Sasakian automorphism φ of W such that M is isomorphic with the Vaisman manifold $M_{\varphi, q}$ above.*

The proof of the theorem is a straightforward application of Theorem 6.2. Indeed, let $\{\gamma_1, \dots, \gamma_k\}$ be a basis of $H^1(M)$, let $\alpha_i = \int_{\gamma_i} \theta$ be the periods of θ and let $A =$

$\langle \alpha_1, \dots, \alpha_k \rangle$ be the abelian group generated by the periods. One clearly has a function $p : M \rightarrow \mathbb{R}/A$. The monodromy of L being \mathbb{Z} , all periods are proportional to a real number α with integer coefficient. Hence $A \cong \alpha \cdot \mathbb{Z}$ and one has the desired map p . Passing to the universal cover, one finds a map $f : \tilde{M} \rightarrow S^1$, $df = \theta$. On the other hand, \tilde{M} is a cone over a Sasakian W . As the monodromy map of \tilde{M} preserves both the complex structure and the Lee field of \tilde{M} and acts as a homothety, it preserves W and induces its desired Sasakian automorphism φ .

Remark 6.2. The same result holds, replacing Sasakian by 3-Sasakian, for locally conformally hyperkähler manifolds. In fact, compact l.c.h.K. manifolds are, in particular, l.c.K. Einstein-Weyl manifolds. As by Theorems 6.8 and 6.7 such manifolds have $b_1 = 1$, the above construction applies. This was the idea of the first proof in [67] for the structure theorem.

One can use Theorem 6.10 to determine the fundamental group of a compact Vaisman manifold:

Proposition 6.4. *The fundamental group of a compact Vaisman manifold is a product $K \times \mathbb{Z}$ where K is the fundamental group of a compact Kähler manifold.*

Indeed, topologically a compact Vaisman manifold is a product of a compact Sasakian manifold with a circle. Again topologically, the Sasakian manifold is a circle bundle over a compact Kähler manifold. Hence the result.

Remark 6.3. Note that, in general, nothing can be said about the fundamental group of a l.c.K. manifold, except the fact that it has to send a non-trivial arrow into \mathbb{R}_+ .

Remark 6.4 ([69]). Another application of Theorem 6.10 is a proof of the vanishing of the d^θ -cohomology on compact Vaisman manifolds, cf. Proposition 6.3. Indeed, this is the cohomology of a locally constant sheaf on M corresponding to the weight bundle L considered with its flat connection (a local system). Note that, as topologically M is $W \times S^1$ and the monodromy of L is \mathbb{Z} , L is the pull-back p^*L' of a local system L' on S^1 . Now, the cohomology of the local system L is the derived direct image $R^iP_*(L)$, where P is a projection onto a point. By the above remark and changing the base, $R_iP_*(L) = R^iP_*p^*(L') = R^i(\mathbb{C} \otimes L')$, where \mathbb{C} is viewed as a trivial local system. From the Künneth formula it follows that $R^i(\mathbb{C})$ is a locally constant sheaf on S^1 , with fiber $H^*(W)$. By the Leray spectral sequence of composition, the hypercohomology of the complex of sheaves $R^*(\mathbb{C} \otimes L')$ converges to $R^iP_*(L)$. Finally, each $R^i(\mathbb{C} \otimes L')$ has zero cohomology, hence the spectral sequence vanishes in E_2 . It then converges to zero.

Remark 6.5. Theorem 6.10 says that, at least in the compact case, Vaisman and Sasakian structures are essentially the same. One of the definitions of a Sasakian structure is the following: a Riemannian manifold together with a unit Killing ξ field satisfying the following second order equation: $R(X, Y)\xi = g(\xi, Y)X - g(X, \xi)Y$. A similar characterization is available also for Vaisman manifolds:

Proposition 6.5 (Kashiwada [40]). *Let (M, g) be a Riemannian manifold with a unit Killing 1-form θ' and a unit parallel 1-form θ orthogonal to θ' . Then (M, g) admits a compatible Vaisman structure if and only if:*

$$R(X, Y)\theta'^{\sharp} = \theta'(Y)X - \theta'(X)Y - (\theta \wedge \theta')(X, Y)\theta^{\sharp}.$$

The Sasakian manifold exhibited in the structure theorem is “the smallest” possible in the following sense:

Proposition 6.6 (Gini-Ornea-Parton-Piccinni [29]). *Let M be a compact Vaisman manifold and W the Sasakian manifold provided by the structure Theorem 6.10. Let W' be any Sasakian manifold whose Riemannian cone covers M . Then W' covers W . In other words, the minimal presentation of a compact Vaisman manifold is (W, \mathbb{Z}) . In particular, the rank of a compact Vaisman manifold is 1.*

The argument is as follows: the homotopy sequence associated to the fibration $W \rightarrow M \rightarrow S^1$ gives the exact sequence

$$0 \rightarrow \pi_1(W) \rightarrow \pi_1(M) \rightarrow \mathbb{Z} \rightarrow 0 .$$

On the other hand, the index of $\pi_1(W')$ cannot be finite in $\pi_1(M)$, as M is compact and the cone $W \times \mathbb{R}$ non-compact. Hence $\text{coker} \pi_1(W')$, which contains \mathbb{Z} , is bigger than $\text{coker} \pi_1(W)$. Then, up to conjugation, the image of $\pi_1(W)$ in $\pi_1(M)$ is bigger than the image of $\pi_1(W')$. From here one may conclude that W' covers W .

6.5. Immersing Vaisman manifolds into Hopf manifolds. The properties of the complexified weight bundle suggest a Kodaira-Nakano type result for compact Vaisman manifolds. The model space will now be the Hopf manifold as described in Section 6.1. The precise statement is:

Theorem 6.11 (Ornea-Verbitsky [54]). *Let M be a compact Vaisman manifold. Then M admits a holomorphic immersion into a diagonal Hopf manifolds H_Λ , preserving the respective Lee fields and inducing a finite covering on the image.*

Although a better result - an embedding theorem - is also available, with a completely different approach, in the framework of l.c.K. manifolds with potential (cf. §7), I think the proof below is interesting in itself, showing how algebraic geometry methods can be used in this non-algebraic context. I thus present the outline of the proof which is divided into two distinct parts: one first assumes M quasi-regular, then shows that any compact Vaisman manifold can be deformed to a quasi-regular one. Note that the deformation must be explicitly exhibited, since the l.c.K. class is not closed under small deformations, cf. [10] (see also the discussion in §7 about small deformations of l.c.K. manifolds).

If M is quasi-regular, the push-forward of L_C over the base orbifold Q is a positive line-bundle (the essential technical step here is to show that L_C is trivial along the fibres of p). Hence, by the orbifold version (cf. [6]) of the Kodaira-Nakano theorem, Q is projective. One thus has an embedding $l : Q \hookrightarrow \mathbb{C}P^{N-1}$, $N = \dim H^0(L_C^k)$ for an appropriate $k > 0$. A basis of this space is easily seen to produce functions $\lambda_i : \widehat{M} \rightarrow \mathbb{C}$ without common zeros on the total space of the weight covering associated to monodromy group of L (which, remember, is \mathbb{Z}). Hence there exists the map $\widehat{\lambda} : \widehat{M} \rightarrow \mathbb{C}^N \setminus 0$. This map proves to be compatible with the action of a generator γ of the monodromy representation and with the multiplication with q^k on \mathbb{C}^N :

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{\lambda}} & \mathbb{C}^N \setminus 0 \\ \gamma \downarrow & & \downarrow \text{mult. by } q^k \\ \widehat{M} & \xrightarrow{\widehat{\lambda}} & \mathbb{C}^N \setminus 0 \end{array}$$

Hence $\widehat{\lambda}$ is the covering of a map $\lambda : M \rightarrow (\mathbb{C}^N \setminus 0)/\langle q^k \rangle$ into a diagonal Hopf manifold. The remainder of the conclusion follows by observing that λ can be included in the following commutative square:

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & (\mathbb{C}^N \setminus 0)/\langle q^k \rangle \\ p \downarrow & & \downarrow \\ Q & \xrightarrow{l} & \mathbb{C}P^{N-1} . \end{array}$$

In order to handle the general case, one needs a deformation argument. The idea is to look at the complexified G_C of the group generated by the Lee flow and at its counter-image \widehat{G}_C by Φ acting on the monodromy covering (cf. Subsection 6.2). The key step consists in proving:

Proposition 6.7. $\widehat{G}_C \cong (\mathbb{C}^*)^l$ for some $l > 0$.

It will be enough to show that (1) $\widehat{G}_{\mathbb{C}}$ is linear and (2) it has a compact real form. By the Remmert-Morimoto theorem, $\widehat{G}_{\mathbb{C}}$ is a product $(\mathbb{C}^*)^l \times \mathbb{C}^m \times T$ where T is a torus. The torus component is easily seen to not appear because the Kähler metric on \widehat{M} has a potential. To prove (2), one proceeds as follows: Let \widehat{G}_K be the Lie subgroup of $\widehat{G}_{\mathbb{C}}$ generated in $\widehat{G}_{\mathbb{C}}$ by the group \widehat{G}_0 of Kähler isometries of \widehat{G} and the flow of $J\theta^{\sharp}$ (which acts by holomorphic Kähler isometries). Clearly $\Phi(\widehat{G}_K)$ consists of isometries of M . Moreover, one proves that $\Phi : \widehat{G}_K \rightarrow \text{Iso}(M)$ is a monomorphism, hence \widehat{G}_K is compact as a subgroup of the isometry group. As $\widehat{G} \cong (S^1)^k \times \mathbb{R}$ is generated by \widehat{G}_0 and $e^{t\theta^{\sharp}}$ and \widehat{G}_K is generated by \widehat{G}_0 and $e^{tJ\theta^{\sharp}}$. Hence the complexifications of \widehat{G}_K and \widehat{G} coincide. All in all one sees that \widehat{G}_K is a real form of $\widehat{G}_{\mathbb{C}}$.

The next step is to deform the given Vaisman structure still remaining in the class of Vaisman structures. To do this, one picks an element γ' sufficiently close in $\widehat{G}_{\mathbb{C}}$ to the generator γ of the monodromy representation $\mathbb{Z} \cong \Gamma \subset \widehat{G}$. Let Γ' be the abelian subgroup generated in $\widehat{G}_{\mathbb{C}}$ by γ' . Recall (cf. [67, Pr. 4.4]) that the Kähler form of \widehat{M} is exact and has a potential $|V|^2$, where V is the lift to \widehat{M} of the Lee field. Then the squared length of $V' = \log \gamma'$ will still be a Kähler potential with corresponding Kähler form ω' . It can be seen that the flow of V' acts by holomorphic conformalities with respect to ω' , hence the quotient \widehat{M}/Γ' is l.c.K. with, by construction, parallel Lee form. This proves:

Proposition 6.8. *The quotient \widehat{M}/Γ' is a compact Vaisman manifold with Lee field proportional to⁶ $V' = \log \gamma' \in \text{Lie}(\widehat{G}_{\mathbb{C}})$.*

What one needs is in fact a quasi-regular deformation. This can be achieved by further restricting the choice of γ' . As $\widehat{G}_{\mathbb{C}}$ has a compact real form, its Lie algebra can be given a rational structure by saying that $\delta \in \text{Lie}(\widehat{G}_K)$ is rational if $e^{t\delta}$ is compact in \widehat{G}_K . Now, a complex subgroup $e^{z\delta}$ of $\widehat{G}_{\mathbb{C}}$ is isomorphic with \mathbb{C}^* if and only if the line $z\delta$ is rational (it contains a rational point). One may prove:

Proposition 6.9. *If the line $\mathbb{C} \log \gamma'$ is rational, then \widehat{M}/Γ' is a quasi-regular Vaisman manifold.*

As the elements γ' satisfying the above assumption are dense in $\widehat{G}_{\mathbb{C}}$, one can find a quasi-regular Vaisman deformation $M' = \widehat{M}/\langle \gamma' \rangle$ of the initial Vaisman structure. Applying the first step, we obtain a map $\hat{\lambda}' : \widehat{M} \rightarrow \mathbb{C}^N \setminus 0$ with the same properties as above. But γ and γ' commute. This gives rise to the following commutative diagram:

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\gamma} & \widehat{M} \\ \hat{\lambda}' \downarrow & & \downarrow \hat{\lambda}' \\ H^0(M', L_{\mathbb{C}}^k) & \xrightarrow{\hat{\lambda}'(\gamma)} & H^0(M', L_{\mathbb{C}}^k) . \end{array}$$

One now chooses γ' sufficiently close to γ in order that the eigenvalues of $\hat{\lambda}'(\gamma)$ be all > 1 . Thus $H := (H^0(M', L_{\mathbb{C}}^k) \setminus 0) / \hat{\lambda}'(\gamma)$ is a (smooth) complex manifold and one has the immersion $M = \widehat{M}/\Gamma \hookrightarrow H$.

The final step is to show that H is indeed a Hopf manifold. This amounts to say that the operator $\hat{\lambda}'(\gamma)$ is semi-simple. But this follows easily from the fact that the vector space $\{f : \widehat{M} \rightarrow \mathbb{C} \mid f \circ \gamma' = q^k f\}$ is finite dimensional. This ends the proof of the immersion theorem.

For the time being, I can only mention a straightforward application of the immersion theorem to Sasakian geometry, namely:

⁶Note that by a result in [62], the Lee field is unique on a compact manifold of Vaisman type.

Theorem 6.12 (Ornea-Verbitsky [54]). *Any compact Sasakian manifold admits a CR immersion, finite covering on the image, into a weighted sphere.*

6.6. Stable bundles on Vaisman manifolds. I quote here a recent result obtained in [68] on stability on Vaisman and, in particular, diagonal Hopf manifolds.

Theorem 6.13 (Verbitsky [68]). *Let M be a compact Vaisman manifold of complex dimension at least 3 and V a stable bundle on M . Then the curvature Θ of the Chern connection satisfies $\Theta(v, *) = 0$ for every v in the canonical foliation \mathcal{F} . In particular, V is equivariant with respect to the complex Lie group generated by the Lie flow and this equivariant structure is compatible with the connection.*

Here stability is understood with respect to the Gauduchon metric.

The proof is divided into two parts. The first one, computational, applies to bundles of degree 0. It follows, in fact, from a more general statement, concerning a Hermitian bundle endowed with a connection and with a primitive closed skew-Hermitian $(1, 1)$ -form $\Theta \in \Lambda^{(1,1)}(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathfrak{u}(V)$ (here primitive means that $\Lambda\Theta = 0$, Λ being the Hermitian adjoint of the operator of tensoring with the Hermitian form).

To pass to bundles of arbitrary degree, observe first that a compact Vaisman manifold admits holomorphic Hermitian bundles of arbitrary degree. Indeed, the complexified weight bundle $L_{\mathbb{C}}$ on M has degree $\delta := \int \omega_0 \wedge \omega^{n-1} > 0$. Now, for any $\lambda \in \mathbb{R}$, one associates a holomorphic Hermitian bundle with connection $\nabla_{triv} - i(\lambda/\delta)\theta^c$. L_{λ} has degree λ . Moreover, L_{λ} is equivariant with respect to the Lee flow.

Now tensoring a stable bundle of arbitrary degree with L_{λ} for appropriate λ produces a stable bundle of degree 0.

With a proof which is too involved to be reported here, this result leads to the following

Theorem 6.14 (Verbitsky [68]). *Any stable holomorphic bundle on a diagonal Hopf manifold is filtrable.*

6.7. Riemannian properties. The Lee field of the Tricerri metric is harmonic and minimal. For Vaisman manifolds, more can be said: the anti-Lee field also defines a harmonic map in the unit tangent bundle endowed with the Sasaki metric g^S . The precise result is:

Proposition 6.10 (Ornea-Vanhecke [52]). *The following properties hold on a Vaisman manifold (M, g) :*

- i) $J\theta^{\sharp}$ is a harmonic and minimal vector field. Moreover, it is a harmonic map from (M, g) into the unit tangent bundle (T_1M, g^S) .
- ii) The bivector $\theta^{\sharp} \wedge J\theta^{\sharp}$, viewed as a map $(M, g) \rightarrow (G_2^{\text{or}}(M), g^S)$ is harmonic and its image is a minimal submanifold.

I also mention that, for a compact M , computing the Hessian of the volume and energy for $J\theta^{\sharp}$, one may also prove (cf. [52]) that this vector field is not stable neither as a harmonic map, nor as a minimal submanifold.

On the other hand, another notion of harmonicity for a foliation \mathcal{F} on a Riemannian manifold (N, g) was introduced and studied in [35] by the equivalence of the following conditions:

- (1) The canonical projection π from TN on the normal bundle of the foliation is harmonic;
- (2) The leaves of \mathcal{F} are minimal submanifolds in (N, g) .

If N is compact, oriented, \mathcal{F} is Riemannian and g is bundle-like with respect to it, the above conditions are also equivalent with $\Delta\pi = 0$.

Using the second characterization and a Bochner-Weitzenböck type formula, the following result was obtained:

Theorem 6.15 (Ichikawa-Noda [34]). *Let \mathcal{F} be a harmonic Riemannian foliation with complex leaves on a compact, oriented l.c.K. manifold. Then \mathcal{F} is stable.*

In particular, the canonical foliation of a Vaisman manifold is stable.

Note that this notion of harmonicity implies the one in [27]. Indeed, Ichikawa and Noda observe that if \mathcal{F} is harmonic in the sense of Kamber-Tondeur, then its Gauss map is harmonic (cf. Ruh-Vilms' theorem) and hence the foliation is harmonic also in Vanhecke's sense. On the other hand, as observed above, the canonical foliation of a Vaisman manifold is not stable as a harmonic map, hence the two notions of stability are not the same.

Another way of looking at the Riemannian properties of a Vaisman manifold is to consider its transversal Kählerian structure guaranteed by Proposition 2.3. This is the viewpoint adopted in [9] where transversally holomorphic maps are studied and the following result is obtained as a corollary of more general properties:

Proposition 6.11 (Barletta-Dragomir [9]). *Let $\varphi : M \rightarrow M'$ be a holomorphic or anti-holomorphic map between compact Vaisman manifolds. Then φ is transversally holomorphic and is an absolute minimum for the energy functional in its foliated homotopy class.*

Of course, here also the energy is considered with respect to the foliated exterior derivative operator etc.

In the same paper [9], the beginning of a theory of transversal l.c.K. structures is sketched. These appear naturally on manifolds which admit submersions over l.c.K. manifolds. I think that the subject still waits to be developed.

6.8. L.c.K. structures with parallel anti-Lee form. Another natural class of l.c.K. manifolds, curiously neglected until recently, is the one defined by the parallelism of the anti-Lee vector field. Although its geometry is completely different from the Vaisman one, I include it in this section.

This class was studied in [41]. I summarize some of the results therein.

The condition $\nabla(J\theta) = 0$ immediately implies $\delta\theta = -(n-1)$, hence, by Green's theorem, the underlying manifold cannot be compact. But the local geometry of such l.c.K. structures is rather rich:

Proposition 6.12 (Kashiwada [41]). *On a l.c.K. manifold with parallel anti-Lee form, the distributions \mathcal{D}_1 and \mathcal{D}_2 locally generated by $J\theta = 0$, respectively $\theta = 0$ and $J\theta = 0$ are integrable. Their leaves are, respectively: non-compact, totally geodesic real hypersurfaces with an induced Kenmotsu structure⁷ and totally umbilical complex hypersurfaces on which the induced structure is Kähler.*

Such l.c.K. structures with parallel $J\theta$ appear naturally on the product of a Kenmotsu manifold with the line. But the converse of this result is still not proved, nor is the result extended to non-trivial bundles.

7. LOCALLY CONFORMALLY KÄHLER MANIFOLDS WITH POTENTIAL

A wider, proper subclass of l.c.K. manifolds, containing strictly the Vaisman one was recently defined in [55]. The main reason for introducing it was the search for a class stable to small deformations. Roughly speaking, we deal with l.c.K. manifolds that admit a Kähler cover having a *global* Kähler potential (and hence non-compact) subject to some further restrictions.

⁷A Kenmotsu structure is a metric almost contact structure satisfying the integrability condition $(\nabla_X\varphi)Y = -\eta(Y)\varphi X - g(X, \varphi Y)\xi$. A typical example is the product of a Kähler manifold L with the line \mathbb{R} , with $\xi = d/dt$, $\eta = dt$, $g = dt^2 + e^{2t}g_L$, $\varphi = \begin{pmatrix} 0 & 0 \\ 0 & J_L \end{pmatrix}$, cf. [42].

7.1. Definition and examples.

Definition 7.1 ([55]). An *l.c.K. manifold with potential* is a manifold which admits a Kähler covering (M, Ω) , and a positive smooth function $\varphi : \tilde{M} \rightarrow \mathbb{R}_+$ (the *l.c.K. potential*) satisfying the following conditions:

- (1) It is proper, *i.e.* its level sets are compact.
- (2) The monodromy map τ acts on it by multiplication with a constant $\tau(\varphi) = \text{const} \cdot \varphi$.
- (3) It is a Kähler potential, *i.e.* $\sqrt{-1} \partial \bar{\partial} \varphi = \Omega$.

Here the monodromy group Γ is simply the deck transformation group of \tilde{M} . If we let $\chi : \Gamma \rightarrow \mathbb{R}_+$ be the homomorphism mapping $\gamma \in \Gamma$ to the number $\gamma(\varphi)/\varphi$, then it can be shown that, eventually choosing another cover with same properties, χ may be supposed injective.

Note that the property in the definition is inherited by any closed complex submanifold. A criterion for the first condition of the definition is contained in the following

Proposition 7.1 (Ornea-Verbitsky [55]). *If the above defined character χ is injective, then the potential φ is proper if and only if the image of χ is discrete in \mathbb{R}_+ .*

Example 7.1. By [67, Pr. 4.4] (cf. also §6.5 above), all compact Vaisman manifolds do have a potential. Indeed, on \tilde{M} , we have $\theta = d\varphi$, where $\varphi = \Omega(\theta^\sharp, J(\theta^\sharp))$. Using the parallelism, one shows that $\Omega = e^{-\varphi} \tilde{\omega}$ is a Kähler form with potential φ .

Example 7.2. The two non-compact l.c.K. manifolds constructed recently in [58] are also examples for the above definition. Here is the outline of the construction which is inspired by the one of the Inoue surfaces.

Let $A = (a_{ij}) \in \text{SL}(n, \mathbb{Z})$ a matrix with n eigenvalues λ_i subject to the condition $\lambda_1 > \lambda_2 \cdots > \lambda_{n-1} > 1 > \lambda_n$ and let $D = \log A$ (real logarithm). Let $v_j = (v_{ij})^t$, $j = 1, \dots, n$, real eigenvectors associated to λ_j , let $P = (v_{ij})$ and let $Q = P^{-1}$. Define a domain V in \mathbb{C}^n by

$$V = \left\{ \sum \alpha_i v_i \mid \alpha_i \in \mathbb{C}, \text{Im } \alpha_i > 0, i = 1, \dots, n-1 \right\} \cong H^{n-1} \times \mathbb{C}.$$

Now, for any field $K = \mathbb{Z}, \mathbb{R}, \mathbb{C}$, Renaud defines the group G_K equal as a set with $K \times K^n$ and endowed with the following multiplication:

$$(z, b) \cdot (z', b') = (z + z', e^{z'D} b + b').$$

Equivalently, in the eigenbasis $\{v_j\}$, this can be written as

$$(z, b_1, \dots, b_n) \cdot (z', b'_1, \dots, b'_n) = (z + z', e^{z'\rho_1} b_1 + b'_1, \dots, e^{z'\rho_n} b_n + b'_n),$$

where $\rho_i = \ln \lambda_i$. One observes that the domain $\tilde{V} = \mathbb{C} \times V$ in $G_{\mathbb{C}}$ is invariant at the right action of $G_{\mathbb{R}}$, hence one can define

$$\mathcal{V} := \tilde{V}/G_{\mathbb{Z}} \hookrightarrow G_{\mathbb{C}}/G_{\mathbb{Z}}.$$

The above manifold can be equivalently defined as follows: let \exp be the map

$$\mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto (e^{-2i\pi z_1}, \dots, e^{-2i\pi z_n}) \in (\mathbb{C}^*)^n$$

and call $\mathcal{R} := \exp(V)$; then \mathcal{V} coincides with the quotient of $\mathbb{C} \times \mathcal{R}$ by the group generated by the automorphism

$$(z, w_1, \dots, w_n) \xrightarrow{\tilde{g}_A} (z + 1, w_1^{a_{11}} w_2^{a_{12}} \cdots w_n^{a_{1n}}, \dots, w_1^{a_{n1}} w_2^{a_{n2}} \cdots w_n^{a_{nn}}).$$

Letting g_A be the restriction \tilde{g}_A to \mathcal{R} , one obtains also the manifold $\mathcal{R}/\langle g_A \rangle$.

To prove that \mathcal{V} and $\mathcal{R}/\langle g_A \rangle$ are l.c.K., Renaud explicitly constructs l.c.K. potentials on their universal covers. Let first $\varphi_i : \mathcal{R} \rightarrow \mathbb{R}$, $\varphi_i(w) = \sum_j q_{ij} \ln |w_j|$. These functions are

p.s.h., strictly positive for $i \neq n$, have non-vanishing differential and behave well under the action of g_A : $\varphi_i \circ g_A = \lambda_i \varphi_i$. With them, one constructs the potential

$$\varphi = \sum_{i=1}^{n-1} \varphi_i^{r_i} + \varphi_n^2,$$

where the powers r_i are chosen so that

$$\lambda_1^{r_1} = \lambda_2^{r_2} = \dots = \lambda_{n-1}^{r_{n-1}} = \lambda_n^2 := \lambda.$$

It follows that r_i are strictly negative. Obviously φ is p.s.h. and $\varphi \circ g_A = \lambda \varphi$. As the monodromy is, by construction, \mathbb{Z} , by Proposition 7.1 the potential is also proper, so $\mathcal{R}/\langle g_A \rangle$ is l.c.K. with potential.

A similar construction is done for \mathcal{V} . Renaud then shows that her manifolds are not of Kähler type. Her proof uses an idea in [44] and consists in (1) showing that $\tilde{\mathcal{V}}$ does not admit any strictly p.s.h. function invariant to the action of $G_{\mathbb{R}}$, then (2) arriving at a contradiction by restricting to the 2-dimensional case. It seems to me that this proof is very similar in spirit with Belgun's for the non-existence of l.c.K. metrics on the 3-rd type Inoue surface.

It is also remarkable that these are new examples of non-Kähler non-compact manifolds.

In complex dimension 2, these examples recover an Inoue-Hirzebruch surface minus its rational curves.

Remark 7.1. I believe that Renaud's examples are not of Vaisman type, but I couldn't verify. Note that without compactness, no general criterion can be applied to decide.

Counterexample 7.1. Although they are defined using a potential on the universal cover, the manifolds constructed in [50] do not have a discrete deck group, hence, by Proposition 7.1, they are not l.c.K. manifolds with potential.

7.2. Stability at small deformations. Let now (M, J, g) be an l.c.K. manifold with potential and (\tilde{M}, J') be a small deformation of the underlying complex manifold (M, J) . Then φ is a proper function on (\tilde{M}, J') satisfying $\gamma(\varphi) = \chi(\gamma)\varphi$. It is strictly plurisubharmonic because a small deformation of strictly plurisubharmonic function is again strictly plurisubharmonic. Therefore, (\tilde{M}, J') is Kähler, and φ is an l.c.K.-potential on (\tilde{M}, J') . This proves

Theorem 7.1 (Ornea-Verbitsky [55]). *The class of compact l.c.K. manifolds with potential is stable under small deformations.*

In particular, any small deformation of a compact Vaisman manifold is still l.c.K. (with potential), but not necessarily Vaisman. This explains *a posteriori* why the construction in [26] worked for deforming the Vaisman structure of a Hopf surface of Kähler rank 1 to a non-Vaisman l.c.K. structure on the Hopf surface of Kähler rank 0 which, moreover, by [10], does not admit any Vaisman metric.

The above theorem also proves that the new defined class is strict: not all l.c.K. manifolds admit an l.c.K. potential. For example, one may consider the blow up in a point of a compact Vaisman manifold. Moreover, the l.c.K. structure of the Inoue surfaces do not admit potential, since they can be deformed to the non-l.c.K. type Inoue surface $S_{n;p,q,r,u}^+$ with $u \in \mathbb{C} \setminus \mathbb{R}$ (cf. [10]).

7.3. Filling the Kähler cover and the embedding theorem. The key step of the following construction is the observation that, leaving apart the surfaces, the Kähler cover \tilde{M} supporting the global potential can be compactified with one point to a Stein variety. Precisely:

Theorem 7.2 (Ornea-Verbitsky [55]). *Let M be a l.c.K manifold with potential, $\dim M \geq 3$, and let \tilde{M} be the corresponding covering. Then \tilde{M} is an open subset of a Stein variety \tilde{M}_c , with at most one singular point. Moreover, $\tilde{M}_c \setminus \tilde{M}$ is just one point.*

The restriction on the dimension is essential to can apply a theorem by Rossi-Andreotti-Siu (cf. [59, Th. 3, p. 245] and [3, Pr. 3.2]) assuring that the set $\tilde{M}(a) = \{x \in \tilde{M} \mid \varphi(x) > a\}$, which is holomorphically concave, can be filled, thus being an open set in a Stein variety \tilde{M}_c with at most isolated singularities. This embedding is then extended to $\tilde{M} \hookrightarrow \tilde{M}_c$. To show that \tilde{M}_c is indeed obtained from \tilde{M} by adding just one point, one looks at the generator γ of the monodromy $\Gamma \cong \mathbb{Z}$ which acts by conformal transformations on \tilde{M}_c . Now, γ can be a contraction or an expansion, let us make the first choice: $\gamma(\Omega) = \lambda \cdot \Omega$, $\lambda < 1$. Letting f be any holomorphic function on \tilde{M}_c , it is possible to show that the sequence $\gamma^n f$ converges to a constant. Now, the set $Z := \tilde{M}_c \setminus \tilde{M}$ is compact and is fixed by Γ . Hence $\sup_Z |\gamma^n f| = \sup_Z |f|$ and $\inf_Z |\gamma^n f| = \inf_Z |f|$. Thus $\inf_Z |f| = \sup_Z |f|$ for any holomorphic function f . Therefore, all holomorphic functions have the same value in all the points of Z . But \tilde{M}_c is Stein, and thus for any distinct x and y there exists a holomorphic f such that $f(x) \neq f(y)$. This proves that $Z = \{z\}$, one point.

The next step is to show that, in the above hypothesis,

Proposition 7.2 (Ornea-Verbitsky [55]). *γ acts with eigenvalues strictly smaller than 1 on the cotangent space $T_z^* \tilde{M}_c$.*

The proof goes as follows (for the tangent space, which is enough). For the smooth case, the argument is the Schwarz lemma. In general, one uses the fact that $\gamma^n f = \text{const.}$ for any holomorphic f (see above). Then, if $\gamma(x) = x$ and $d_x \gamma(v) = \lambda v$ and $d_x f(v) \neq 0$, then $d_x(\gamma^n f)(v) = \lambda^n d_x f(v) \rightarrow 0$ because $\gamma^n f$ converges to a constant. Hence $|\lambda| < 1$.

This implies that the formal logarithm of γ converges. A theorem in [66] assures that in these conditions γ acts with finite Jordan blocks on the formal completion $\hat{\mathcal{O}}_z$ of \mathcal{O}_z (the local ring of analytic functions in $z \in \tilde{M}_c$). But one can show that on a Stein variety S , for a holomorphic flow with eigenvalues smaller than 1 on $T_s S$ for some s , there exists a sequence of $V_n \subset \mathcal{O}_s S$ of finite dimensional subspaces such that the s -adic completion of $\oplus V_n$ is exactly the completion of $\mathcal{O}_s S$, each V_n being preserved by the flow which acts by linear transformations on it. For the proof, one first observes that S can be assumed smooth, in fact an open ball in \mathbb{C}^n : S can be seen as an analytic subvariety of an open ball and the holomorphic flow can be extended to that ball. Now it is possible to apply an old theorem of Poincaré and Dulac (cf. [5, p. 181]) which gives the normal form of such a flow: $\lambda_i x_i + P(x_{i+1}, \dots, x_n)$, where P are resonant polynomials corresponding to the eigenvalues λ_i . Each V_{λ_i} generated by the coordinate function x_i is preserved by the flow. Hence, picking some $q_1 \in V_{\lambda_{i_1}}, \dots, q_k \in V_{\lambda_{i_k}}$, the flow will preserve the space $V_{\lambda_{i_1} \dots \lambda_{i_k}}$ they generate. On the other hand, the completion of all these $V_{\lambda_{i_1} \dots \lambda_{i_k}}$ is precisely $\hat{\mathcal{O}}_s S$.

All in all, we find that it is possible to choose enough holomorphic functions in the eigenspaces V_n which, together, give an embedding of the cover \tilde{M}_c in a \mathbb{C}^N in such a way that the monodromy Γ acts equivariantly on \mathbb{C}^N , with eigenvalues smaller than 1. Hence M embeds in $(\mathbb{C}^N \setminus 0)/\Gamma$. Such a quotient was called a linear *Hopf manifold*, generalizing both class 0 and class 1 Hopf surfaces of Kodaira⁸. As in [26], such a Hopf manifold is l.c.K., but non-Vaisman in general.

Remark 7.2. 1) The Hopf manifolds just constructed are the appropriate generalization to arbitrary dimension of the Hopf surfaces of class 0.

2) Using another approach, namely exploiting Sternberg's normal form of holomorphic contractions, Belgun also showed that Hopf manifolds are l.c.K. and classified the Vaisman ones among them: they correspond to contractions whose normal form is resonance free, see (and mainly listen to) [12].

Summing up, we arrive at the following result which improves a lot the immersion Theorem 6.11:

⁸The terminology is misleading, but belongs to Kodaira. In fact, the group that defines the class 0 Hopf surfaces does not contain linear transformation except when the transformation is a Jordan cell.

Theorem 7.3 (Ornea-Verbitsky [55]). *Any compact l.c.K. manifold with potential, of complex dimension at least 3, admits an embedding in a linear Hopf manifold. If M is Vaisman, it can be embedded in a Vaisman-type Hopf manifold $(\mathbb{C}^N \setminus 0)/\langle A \rangle$, where A is a diagonal linear operator with eigenvalues strictly smaller than 1.*

The first statement was already proved. When M is Vaisman, the idea is to first identify the L^2 -completion of the space V of CR-holomorphic functions on S with the space of L^2 -integrable holomorphic functions on a pseudoconvex domain bounded by a level set S_a of the potential function. This is assured by a result in [48]. One next considers the flow X of the Lee field, which is holomorphic and naturally acts on V . As X acts by homotheties on any Kähler cover, it acts on V as a self-dual operator, hence it is diagonal any finite-dimensional eigenspace. In particular, this holds for the subspaces $V_{\lambda_{i_1} \dots \lambda_{i_k}}$ constructed above which were used to construct the embedding of \tilde{M}_c in \mathbb{C}^N .

As a straightforward consequence, one obtains the embedding result for Sasakian manifolds:

Theorem 7.4 (Ornea-Verbitsky [55]). *Any compact Sasakian manifold admits a CR-embedding into a Sasakian weighted sphere, preserving the respective Reeb fields.*

8. LOCALLY CONFORMALLY KÄHLER REDUCTION

Symplectic reduction (at 0) was easily extended to the Kählerian case. In fact, it was enough to verify that if the Hamiltonian action of a group preserves also the complex structure, then this is projectable on the symplectic quotient and still compatible with the symplectic form. But the passage from holomorphic isometric actions to holomorphic *conformal* actions was more difficult. Among the main problems one has to solve are:

- (1) producing a good definition of specific action on l.c.K. manifolds, producing a momentum map;
- (2) producing a reduction scheme compatible with the reduction of the related structures (Kähler, Sasakian);
- (3) finding conditions for the quotient of a Vaisman manifold to be Vaisman.

One sees that a major difficulty is that on l.c.K. manifolds one should act with conformalities, whereas for the other structures one uses isometries.

The first results were obtained in [14]. This paper essentially contains all that is needed to define and perform l.c.K. reduction, the theory being sketched in the language of conformal geometry. The symplectic version of this result, namely the locally conformally symplectic reduction, was independently discovered in [32] and presented in a local language. Then, in [28], locally conformally symplectic reduction is shown to be compatible with the complex frame and, on the other hand, the reduction thus obtained is shown to be equivalent with the Biquard-Gauduchon one. Moreover, all the above problems are addressed and solved. Further developments are given in [29].

To begin with, using the twisted differential d^θ , one defines a twisted Poisson bracket by $\{f_1, f_2\}^\theta = \omega(\sharp d^\theta f_1, \sharp d^\theta f_2)$ which is easily seen to satisfy the Jacobi identity. Now an action of a subgroup $G \subseteq \text{Aut}(M)$ is said *twisted Hamiltonian* if there exists a Lie algebra homomorphism (with respect to the twisted Poisson bracket) $\mu : \mathfrak{g} \rightarrow \mathcal{C}^\infty(M)$ such that $i(X_M)\omega = d^\theta \mu^{X_M}$ for any vector field⁹ $X \in \mathfrak{g}$.

An essential property of a twisted Hamiltonian action is its conformal invariance in the following sense: by direct computation one sees that to the conformal change $\omega' = e^\alpha \omega$ correspond the following “conformal” changes:

$$\sharp_{\omega'} d^{\theta'}(e^\alpha f) = \sharp_\omega d^\theta f, \quad \{e^\alpha f_1, e^\alpha f_2\}^{\omega'} = e^\alpha \{f_1, f_2\}^\omega.$$

This implies that multiplication with e^α induces an isomorphism of Lie algebras between $(\mathcal{C}^\infty(M), \{, \}^\omega)$ and $(\mathcal{C}^\infty(M), \{, \}^{\omega'})$ which commutes with taking Hamiltonians.

⁹I denote with X_M the fundamental field associated to X , namely whose flow is $(d/dt) e^{tX} \cdot x$.

When an action is twisted Hamiltonian, μ is called a momentum map for the action of G . It is important to note that the definition is consistent with the conformal framework: if $g' = e^\alpha g$, then $\mu' = e^\alpha \mu$ and, in particular, the level set at 0 is well-defined. That is why, as in the classical contact (and Sasakian) case, one performs, for the moment at least, only l.c.K. reduction *at zero*.

The l.c.K. reduction can now be stated in:

Theorem 8.1 (Gini-Ornea-Parton [28]). *Let (M, J, g) be a locally conformally Kähler manifold and G a subgroup of $\text{Aut}(M)$ whose action is twisted Hamiltonian. Suppose that 0 is a regular value for the associated momentum map μ and that the action of G is free and proper on $\mu^{-1}(0)$. Then there exists a locally conformally Kähler structure (J_0, g_0) on $M_0 = \mu^{-1}(0)/G$, uniquely determined by the condition $\pi^*g_0 = i^*g$ where $i : \mu^{-1}(0) \rightarrow M_0$ is the canonical projection.*

The striking property, cf. [28], of twisted Hamiltonian actions is that they lift to Hamiltonian (in particular isometric) actions with respect to the Kähler metric of any globally conformally Kähler covering: indeed, the lifted action is twisted Hamiltonian with respect to the lifted l.c.K. metric; but this one is globally conformal to a Kähler metric with respect to which the lifted action is still twisted Hamiltonian (see the above remark on the conformal invariance). As the Lee form of a Kähler metric is zero, the notion of twisted Hamiltonian coincides with that of Hamiltonian action. This reduces l.c.K. reduction to Kähler reduction of any Kähler covering:

Theorem 8.2 (Gini-Ornea-Parton [28]). *Let (M, J, g) be a locally conformally Kähler manifold and let $G \subset \text{Aut}(M)$ satisfy the hypothesis of the above reduction theorem. Let \tilde{G} be a lift of G to the universal covering \tilde{M} of M . Then the Kähler reduction is defined, with momentum map denoted by $\mu_{\tilde{M}}$, \tilde{G} commutes with the action of $\pi_1(M)$, and the following equality of locally conformally Kähler structures holds:*

$$(8.1) \quad \mu^{-1}(0)/G \cong (\mu_{\tilde{M}}^{-1}(0)/\tilde{G})/\pi_1(M) .$$

Conversely, let \tilde{G} be a subgroup of isometries of a Kähler manifold $(\tilde{M}, \tilde{g}, J)$ of complex dimension bigger than 1 satisfying the hypothesis of Kähler reduction and commuting with the action of a subgroup $\Gamma \subset \mathcal{H}(\tilde{M})$ of holomorphic homotheties acting freely and properly discontinuously and such that $\rho(\Gamma) \neq 1$. Moreover, assume that Γ acts freely and properly discontinuously on $\mu_{\tilde{M}}^{-1}(0)$. Then \tilde{G} induces a subgroup G of $\text{Aut}(M)$, M being the locally conformally Kähler manifold \tilde{M}/Γ , which satisfies the hypothesis of the reduction theorem, and the isomorphism (8.1) holds.

This construction applies in particular to the Riemannian cone which covers a (compact) Vaisman manifold and permits the link between l.c.K. and Sasakian reductions (for the Sasakian reduction, see [30]). The precise result reads:

Theorem 8.3 (Gini-Ornea-Parton [28]). *Let W be a Sasakian manifold and let Γ be a group of Sasakian automorphisms inducing holomorphic homotheties on the cone $W \times \mathbb{R}$. Let $M = W \times \mathbb{R}/\Gamma$ be the corresponding Vaisman manifold. Let $G \subset \text{Iso}(W)$ be a subgroup satisfying the hypothesis of Sasakian reduction. Then G can be considered as a subgroup of $\mathcal{H}(W \times \mathbb{R})$. Assume that the action of G commutes with that of Γ , and that Γ acts freely and properly discontinuously on the Kähler cone $(\mu_W^{-1}(0)/G) \times \mathbb{R}$.*

Then G induces a subgroup of $\text{Aut}(M)$ satisfying the hypothesis of the reduction theorem, and the reduced manifold is isomorphic with $((\mu_W^{-1}(0)/G) \times \mathbb{R})/\Gamma$. In particular the reduced manifold is Vaisman.

In the compact case, this situation can be, rather unexpectedly, reversed, hence l.c.K. reduction of compact Vaisman manifolds is completely equivalent to the Sasakian reduction. In general, if one starts with a Kähler action on the cone over a Sasakian manifold, this action does not come from a Sasakian one, namely the action does not necessarily

commute with the translations along the generators of the cone. But if the Sasakian manifold is compact, then:

Proposition 8.1 (Gini-Ornea-Parton-Piccinni [29]). *Any Kähler automorphism of the cone $W \times \mathbb{R}$ over a compact Sasakian manifold W is of the form (f, Id) , where f is a Sasakian automorphism of W .*

The proof is based on showing that, as a metric space – with the distance induced from the cone Riemannian metric – the cone can be completed with only one point. This allows extending any isometry to the completion by fixing the new added point which, in turn, implies that the isometry preserves the horizontal sections of the cone, so it comes from an isometry f of W . The fact that f preserves the Sasakian structure is an easy consequence of the Kählerian character of the initial automorphism.

Remark 8.1. Due to the fact that the above proof is, in its essence, Riemannian, the result also works for almost Kähler cones over compact K -contact manifolds and, in general, for the Riemannian cone over a compact contact metric manifold.

As the universal cover of a compact Vaisman manifold is a Riemannian cone over a compact Sasakian manifold, due to the above proven compatibility of reductions, one obtains:

Theorem 8.4 (Gini-Ornea-Parton-Piccinni [29]). *The l.c.K. reduction of a compact Vaisman manifold is a Vaisman manifold.*

This mechanism allows the obtaining of examples of Vaisman manifolds by merely reducing weighted Sasakian spheres (cf. [28]). In fact, let S^1 act on the weighted sphere S^{2n-1} endowed with the contact form η_A and the metric described in paragraph 6.1 by

$$(8.2) \quad (z_1, \dots, z_n) \mapsto (e^{i\alpha_1 t} z_1, \dots, e^{i\alpha_n t} z_n).$$

This action is by Sasakian automorphisms, independently on the weights $A = (a_1, \dots, a_n)$. The corresponding Sasakian momentum map reads:

$$\mu(z) = \frac{1}{2(\sum \alpha_i |z_i|^2)} \sum \alpha_i |z_i|^2.$$

Its zero level set is non-empty as soon as the coefficients α_i have not the same sign. Hence, it is no loss of generality to suppose the first k α -s negative and the others positive. Then $\mu^{-1}(0)$ is diffeomorphic with $S^{2k-1} \times S^{2n-2k-1}$. To have a good quotient, the action of the circle on $\mu^{-1}(0)$ has to be free and proper. A simple computation shows that the necessary and sufficient conditions is that all the positive α -s be coprime with the negative ones. Applying the Sasakian reduction as in [30], this provides Sasakian quotients, call them $S(\alpha)$, of the weighted sphere. So, the above discussion yields:

Proposition 8.2 (Gini-Ornea-Parton [28]). *For any $(\alpha_1, \dots, \alpha_n)$ as above, for any $(a_1, \dots, a_n) \in \mathbb{R}^n$, $0 < a_1 \leq \dots \leq a_n$, for any $(c_1, \dots, c_n) \in (S^1)^n$, there exists a Vaisman structure on $S(\alpha) \times S^1$ which is the l.c.K. reduction of the Vaisman structure of the Hopf manifold associated to the couple $((a_1, \dots, a_n), (c_1, \dots, c_n))$ with respect to the S^1 action (8.2).*

For example, if $n = 2k = 4s$, then $\mu^{-1}(0)$ is diffeomorphic to $S^{4s-1} \times S^{4s-1}$ and the Sasakian quotient can be seen to be diffeomorphic to $S^{4s-1} \times \mathbb{C}P^{2s-1}$. This leads to Vaisman structures on $S^{4s-1} \times \mathbb{C}P^{2s-1} \times S^1$.

Let me stress that when reducing a compact Vaisman manifold M , one in fact reduces the simply connected Sasakian manifold whose cone is the universal cover of M . But this Sasakian reduction need not be simply connected, hence the cone over it is a cover, but not necessarily the universal cover of the reduced Vaisman manifold. By contrast, as the Sasakian manifold which is the fibre of the fibration $M \rightarrow S^1$ is the smallest whose cone covers M (see Proposition 6.6):

Proposition 8.3 (Gini-Ornea-Parton-Piccinni [29]). *The reduction procedure is compatible with the structure Theorem 6.10.*

But something more general is true: the minimal presentation of a l.c.K. manifold is compatible with the reduction. Indeed one has:

Proposition 8.4 (Gini-Ornea-Parton-Piccinni [29]). *The minimal presentation of a l.c.K. reduction $M//G$ is given by*

$$(K_{\min} // G_{\min}^{\circ}, \Gamma_{\min}).$$

In particular, the rank is preserved under reduction:

$$r_{M//G} = r_M.$$

9. OPEN PROBLEMS

I shall end these notes with mentioning some open questions that I find worth thinking about.

- (1) The following 3 questions were asked in [54] in connection with the algebraic geometry of compact Vaisman manifolds:
 - (a) Determine the moduli spaces of Vaisman structures. One should note that complex analytic deformations do not preserve the Vaisman class.
 - (b) Given a singular Vaisman variety (in view of the embedding theorem, this is a sub-variety of a diagonal Hopf manifold), does there exist a resolution of singularities within the Vaisman category? Determine the birational maps of Vaisman varieties.
 - (c) Let M^n be a compact Vaisman manifold with canonical bundle isomorphic to $L^{\otimes n}$, where L is the weight bundle (this happens when M is Einstein-Weyl). Does M carry an Einstein-Weyl structure? A positive answer will give a Vaisman analogue of Calabi-Yau theorem.
- (2) In connection with the last question and with the embedding theorem: Suppose that a compact Hermite-Einstein-Weyl manifold (in particular Vaisman) is isometrically immersed into a Hopf manifold. What can be said about the Weyl-Ricci curvature? A recent result concerning this problem in the Kähler case can be found in [33].
- (3) As regards the non-Kähler compact surfaces which do not appear in Belgun's list, the most important question is: Which compact complex surfaces with non-zero Euler-Poincaré characteristic do admit l.c.K. metrics?
- (4) Among the known examples of l.c.K. manifolds, some have discrete fundamental group, some have non-abelian one. The fundamental group of a compact Vaisman manifold is determined in Proposition 6.4. But in general, which are the groups that can be fundamental groups for a (compact) l.c.K. manifold?
- (5) Related to the reduction scheme: is it possible to obtain a Vaisman quotient out of a non-Vaisman manifold?
- (6) As the (compact) Vaisman class is preserved by l.c.K. reduction and Vaisman manifolds are l.c.K. with potential, is the (compact) l.c.K. with potential class preserved by l.c.K. reduction?
- (7) Is it possible to construct a convexity theory for l.c.s. and l.c.K. manifolds? The difficulty is that one cannot use Morse theory using the d operator, whereas the cohomology of d^{θ} is trivial on compact Vaisman manifolds.
- (8) What is the structure of compact toric l.c.K. and, in particular, Vaisman manifolds?
- (9) Not in the very framework of l.c.K. geometry, but still related to it: what is the subclass of locally conformally symplectic manifolds corresponding to Vaisman ones? A natural analogy would be to consider those l.c.s. manifolds which universal cover is globally conformal to a symplectic cone over a contact manifold.

Does there exist an intrinsic characterization of this class, eventually in terms of transformation groups as in Theorem 5.2?

- (10) Find a spinorial characterization of l.c.K. (or, at least, Vaisman) manifolds. On one hand, one can try to characterize l.c.K. manifolds in the Hermitian class as the limiting case of an estimate for the eigenvalues of the Dirac operator. Maybe more at hand would be to start with a l.c.K. manifold endowed with the Gauduchon metric, to find a good notion of a “twisted Killing spinor”, to obtain a Hijazi type inequality whose limiting case to force the Lee form to be parallel. Such an inequality should refine the one obtained in [49], where the complex structure plays no role.

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