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On the sharp approach in theorems of Fatou type

by Fausto Di Biase

Abstract[1]. We consider the almost everywhere boundary behaviour of bounded holomorphic functions on a domain, along certain approach regions that are optimal, in a sense to be made precise. The work[2] we describe is in part in collaboration with A. Stokolos, O. Svensson and T. Weiss [8].

1. How sharp is the Stolz approach?

1.1. **The sharpness of the Stolz approach in the unit disc.** In 1906, P. Fatou [10] proved that every bounded holomorphic functions defined on the unit disc D in the plane admit boundary values, for almost every points in the boundary, provided we approach the boundary ∂D of the unit disc in a nontangential way. Moreover, these functions are uniquely determined by their nontangential boundary values.

If $w \in \partial D$ then the set

$$\Gamma_\alpha(e^{i\theta}) = \left\{ z \in D : |z - e^{i\theta}| < (1+\alpha)(1-|z|) \right\} \tag{1.1}$$

is called the Stolz (nontangential) approach at w. Denote by H^∞ the space of all bounded holomorphic functions defined on D. The Stolz approach at w does not contain any curve ending at w and tangential to the boundary.

How sharp is the Stolz (nontangential) approach for the a. e. boundary convergence of H^∞ functions? In other words: Is there an approach that is essentially larger than Γ_α and along which all bounded holomorphic functions do converge almost everywhere to their nontangential boundary values? This issue was first addressed in 1927 by Littlewood [22], who proved that there is no **rotation invariant** approach by tangential curves along which all bounded holomorphic functions converge a. e. to their nontangential boundary values. Indeed, a family $\gamma = \{\gamma(\theta)\}_{\theta\in[0,2\pi)}$ of subsets of D, called an *approach*, may have the following properties:

c: each $\gamma(\theta)$ is a curve in D ending at $e^{i\theta}$;

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[1]Author's address: F. Di Biase, Università degli Studi "G. d'Annunzio", Dipartimento di Scienze, Viale Pindaro 42, 65127 Pescara, Italy; e-mail: `fdibiase@unich.it` .

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tg: each $\gamma(\theta)$ ends tangentially at $e^{i\theta}$;
aecv: each $h \in \mathrm{H}^\infty$ converges a. e. along $\gamma(\theta)$ to its Stolz boundary values.

A precise form of the question about the sharpness of the Stolz approach for H^∞ functions is the following claim, called the STRONG SHARPNESS CLAIM.

(SSC) *There is no approach γ satisfying* (**c**)&(**tg**)&(**aecv**).

This claim is coherent with a principle — implicit in Fatou [10] — whose first rendition is found in Littlewood [22], who showed that there is no **rotation invariant** approach γ satisfying (**c**)&(**tg**)&(**aecv**). Another rendition of this principle (with stronger conclusions) has been given by Aikawa [1], who proved that, if (**u**) is the condition:

u: the curves $\{\gamma(\theta)\}_\theta$ are uniformly bi-Lipschitz equivalent;

then there is no approach γ satisfying **(u)** and (**c**)&(**tg**)&(**aecv**).

Our first result[3] is a theorem of Littlewood type where the tangential curve is allowed to vary its shape, and we do not require uniformity in the order of tangency. Moreover, we show that, in a precise sense, Theorem 1.1 is sharp.

Theorem 1.1 (A sharp Littlewood type theorem [8])**.** *Let $\gamma : [0, 2\pi) \to 2^D$ such that*

(c⋆): *for each $\theta \in [0, 2\pi)$, the set $\{e^{i\theta}\} \cup \gamma(\theta)$ is connected;*
(tg): *for each $\alpha > 0$ and $\theta \in [0, 2\pi)$ there exists $\delta > 0$ such that if $z \in \gamma(\theta) \cap \Gamma_\alpha(e^{i\theta})$ then $|z - e^{i\theta}| > \delta$;*
(reg): *for each open subset O of D the set*

$$\{\theta \in [0, 2\pi) : \gamma(\theta) \cap O \neq \emptyset\}$$

is a measurable subset of $[0, 2\pi)$.

Then there exists $h \in \mathrm{H}^\infty$ with the property that, for almost every $\theta \in [0, 2\pi)$, the limit of $h(z)$ as $z \to e^{i\theta}$ and $z \in \gamma(\theta)$ does not exist.

• Condition (**c⋆**) is strictly weaker than (**c**) but it *cannot* be relaxed to the minimal condition one may ask for:

(apprch): $e^{i\theta}$ belongs to the closure of $\gamma(\theta)$ for all $\theta \in [0, 2\pi)$;

since Nagel and Stein [24] showed that there is a rotation invariant approach γ satisfying (**apprch**) and (**tg**)&(**aecv**). This discovery disproved a conjecture of Rudin [27], prompted by his construction of a highly oscillating inner function in D. Thus, (**c⋆**) identifies the property of *curves* relevant to a theorem of Littlewood type.

• It is not easy to see **(reg)** fail. The images of radii by an inner function satisfy **(reg)**: this example prompted Rudin [27] to ask about the truth value of (SSC). Observe that **(reg)** is a qualitative condition, while **(u)** is quantitative. The former is perhaps more commonly met than the latter. They are independent of each other.

• Since our hypothesis do not impose any smoothness, neither on $\gamma(\theta)$ nor on the domain, a version of our theorem can be formulated, and proved as well, for domains with rough boundary, such as NTA domains in $\mathbb{R}^n$; see Theorem 2.1.

• Is it possible to prove Theorem 1.1 without assuming **(reg)**? Several theorems in Analysis do fail if we omit some regularity conditions, while others (typically

[3] A preliminary version of this result was announced in Di Biase et al. [7].

those involving null sets) remain valid without 'regularity' hypothesis[4]. This question brings us back to the truth value of (SSC), and we prove the following result.

Theorem 1.2 ([8]). *It is neither possible to prove the Strong Sharpness Claim, nor to disprove it.*

The proof uses a combination of methods of modern logic (developed after 1929) and harmonic analysis, based upon an insight about the location of the link that makes the combination possible. See Theorem 3.1, Theorem 3.2 and Theorem 3.3.

Remark 1.1. We quote a remark made by K. Gödel in [12] about the Continuum Hypothesis, or Cantor's conjecture.

> Only someone who [...] denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as knows today can only mean that these axioms do not contain a complete description of this reality; and such a belief is by no means chimerical, since it is possible to points out ways in which a decision of the question, even if it is undecidable from the axioms in their present form, might nevertheless be obtained.

It seems to us that Gödel's remark applies equally well to (SSC), for those who share the Platonist viewpoint of Gödel.

2. The sharpness of the corkscrew approach on NTA domains

Let h^∞ be the space of bounded harmonic functions on a bounded domain $D \subset \mathbb{R}^n$. Assume that D is NTA, as defined and studied by Jerison and Kenig [17]. Jerison and Kenig proved a Fatou type theorem for functions in h^∞ (as well as in the Hardy spaces). Indeed, they proved that if $f \in \mathrm{h}^\infty$ then for almost every point in the boundary, with respect to harmonic measure, f admits boundary values taken along the corkscrew approach, defined, for $w \in \partial D$, by

$$\Gamma_\alpha(w) \stackrel{\text{def}}{=} \{z \in D : |z - w| < (1+\alpha)\text{dist}(z, \partial D)\} \tag{2.1}$$

Observe that D may be *twisting* a. e. relative to harmonic measure. In this case, the 'corkscrew' approach (2.1) does not look like a sectorial angle at all.

How sharp is the *corkscrew approach* for the boundary convergence for h^∞ functions, a. e. relative to harmonic measure? Theorem 1.1 lends itself to the task of formulating[5] the appropriate sharpness statement for NTA domains, without any further restrictions on the domain.

Theorem 2.1. *If D is an NTA domain in $\mathbb{R}^n$ and $\gamma = \{\gamma(w)\}_{w\in\partial D}$ is a family of subsets of D such that*

(c⋆): *for each $w \in \partial D$, $\gamma(w) \cup \{w\}$ is connected;*

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[4]A *regularity hypothesis* in a theorem is one which is not (formally) necessary to give meaning to the conclusion of the theorem. *A priori* it is not clear which theorems belong to which group. Egorov's theorem on pointwise convergence belongs to the first; see Bourbaki [2], p. 198. One example in the second group can be found in Stein [28], p. 251.

[5]In formulating (and proving) our Theorem 1.1 we also had this goal in mind.

(tg): *for each $\alpha > 0$ and $w \in \partial D$ there exists $\delta > 0$ such that if $z \in \gamma(w) \cap \Gamma_\alpha(w)$ then $|z - w| > \delta$;*

(reg): *for each open subset O of D the set*

$$\{w \in \partial D : \gamma(w) \cap O \neq \emptyset\}$$

is a measurable subset of ∂D (i. e. its characteristic function is resolutive);

then there exists $h \in \mathrm{h}^\infty$ such that for almost every $w \in \partial D$, with respect to harmonic measure, the limit of $h(z)$ as $z \to w$ and $z \in \gamma(w)$ does not exist.

• A condition such as rotation invariance, in place of **(reg)**, would have no meaning, since in this context there is no group suitably acting, not even locally.

• Observe that **(c⋆)** *cannot* be relaxed to the condition

$$w \text{ belongs to the closure of } \gamma(w) \tag{2.2}$$

(the minimal one needed to take boundary values). Indeed, Di Biase showed the existence, for NTA domains in $\mathbb{R}^n$, of an approach γ, satisfying (2.2) and **(tg)**, along which all h^∞ functions converge to their boundary values taken along (2.1), a. e. relative to harmonic measure[6].

3. Notation and Preliminary Results

The core of the problem belongs to harmonic analysis, so we restrict ourselves, without loss of generality, to the space h^∞ of bounded harmonic functions on D.

Much of the following notation applies not only when D is the open unit disc in the plane, but, more generally, when D is a bounded open subset of $\mathbb{R}^n$. If γ is a subset of $D \times \partial D$ and $w \in \partial D$, *the shape of γ at w* is the set

$$\gamma(w) \stackrel{\text{def}}{=} \{z \in D : (z, w) \in \gamma\} \subset D \ .$$

An *approach* is a subset γ of $D \times \partial D$ such that the following condition:

(apprch): w belongs to the closure of $\gamma(w)$

holds for all $w \in \partial D$. One may think of γ as a family $\{\gamma(w)\}_{w \in \partial D}$ of subsets of D. If γ an approach and $u : D \to \mathbb{R}$ a function on D, the function on ∂D given by

$$\gamma^\star(u)(w) \stackrel{\text{def}}{=} \sup\{|u(z)| : z \in \gamma(w)\tag{3.1}$$

is called the *maximal function of u along γ* at $w \in \partial D$.

Lemma 3.1. *The following properties of an approach γ are equivalent:*

(a) *$\gamma^\star$ maps all continuous functions (on D) to measurable functions (on ∂D);*

(b) *for every open $Z \subset D$, the boundary subset*

$$\gamma^\downarrow(Z) \stackrel{\text{def}}{=} \{w \in \partial D : Z \cap \gamma(w) \neq \emptyset\}$$

is a measurable subset of ∂D.

The subset in (b) is called the *shadow projected by Z along γ*. The proof of Lemma 3.1 is left to the reader[7]. The approach γ is called: *regular* if it satisfies (a) or (b) in Lemma 3.1.

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[6]In Di Biase [5], the existence is showed by reducing the problem to the discrete setting of a (not-necessarily-homogeneous) tree, rather than on the action of a group on the space. In general, in this context, there is no group suitably acting on the space.

[7]This circle of ideas is based on the work of E. M. Stein. Cf. Fefferman and Stein [11].

If D is the open unit disc in the plane then the boundary of D, denoted by ∂D, is naturally identified to the quotient group $\mathbb{R}/2\pi\mathbb{Z}$, from which it inherits the *Lebesgue measure* m; thus, $m(\partial D) = 2\pi$. If $h \in \mathrm{h}^\infty$, the *Fatou set of* h, denoted by $\mathcal{F}(h) \subset \partial D$, is the set of points $w \in \partial D$, such that *the limit of* $h(z)$ *as* $z \to w$ *and* $z \in \Gamma_\alpha(w)$ *exists for all* $\alpha > 0$; this limit is denoted $h_\flat(w)$. Now, $m(\mathcal{F}(h)) = 2\pi$ and $h_\flat \in L^\infty(\partial D)$; see Fatou [10]. The *Poisson extension* $P : L^\infty(\partial D) \to \mathrm{h}^\infty$ recaptures h from $h_\flat$, since $h = P[h_\flat]$. If $h \in \mathrm{h}^\infty$ and γ is an approach, then define the following two subsets of ∂D: $\mathrm{C}(h, \gamma)$ is the set

$$\{w \in \mathcal{F}(h);\ h(z) \text{ converges to } h_\flat(w) \text{ as } z \to w \text{ and } z \in \gamma(w)\}$$

and $\mathrm{D}(h, \gamma)$ is the subset

$$\{w \in \partial D;\ h(z) \text{ does not have any limit as } z \to w \text{ and } z \in \gamma(w)\}\ .$$

The approach γ is called *rotation invariant* if $(z, w) \in \gamma$ implies $(e^{i\theta} z, e^{i\theta} w) \in \gamma$ for all θ, z, w. A rotation invariant approach is regular. If $h : D \to D$ is an inner function, then the set

$$\{(z, w) \in D \times \partial D; z = f(ru) \text{ for some } u \in \mathcal{F}(h), h_\flat(u) = w, 0 \le r < 1\}$$

is a (not necessarily rotation invariant) regular approach whose shape, given by the images of radii by h, may be empty over a null set only; see Rudin [27].

3.1. **The Independence Theorem.** Modern logic gives us tools that show that some statements can be neither proved nor disproved. The basic idea is familiar: if different models (or 'concrete' representations) of some axioms exhibit different properties, then these properties do not follows from those axioms. For example, the existence of a single, 'concrete' non commutative group shows that commutativity can not be derived from the group axioms, and the existence of different models of geometry shows that Euclid's Fifth Postulate does not follow from the others. Since the currently adopted system of axioms for Mathematics is ZFC[8], to prove a theorem amounts to deduce the statement from ZFC. A *model* of ZFC stands to ZFC as, say, a 'concrete' group stands to the axioms of groups. If ZFC is consistent, then it has several, different models. K. Gödel showed, in his *completeness theorem*, that a statement can be deduced from ZFC if and only if it holds in every model of ZFC; in particular, if it holds in some models but not in others, then it follows that it *can be neither proved nor disproved.* The tangential boundary behaviour of h^∞ functions is radically different in different models of ZFC[9].

Theorem 3.1 ([8])**.** *There is a model of* ZFC *in which there exists an approach* γ *satisfying* (**c**) *and* (**tg**) *and such that* $C(h, \gamma)$ *has measure equal to* 2π *for every* $h \in \mathrm{h}^\infty$*.*

Theorem 3.2 ([8])**.** *There is a model of* ZFC *in which for every approach satisfying* (**c**$\star$) *and* (**tg**) *there exists* $h \in \mathrm{h}^\infty$ *such that* $D(h, \gamma)$ *has outer measure equal to* 2π*.*

The following result shows that Theorem 3.2 cannot be improved[10].

[8]Acronym for Zermelo, Fraenkel and the Axiom of Choice. See Cohen [4], Drake [9], Jech [16], Kunen [21].

[9]Since an approach is a fairly arbitrary subset of $D \times \partial D$, *in retrospect* this result can be rationalized, but other examples in Analysis show that this rationalization is not *a priori* infallible.

[10]Theorem 3.3 in itself does not say whether (SSC) can be proved or not.

Theorem 3.3 (A theorem in ZFC [8])**.** *There exists an approach γ satisfying* (**c**) *and* (**tg**) *such that for each $h \in \mathrm{h}^\infty$, the set $C(h, \gamma)$ has outer measure equal to 2π.*

4. How un-Stolz are the sharp approach regions in $\mathbb{C}^n$?

Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with smooth boundary and let H^∞ be the space of all bounded holomorphic functions defined on D. The sharp approach, along which all functions in H^∞ converge almost everywhere to their boundary values, has been so far been sufficiently understood in a few cases only. In the few cases that are sufficiently understood (listed below), two features have been observed. The first one is that the order of contact of the boundary, with the section of the shape of the approach at a given point, taken along a direction tangential to the boundary, depends on the direction itself and may vary even among different complex tangential directions (unless the domain has certain symmetries, like the unit ball in $\mathbb{C}^n$); the second feature is that the shape of the approach changes near weakly pseudoconvex points.

(1) If D is the unit ball in $\mathbb{C}^n$ then Korányi [18] has considered the approach $\mathcal{K}$ that can be essentially described by inequalities of the following form

$$\frac{\operatorname{dist}(z, \partial D)}{\operatorname{dist}(z, w + T^c_w(\partial D))} \geq C > 0$$

where $T^c_w(\partial D)$ is the complex tangent space at $w \in \partial D$ and $z \in D$. He proved that all functions in H^∞ admit a. e. boundary values along $\mathcal{K}$. A notable feature of this approach is that the section of its shape at $w \in \partial D$, taken along a tangential direction, depends on the direction itself. Indeed, it is nontangential along the complex normal direction and tangential along any complex tangential directions. Its degree of tangency to the boundary depends upon the degree of contact of the boundary with the complex tangent space and therefore it is, in the case of the unit ball, isotropic along all complex tangential directions.

How sharp is $\mathcal{K}$ for the unit ball in $\mathbb{C}^n$? This question has been first addressed in 1983 by Hakim and Sibony [13] and more recently by Kentaro Hirata [14]). Presumably, one should be able to obtain a Littlewood type theorem along the lines of Theorem 1.1 in the case of the unit ball.

(2) If $D \subset \mathbb{C}^n$ is a bounded domain with smooth boundary, then an approach along which all functions in H^∞ admit a. e. boundary values has been defined and studied in 1972 by Stein [29]. We denote this approach by $\mathcal{S}$.

If D is strongly pseudoconvex, then the approach $\mathcal{S}$ should be sharp, in the sense that a Littlewood type theorem along the lines of Theorem 1.1 should hold in this context as well, but so far a proof of this sharpness statement has not appeared in the literature in this form; such a sharpness result for $\mathcal{S}$ may very well hold for any bounded domain with smooth boundary, but nevertheless, if D is not strongly pseudoconvex, then the approach $\mathcal{S}$ does not appear to be sharp any longer, in a certain sense, as we shall see in the next class of examples.

(3) If $D \subset \mathbb{C}^n$ is a bounded pseudoconvex domain with smooth boundary and of finite type in the sense of commutators of vector fields, then an approach, denoted by $\mathcal{A}$, along which all functions in H^∞ admit a. e. boundary values has been described in 1981 by Nagel et al. [25]. See also [26] and [23]. In

this context, the following additional feature appears: the shape of this approach at w does change, and gets wider, when w gets close to weakly pseudoconvex points.

How sharp is the approach $\mathcal{A}$? Suppose first that $n = 2$. Then, since the shape of $\mathcal{A}$ is asymptotically equal to that of $\mathcal{S}$ at strongly pseudoconvex points (thus, on a set of full measure), the appropriate sharpness statement for this approach should have the following form:

Claim 4.1. *If $D \subset \mathbb{C}^n$ is a bounded pseudoconvex domain with smooth boundary and of finite type and if $n = 2$ then there is no natural approach γ that is essentially larger that $\mathcal{A}$ and whose maximal operator, defined in* (3.1)*, satisfies the following inequality, for all $f \in H^p$*

$$\|\gamma^\star(f)\|_{L^p} \leq C_p \|f\|_{H^p} \ .$$

(An approach is *natural* if the shadow it projects by points in the domain is uniformly comparable to balls in the boundary; this condition is needed to take into account the Nagel-Stein phenomenon; see [5] for the precise definition). Indeed, the inequality appearing in the claim holds for $\gamma = \mathcal{A}$ and it is a quantitative form of Fatou's theorem. The inequality appearing in the claim, for $\gamma = \mathcal{A}$, does not follow from the corresponding statement for $\mathcal{S}$, because the distribution function of $\mathcal{A}^\star(f)$ is not controlled by the distribution function of $\mathcal{S}^\star(f)$.

If $n > 2$, then the approach $\mathcal{A}$ does not appear to be sharp any longer, in a certain sense, as we shall see by looking at smoothly bounded domains in $\mathbb{C}^n$ that are convex and of finite type.

(4) If $D \subset \mathbb{C}^n$ is a convex, smoothly bounded domain of finite type, then an approach along which all functions in H^∞ admit a. e. boundary values has been described in 1998 by Di Biase and Fischer [6]. We denote this approach by $\mathcal{C}$. In this case, another feature appears: the section of the shape of $\mathcal{C}$ is anisotropic along the complex tangential directions, and it is appropriately larger than that of $\mathcal{A}$ near weakly pseudoconvex points; in contrast, the approach $\mathcal{A}$ is isotropic among all complex tangential directions (unless of course $n = 2$). See [6]. In particular, we see that the notion of finite type based on commutators of vector fields does not capture the optimal order of contact with the boundary for the sharp approach.

How sharp is $\mathcal{C}$? Since the shape of $\mathcal{C}$ is asymptotically equal to that of $\mathcal{S}$ near strongly pseudoconvex points, the appropriate sharpness statement for this approach should have the following form:

Claim 4.2. *If $D \subset \mathbb{C}^n$ is a convex, smoothly bounded domain of finite type, then there is no natural approach γ that is essentially larger that $\mathcal{C}$ and whose maximal operator satisfies the inequality:*

$$\|\gamma^\star(f)\|_{L^p} \leq C_p \|f\|_{H^p}$$

for all $f \in H^p$.

Indeed, the inequality appearing in the claim holds for $\gamma = \mathcal{C}$ and it is a quantitative form of Fatou's theorem. The inequality appearing in the claim, for $\gamma = \mathcal{C}$, does not follow from the corresponding statement for $\mathcal{A}$, because the distribution function of $\mathcal{C}^\star(f)$ is not controlled by the distribution function of $\mathcal{A}^\star(f)$.

Once a precise (possibly intrinsic) description of the sharp approach, together with a precise statement and proof of its sharpness, will have been given in the desidered degree of generality (including bounded pseudoconvex domains in $\mathbb{C}^n$), then one should also establish further quantitative results related to the area function and the maximal function, both evaluted along the sharp approach, and the L^p estimates relating these operators to each other, and so forth. See [25] and [20].

In the few cases that are sufficiently understood, a family of balls in the boundary, having certain covering and doubling properties, plays an important role in the theory; see Hörmander [15], Nagel et al. [26], Stein [29]. However, in general, this structure seems to be missing; see Cirka [3] (where the result about a. e. convergence appears to have a conditional nature, i.e. conditional upon the occurrence of certain covering and doubling properties of certain boundary balls, that are rather difficult to verify). See also [19].

References

[1] H. Aikawa, *Harmonic functions and Green potentials having no tangential limits*, J. London Math. Soc., (2)43(1991), 125-136.

[2] N. Bourbaki, *Intégration, Chap. I-IV*, Herman, 1952.

[3] E. Chirka, *The theorems of Lindelöf and Fatou in* $\mathbb{C}^n$, Math. Sb., 92(1973), 622-644.

[4] P. J. Cohen, *Set theory and the continuum hypothesis*, Benjamin, 1966.

[5] F. Di Biase, *Fatou type theorems. Maximal functions and approach regions*, Birkhäuser, 1998.

[6] F. Di Biase & B. Fischer, *Boundary behaviour of* H^p *functions on convex domains of finite type in* $\mathbb{C}^n$, Pacific J. Math., 183(1998), 25-38.

[7] F. Di Biase, A. Stokolos, O. Svensson & T. Weiss, *Tangential boundary behaviour of bounded harmonic functions in the unit disc*, Geometry Seminars 1996-97(1998), 63-68.

[8] F. Di Biase, A. Stokolos, O. Svensson & T. Weiss, *On tangential approach regions for bounded harmonic functions*, Preprint, 2003.

[9] F. R. Drake, *Set theory.*, North-Holland., 1974.

[10] P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math., 30(1906), 335-400.

[11] C. Fefferman & E. M. Stein, *Some maximal inequalities*, Amer. J. Math., 93(1971), 107-115.

[12] K. Gödel, *What is Cantor's continuum problem?*, Amer. Math. Monthly, 54(1947), 515-525.

[13] M. Hakim & N. Sibony, *Fonctions holomorphes bornées et limites tangentielles*, Duke Math. J., 50(1983), 133-141.

[14] K. Hirata, *Sharpness of the Korányi approach regions*, to appear, Proc. Amer. Math. Soc.

[15] L. Hörmander, L^p *estimates for (pluri-)subharmonic functions*, Math. Scand., 20(1967), 65-78.

[16] T. Jech, *Set theory*, Academic Press, 1978.

[17] D. Jerison & C. Kenig, *Boundary behaviour of harmonic functions in non-tangentially accessible domains*, Adv. Math., 46(1982), 80-147.

[18] A. Korányi, *Harmonic functions on Hermitian hyperbolic space*, Trans. Amer. Math. Soc., 135(1969), 507-516.

[19] S. G. Krantz, *Invariant metrics and the boundary behavior of holomorphic functions on domains in* $\mathbb{C}^n$, J. Geom. Anal., (2)1(1991), 71–97.

[20] S. G. Krantz % S.-Y. Li, *Area integral characterizations of functions in Hardy spaces on domains in* $\mathbb{C}^n$, Complex Variables Theory Appl., (4)32(1997), 373-399.

[21] K. Kunen, *Set theory. An introduction to independence proofs*, North-Holland, 1980.

[22] J. E. Littlewood, *Mathematical notes (4): On a theorem of Fatou*, J. London Math. Soc., 2(1927), 172-176.

[23] A. Nagel, *Nonisotropic metrics on boundaries of domains of finite type*, Topics in several complex variables (Mexico, 1983), Res. Notes in Math., 112, Pitman, Boston, MA, 1985, 26-135.

[24] A. Nagel & E. M. Stein, *On certain maximal functions and approach regions*, Adv. Math., 54(1984), 83-106.

[25] A. Nagel, E. M. Stein & S. Wainger, *Boundary behavior of functions holomorphic in domains of finite type*, Proc. Nat. Acad. Sci. U.S.A., (11)78(1981), 6596-6599.

[26] A. Nagel, E. M. Stein & S. Wainger, *Balls and metrics defined by vector fields I: Basic properties*, Acta Math., 155(1985), 103-147.

[27] W. Rudin, *Inner function images of radii*, Math. Proc. Cambridge Philos. Soc., (2)85(1979), 357-360.

[28] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, N.J., 1970.

[29] E. M. Stein, *Bounday behaviour of holomorphic functions of several complex variables*, Princeton University Press, Princeton, N.J., 1972.