Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 14(2017), pp. 1 – 15.

2–Factors of regular graphs: an updated survey

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Abstract. A 2–factor of a graph G is a 2–regular spanning subgraph of G. We give an updated survey on results on the structure of 2–factors in regular graphs obtained in the last years by several authors.

1. Introduction

All graphs considered are finite and simple (without loops or multiple edges). We shall use the term multigraph when multiple edges are permitted. For definitions and notations not explicitly stated the reader may refer to Bondy and Murty's book [13].

Several authors have considered the number of Hamilton circuits in k–regular graphs and there are interesting and beautiful results and conjectures in the literature. In particular, C.A.B. Smith (1940, cf. Tutte [45]) proved that each edge of a 3–regular multigraph lies in an even number of Hamilton circuits. This result was extended to multigraphs in which each vertex has odd degree by Thomason [42].

A multigraph with exactly one Hamilton circuit is said to be *uniquely hamiltonian*. Thomason's result implies that there are no regular uniquely hamiltonian multigraphs of odd degree. In 1975, Sheehan [40] posed the following famous conjecture:

Conjecture 1.1. There are no uniquely hamiltonian k–regular graphs for all integers $k \geq 3$.

It is well known that it is enough to prove it for $k = 4$. This conjecture has been verified by Thomassen for bipartite graphs, [43] (under the weaker hypothesis that G has minimum degree 3), and for k–regular graphs when $k \geq 300$, [44]. This value has been improved by Ghandehari and Hatami for $k \geq 48$ [21] and, recently, by Haxell, Seamone and Verstraete [24] for $k > 22$.

In this context, several recent papers addressed the problem of characterizing families of graphs (particularly regular graphs) which have certain conditions imposed on their 2–factors. In this survey we present the main results obtained in the

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This research was carried out within the activity of I.N.d.A.M.-G.N.S.A.G.A. and supported by the Italian Ministry M.I.U.R.

Keywords. 2-factors, snarks, cycles.

AMS Subject Classification. 05C70, 05C75, 05C35, 05C45.

last fifteen years and we give an update of the main open problems in this field. We will also discuss relations of these problems with a particular class of snarks i.e. the *odd 2–factored snarks*.

2. Preliminaries

An *r–factor* of a graph G is an r–regular spanning subgraph of G. Thus a 2–*factor* of a graph G is a 2–regular spanning subgraph of G. A 1–*factorization* of G is a partition of the edge set of G into 1–factors.

Let G be a bipartite graph with bipartition (X, Y) such that $|X| = |Y|$, and A be its adjacency matrix. In general $0 \leq |det(A)| \leq per(A)$. We say that G is *det–extremal* if $|det(A)| = per(A)$. Let $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$ be the bipartition of G. For F a 1–factor of G define the *sign* of F, $sgn(F)$, to be the sign of the permutation of $\{1, 2, \cdots, n\}$ corresponding to F. (Thus G is det–extremal if and only if all 1–factors of G have the same sign.) The following elementary result is a special case of [32, Lemma 8.3.1].

Lemma 2.1. *Let F_1, F_2 be 1-factors in a bipartite graph G and t be the number of circuits in $F_1 \cup F_2$ of length congruent to zero modulo four. Then $sgn(F_1)sgn(F_2) = (-1)^t$.*

Let G, G_1, G_2 be graphs such that $G_1 \cap G_2 = \emptyset$. Let $y \in V(G_1)$ and $x \in V(G_2)$ such that $d_{G_1}(y) = 3 = d_{G_2}(x)$. Let x_1, x_2, x_3 be the neighbours of y in G_1 and y_1, y_2, y_3 be the neighbours of x in G_2. If $G = (G_1 - y) \cup (G_2 - x) \cup \{x_1y_1, x_2y_2, x_3y_3\}$, then we say that G is a *star product* of G_1 and G_2 and write $G = (G_1, y) * (G_2, x)$.

The *Heawood graph* H_0 is the bipartite graph associated with the point/ line incidence matrix of the Fano plane $PG(2, 2)$. Let $\mathcal{H}$ be the class of graphs obtained from the Heawood graph by repeated star products.

These graphs were used by McCuaig in [35] to characterise the 3–connected cubic det-extremal bipartite graphs:

Theorem 2.2 ([35])**.** *A 3-connected cubic bipartite graph is det-extremal if and only if it belongs to $\mathcal{H}$.*

Note (i) Theorem 2.2 has been improved for connectivity 2 graphs by Funk, Jackson, Labbate and Sheehan in [17]

(ii) Bipartite graphs G with the more general property that some of the entries in the adjacency matrix A of G can be changed from 1 to -1 in such a way that the resulting matrix A^* satisfies $per(A) = det(A^*)$ have been characterised in [31, 34, 36].

3. 2–factor hamiltonian graphs

A graph with a 2–factor is said to be 2–*factor hamiltonian* if all its 2–factors are Hamilton cycles. Examples of such graphs are K_4, K_5, $K_{3,3}$, the Heawood graph H_0, and the cubic graph of girth five obtained from a 9-circuit by adding three vertices, each joined to three vertices of the 9-circuit. (The latter graph is the 'Triplex graph' of Robertson, Seymour and Thomas.)

The following property is easy to prove and, at the same time, important for approaching a characterization of this family of graphs.

Proposition 3.1 ([18]). *If a bipartite graph G can be represented as a star product $G = (G_1, y) * (G_2, x)$, then G is 2–factor hamiltonian if and only if G_1 and G_2 are 2–factor hamiltonian.*

Note that $K_4 * K_4$ shows that the star products of non–bipartite 2–factor hamiltonian graphs is not necessarily 2–factor hamiltonian.

Using Proposition 3.1, Funk, Jackson, Labbate and Sheehan constructed an infinite family of 2–factor hamiltonian cubic bipartite graphs by taking iterated star products of $K_{3,3}$ and H_0 [18]. They conjecture that these are the only non-trivial 2–factor hamiltonian regular bipartite graphs.

Conjecture 3.2 ([18]). *Let G be a 2–factor hamiltonian k-regular bipartite graph. Then either $k = 2$ and G is a circuit or $k = 3$ and G can be obtained from $K_{3,3}$ and H_0 by repeated star products.*

If proved, Conjecture 3.2 will allow to completely characterize the family of 2–factor hamiltonian regular bipartite graphs. In the 80's Sheehan posed the following

Conjecture 3.3. *There are no 2–factor hamiltonian k–regular bipartite graphs for all integers $k \geq 4$.*

The following properties have been proved by Labbate [28, 29] for an equivalent family of cubic graphs (cf. subsection 3.1), and then by Funk, Jackson, Labbate and Sheehan [18] for 2–factor hamiltonian graphs:

Lemma 3.4 ([29, 28, 18]). *Let G be a 2–factor hamiltonian cubic bipartite graph. Then G is 3–connected and $|V(G)| \equiv 2 \pmod 4$.*

A graph H is *'maximally'* 2–factor hamiltonian if the multigraph G obtained by adding an edge e with endvertices u, v to H has a disconnected 2–factor containing e.

Lemma 3.5 ([18, Lemma 3.4 (a)(i)]). *Graphs obtained by taking star products of H_0 are maximally 2–factor hamiltonian.*

Funk, Jackson, Labbate and Sheehan in [18] proved Conjecture 3.3 applying Lemmas 2.1, 3.4, 3.5 and Theorem 2.2:

Theorem 3.6 ([18]). *Let G be a 2–factor hamiltonian k–regular bipartite graph. Then $k \leq 3$.*

Theorem 3.6 has inspired further results by Faudree, Gould and Jacobsen [15] that determined the maximum number of edges in both 2–factor hamiltonian graphs and 2–factor hamiltonian bipartite graphs. In particular they proved the following:

Theorem 3.7 ([15]). *If G is a bipartite 2–factor hamiltonian graph of order n then*

$$|E(G)| \leq \begin{cases} n^2/8 + n/2 & \text{if } n \equiv 0 \mod 4\ , \\ n^2/8 + n/2 + 1/2 & \text{if } n \equiv 2 \mod 4\ , \end{cases}$$

and the bound is sharp.

Theorem 3.8 ([15]). *If G is a 2–factor hamiltonian graph of order n then*

$$|E(G)| \leq \lceil n^2/4 + n/4 \rceil$$

and the bound is sharp for all $n \geq 6$.

In addition, Diwan [14] has shown that

Theorem 3.9. *K_4 is the only 3–regular 2–factor hamiltonian planar graph.*

Conjecture 3.2 has been partially solved in terms of minimally 1–factorable cubic bipartite graphs as we explain in the following subsection.

3.1. **Minimally 1-factorable graphs.** Let G be a k–regular bipartite graph. We say that G is *minimally 1–factorable* if every 1–factor of G is contained in a unique 1–factorization of G.

The results cited above by Funk, Jackson, Labbate and Sheehan were inspired by results on minimally 1-factorable graphs obtained in [19, 28, 29, 30]. It can be seen that:

Proposition 3.10 ([18])**.** *If G is minimally 1–factorable then G is 2–factor hamiltonian. If $k = 2, 3$, then G is minimally 1–factorable if and only if G is 2–factor hamiltonian.*

Theorem 3.6 extends the result of [19] that minimally 1–factorable k–regular bipartite graphs exist only when $k \leq 3$.

Furthermore, Labbate in [30] proved the following characterization:

Theorem 3.11 ([30])**.** *Let G be a minimally 1–factorable k–regular bipartite graph of girth 4. Then either $k = 2$ and G is a circuit or $k = 3$ and G can be obtained from $K_{3,3}$ by repeated star products.*

Hence, it follows from results in [29] that a smallest counterexample to Conjecture 3.2 is cubic and cyclically 4-edge connected, and from Theorem 3.11 that it has girth at least six. Thus, to prove the conjecture, it would suffice to show that the Heawood graph is the only 2-factor hamiltonian cyclically 4–edge connected cubic bipartite graph of girth at least six.

This seems a very difficult task to achieve at least with the techniques used so far. We have obtained, jointly with Sheehan in [5], partial results using *irreducible Levi graphs* (cf. Section 5.1 and Theorem 5.10).

4. 2–factor isomorphic graphs

The family of 2–factor hamiltonian k–regular graphs can be extended to the family of connected k–regular graphs with the more general property that all their 2–factors are isomorphic, i.e. the family of *2–factor isomorphic* k–regular bipartite graph.

Examples of such graphs are given by all the 2–factor hamiltonian and the Petersen graph (which is 2–factor isomorphic since it has all its 2–factors of length $(5, 5)$ but it is not 2-factor hamiltonian). Note that star product preserves also 2–factor isomorphic regular graphs.

In [7] Aldred, Funk, Jackson, Labbate and Sheehan proved the following existence theorem

Theorem 4.1 ([7])**.** *Let G be a 2–factor isomorphic k–regular bipartite graph. Then $k \leq 3$.*

They also conjecture that the family of 2–factor isomorphic and the one of 2–factor hamiltonian k–regular bipartite graphs are, in fact, the same.

Conjecture 4.2 ([7]). *Let G be a connected k-regular bipartite graph. Then G is 2–factor isomorphic if and only if G is 2–factor hamiltonian.*

Abreu, Diwan, Jackson, Labbate and Sheehan proved in [3] that Conjecture 4.2 is false applying the following construction:

Proposition 4.3 ([3]). *Let G_i be a 2-factor hamiltonian cubic bipartite graph with k vertices and $e_i = u_i v_i \in E(G_i)$ for $i = 1, 2, 3$. Let G be the graph obtained from the disjoint union of the graphs $G_i - e_i$ by adding two new vertices w and z and new edges wu_i and zv_i for $i = 1, 2, 3$. Then G is a non-hamiltonian connected 2-factor isomorphic cubic bipartite graph of edge-connectivity two.*

Given a set $\{G_1, G_2, \cdots, G_k\}$ of 3–edge–connected cubic bipartite graphs let $\mathcal{SP}(G_1, G_2, \cdots, G_k)$ be the set of cubic bipartite graphs which can be obtained from $G_1, G_2, \cdots, G_k$ by repeated star products. In Section 3 we have seen that it was shown in [18] that all graphs in $\mathcal{SP}(K_{3,3}, H_0)$ are 2–factor hamiltonian. Thus we may apply Proposition 4.3 by taking $G_1 = G_2 = G_3$ to be any graph in $\mathcal{SP}(K_{3,3}, H_0)$ to obtain an infinite family of 2–edge–connected non–hamiltonian 2–factor isomorphic cubic bipartite graphs. This family gives counterexamples to the Conjecture 4.2. Note, however, that Conjecture 4.2 can be modified as follows:

Conjecture 4.4 ([3]). *Let G be a 3–edge–connected 2–factor isomorphic cubic bipartite graph. Then G is a 2–factor hamiltonian cubic bipartite graph.*

In [1, 2] Abreu, Aldred, Funk, Jackson and Sheehan proved existence theorems also for the directed and non–bipartite graphs case as follows:

For v a vertex of a digraph D, let $d^+(v)$ and $d^-(v)$ denote the out–degree and in–degree of v. We say that D is k–diregular if for all vertices v of G, we have $d^+(v) = d^-(v) = k$.

Theorem 4.5 ([1, 2]). *Let D be a digraph with n vertices and X be a directed 2-factor of D. Suppose that either*

(*a*) $d^+(v) \geq \lfloor \log_2 n \rfloor + 2$ *for all* $v \in V(D)$, *or*

(*b*) $d^+(v) = d^-(v) \geq 4$ *for all* $v \in V(D)$.

Then D has a directed 2-factor Y with $Y \ncong X$.

Corollary 4.6 ([1]). *Let G be a k–diregular directed graph. Then $k \leq 3$.*

Theorem 4.7 ([1, 2]). *Let G be a graph with n vertices and X be a 2-factor of G. Suppose that either*

(*a*) $d(v) \geq 2(\lfloor \log_2 n \rfloor + 2)$ *for all* $v \in V(G)$, *or*

(*b*) *G is a $2k$-regular graph for some $k \geq 4$.*

Then G has a 2-factor Y with $Y \ncong X$.

They have also posed the following open problems and conjecture:

Question 4.8 ([2]). *Do there exist 2–factor isomorphic bipartite graphs of arbitrarily large minimum degree?*

Question 4.9 ([2]). *Do there exist 2–factor isomorphic regular graphs of arbitrarily large degree?*

Conjecture 4.10 ([1]). *The graph K_5 is the only 2–factor hamiltonian 4–regular non–bipartite graph.*

5. Pseudo 2–factor isomorphic graphs

In [3] Abreu, Diwan, Jackson, Labbate and Sheehan extended the above mentioned results on regular 2–factor isomorphic bipartite graphs to the more general family of *pseudo 2–factor isomorphic graphs* i.e. graphs G with the property that the parity of the number of circuits in a 2–factor is the same for all 2–factors of G.

Examples of such graphs are given by all the 2–factor isomorphic regular graphs and the Pappus graph (i.e. the point/line incidence graph of the Pappus configuration). The family of pseudo 2–factor isomorphic is wider than the one of 2–factor isomorphic regular bipartite graphs:

Proposition 5.1 ([3]). *The Pappus graph P_0 is pseudo 2–factor isomorphic but not 2–factor isomorphic.*

In [3] Abreu, Diwan, Jackson, Labbate and Sheehan proved the following existence theorem:

Theorem 5.2 ([3]). *Let G be a pseudo 2–factor isomorphic k–regular bipartite graph. Then $k \in \{2, 3\}$.*

They have also shown that there are no planar pseudo 2–factor isomorphic cubic bipartite graphs.

Theorem 5.3 ([3]). *Let G be a pseudo 2–factor isomorphic cubic bipartite graph. Then G is non–planar.*

Star products preserve also the property of being pseudo 2–factor isomorphic in the family of cubic bipartite graphs.

Lemma 5.4 ([3]). *Let G be a star product of two pseudo 2–factor isomorphic cubic bipartite graphs G_1 and G_2. Then G is also pseudo 2–factor isomorphic.*

Thus $K_{3,3}$, H_0 and P_0 can be used to construct an infinite family of 3–edge–connected pseudo 2–factor isomorphic cubic bipartite graphs.

Given a set $\{G_1, G_2, \cdots, G_k\}$ of 3–edge–connected cubic bipartite graphs let $\mathcal{SP}(G_1, G_2, \cdots, G_k)$ be the set of cubic bipartite graphs which can be obtained from $G_1, G_2, \cdots, G_k$ by repeated star products. Lemma 5.4 implies that all graphs in $\mathcal{SP}(K_{3,3}, H_0, P_0)$ are pseudo 2–factor isomorphic. In [3] Abreu, Diwan, Jackson, Labbate and Sheehan conjectured that these are the only 3–edge–connected pseudo 2–factor isomorphic cubic bipartite graphs.

Conjecture 5.5 ([3]). *Let G be a 3–edge–connected cubic bipartite graph. Then G is pseudo 2–factor isomorphic if and only if G belongs to $\mathcal{SP}(K_{3,3}, H_0, P_0)$.*

Recall that McCuaig [35] has shown that a 3–edge–connected cubic bipartite graph G is det-extremal if and only if $G \in \mathcal{SP}(H_0)$.

Let G be a graph and E_1 be an edge-cut of G. We say that E_1 is a *non-trivial edge-cut* if all components of $G - E_1$ have at least two vertices. The graph G is *essentially 4–edge–connected* if G is 3–edge–connected and has no non-trivial 3–edge–cuts. Let G be a cubic bipartite graph with bipartition (X, Y) and K be a non-trivial 3–edge–cut of G. Let H_1, H_2 be the components of $G - K$. We have seen that G can be expressed as a star product $G = (G_1, y_K) * (G_2, x_K)$ where

$G_1 - y_K = H_1$ and $G_2 - x_K = H_2$. We say that y_K, respectively x_K, is the *marker vertex* of G_1, respectively G_2, *corresponding to the cut* K. Each non-trivial 3–edge–cut of G distinct from K is a non-trivial 3–edge–cut of G_1 or G_2, and vice versa. If G_i is not essentially 4–edge–connected for $i = 1, 2$, then we may reduce G_i along another non-trivial 3-edge-cut. We can continue this process until all the graphs we obtain are essentially 4–edge–connected. We call these resulting graphs the *constituents* of G. It is easy to see that the constituents of G are unique i.e. they are independent of the order we choose to reduce the non-trivial 3–edge–cuts of G.

It is also easy to see that Conjecture 5.5 holds if and only if Conjectures 5.6 and 5.7 below are both valid.

Conjecture 5.6 ([3])**.** *Let G be an essentially 4–edge–connected pseudo 2–factor isomorphic cubic bipartite graph. Then $G \in \{K_{3,3}, H_0, P_0\}$.*

Conjecture 5.7 ([3])**.** *Let G be a 3–edge–connected pseudo 2–factor isomorphic cubic bipartite graph and suppose that $G = G_1 * G_2$. Then G_1 and G_2 are both pseudo 2–factor isomorphic.*

In [3] Abreu, Diwan, Jackson, Labbate and Sheehan obtained partial results on Conjectures 5.6 and 5.7 as follows:

Theorem 5.8 ([3])**.** *Let G be an essentially 4–edge–connected pseudo 2–factor isomorphic cubic bipartite graph. Suppose G contains a 4–circuit. Then $G = K_{3,3}$.*

They used Theorem 5.8 to deduce some evidence in favour Conjecture 5.5.

Theorem 5.9 ([3])**.** *Let G be a 3–edge-connected pseudo 2–factor isomorphic bipartite graph. Suppose G contains a 4–cycle C. Then C is contained in a constituent of G which is isomorphic to $K_{3,3}$.*

Note that Theorem 5.9 generalizes Theorem 3.11 obtained by Labbate in [30] for minimally 1–factorable bipartite cubic graphs (or equivalently 2–factor hamiltonian cubic bipartite graphs) to the family of pseudo 2–factor isomorphic bipartite graph. Furthermore, Theorem 5.9 leaves the characterization of pseudo 2–factor isomorphic bipartite graph open for girth ≥ 6.

Recently, Abreu, Labbate and Sheehan [6] gave a partial solution to this open case in terms of irreducible configuration of Levi graphs as described in the next subsection.

5.1. **Irreducible pseudo 2-factor isomorphic cubic bipartite graphs.** An incidence structure is *linear* if two different points are incident with at most one line. A *symmetric configuration* n_k (or n_k *configuration*) is a linear incidence structure consisting of n points and n lines such that each point and line is respectively incident with k lines and points. Let $\mathcal{C}$ be a symmetric configuration n_k, its *Levi graph* $G(\mathcal{C})$ is a k–regular bipartite graph whose vertex set are the points and the lines of $\mathcal{C}$ and there is an edge between a point and a line in the graph if and only if they are incident in $\mathcal{C}$. We will indistinctly refer to Levi graphs of configurations as their *incidence graphs*.

It follows from Theorem 5.9 that an essentially 4–edge–connected pseudo 2–factor isomorphic cubic bipartite graph of girth greater than or equal to 6 is the Levi graph of a symmetric configuration n_3.

In 1886 V. Martinetti [33] characterized symmetric configurations n_3, showing that they can be obtained from an infinite set of so called *irreducible* configurations, of which he gave a list. Recently, Boben proved that Martinetti's list of irreducible configurations was incomplete and completed it [8]. Boben's list of irreducible configurations was obtained characterizing their Levi graphs, which he called *irreducible Levi graphs*.

In [6] Abreu, Labbate and Sheehan characterized *irreducible* pseudo 2–factor isomorphic cubic bipartite graphs (and hence gave a further partial answer to Conjecture 5.5) as follows:

Theorem 5.10 ([6]). *The Heawood and the Pappus graphs are the only irreducible Levi graphs which are pseudo 2–factor isomorphic.*

This approach is not feasible to prove Conjecture 5.5 and hence our main Conjecture 3.2 by studying the 2–factors of reducible configurations from the set of 2–factors of their underlying irreducible ones as the following discussion shows.

It is well known that the 7_3 configuration, whose Levi graph is the Heawood graph, is not Martinetti extendible and that the Pappus configuration is Martinetti extendible in a unique way; it is easy to show that this extension is not pseudo 2-factor isomorphic. Let $\mathcal{C}$ be a symmetric configuration n_3 and $\mathcal{C}$ be a symmetric configuration $(n+1)_3$ obtained from $\mathcal{C}$ through a Martinetti extension. It can be easily checked that there are 2–factors in $\mathcal{C}$ that cannot be reduced to a 2–factor in $\mathcal{C}$. On the other hand, all of its Martinetti reductions are no longer pseudo 2–factor isomorphic (for further details cf. [6]).

In the next section we will see that Conjecture 5.5 has been disproved while our main Conjecture 3.2 still holds.

6. A counterexample to the pseudo 2–factor's conjecture

In this section we present the counterexample by J. Goedgebeur to the pseudo 2–factor isomorphic bipartite's Conjecture 5.5 obtained using an exhaustive research via parallel computers partially described below (for details refer to [23]).

Using the program *minibaum* [9], he generated all cubic bipartite graphs with girth at least 6 up to 40 vertices and all cubic bipartite graphs with girth at least 8 up to 48 vertices. The counts of these graphs can be found in [23, Table 1]. Some of these graphs can be downloaded from http://hog.grinvin.org/Cubic i.e. the House of graphs [10]. He then implemented a program which tests if a given graph is pseudo 2–factor isomorphic and applied it to the generated cubic bipartite graphs. This yielded the following results:

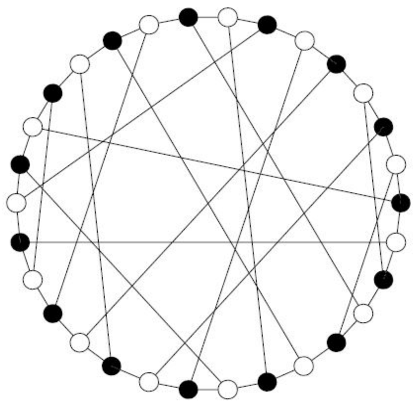
Remark 6.1 ([23]). There is exactly one essentially 4–edge–connected pseudo 2–factor isomorphic graph different from the Heawood graph and the Pappus graph among the cubic bipartite graphs with girth at least 6 with at most 40 vertices.

Remark 6.2 ([23]). There is no essentially 4–edge–connected pseudo 2–factor isomorphic graph among the cubic bipartite graphs with girth at least 8 with at most 48 vertices.

This implies that Conjecture 5.5 (and consequently also Conjecture 5.6) is false.

The counterexample has 30 vertices and there are no additional counterexamples up to at least 40 vertices and also no counterexamples among the cubic bipartite graphs with girth at least 8 up to at least 48 vertices.

The counterexample (which we will denote by $\mathcal{G}$) is shown in Figure below. $\mathcal{G}$ can also be obtained from the House of Graphs [10] by searching for the keywords *pseudo 2-factor isomorphic *counterexample* where it can be downloaded and several of its invariants can be inspected.



$\mathcal{G}$ has cyclic edge–connectivity 6, automorphism group size 144, is not vertex–transitive, has 312 2–factors and the cycle sizes of its 2–factors are: $(6, 6, 18)$, $(6, 10, 14)$, $(10, 10, 10)$ and (30). Since all 2–factor hamiltonian graphs are pseudo 2–factor isomorphic and $\mathcal{G}$ is not 2–factor hamiltonian, this implies the following:

Remark 6.3 ([23])**.** Conjecture 3.2 holds up to at least 40 vertices and holds for cubic bipartite graphs with girth at least 8 up to at least 48 vertices.

7. Strongly pseudo 2-factor isomorphic graphs

The authors and Sheehan in [5] have extended the above mentioned results on regular pseudo 2–factor isomorphic bipartite graphs to the not necessarily bipartite case introducing the family of strongly pseudo 2–factor isomorphic graphs:

Definition 7.1. Let G be a graph which has a 2–factor. For each 2–factor F of G, let $t_i^*(F)$ be the number of cycles of F of length $2i$ modulo 4. Set t_i to be the function defined on the set of 2–factors F of G by:

$$t_i(F) = \begin{cases} 0 & \text{if } t_i^*(F) \text{ is even} \\ 1 & \text{if } t_i^*(F) \text{ is odd} \end{cases} \qquad (i = 0, 1)\ .$$

Then G is said to be *strongly pseudo* 2*–factor isomorphic* if both t_0 and t_1 are constant functions. Moreover, if in addition $t_0 = t_1$, set $t(G) := t_i(F)$, $i = 0, 1$.

By definition, *if G is strongly pseudo* 2*–factor isomorphic then G is pseudo* 2*–factor isomorphic.* On the other hand there exist graphs such as the Dodecahedron which are pseudo 2–factor isomorphic but not strongly pseudo 2–factor isomorphic: the 2–factors of the Dodecahedron consist either of a cycle of length 20 or of three cycles: one of length 10 and the other two of length 5.

In the bipartite case, pseudo 2–factor isomorphic and strongly pseudo 2–factor isomorphic are equivalent.

In what follows we will denote by HU, U, SPU and PU the sets of 2–factor hamiltonian, 2–factor isomorphic, strongly pseudo 2–factor isomorphic and pseudo 2–factor isomorphic graphs, respectively. Similarly, $HU(k)$, $U(k)$, $SPU(k)$, $PU(k)$ respectively denote the k–regular graphs in HU, U, SPU and PU.

Theorem 7.2 ([5]). *Let D be a digraph with n vertices and X be a directed 2–factor of D. Suppose that either*

(*a*) $d^+(v) \geq \lfloor log_2 n \rfloor + 2$ *for all* $v \in V(D)$, *or*

(*b*) $d^+(v) = d^-(v) \geq 4$ *for all* $v \in V(D)$.

Then D has a directed 2–factor Y with a different parity of number of cycles from X.

Let $DSPU$ and DPU be the sets of digraphs in SPU and PU, i.e. strongly pseudo and pseudo 2–factor isomorphic digraphs, respectively. Similarly, $DSPU(k)$ and $DPU(k)$ respectively denote the k–diregular digraphs in $DSPU$ and DPU.

Corollary 7.3 ([5]).

(*i*) $DSPU(k) = DPU(k) = \emptyset$ *for* $k \geq 4$;

(*ii*) *If* $D \in DPU$ *then D has a vertex of out–degree at most* $\lfloor log_2 n \rfloor + 1$.

Theorem 7.4 ([5]). *Let G be a graph with n vertices and X be a 2–factor of G. Suppose that either*

(*a*) $d(v) \geq 2(\lfloor log_2 n \rfloor + 2)$ *for all* $v \in V(G)$, *or*

(*b*) *G is a $2k$–regular graph for some* $k \geq 4$.

Then G has a 2–factor Y with a different parity of number of cycles from X.

Corollary 7.5 ([5]).

(*i*) *If* $G \in PU$ *then G contains a vertex of degree at most* $2\lfloor log_2 n \rfloor + 3$;

(*ii*) $PU(2k) = SPU(2k) = \emptyset$ *for* $k \geq 4$.

We know that there are examples of graphs in $PU(3)$, $SPU(3)$, $PU(4)$ and $SPU(4)$, hence they are not empty and we have seen (cf. Conjecture 4.10) that it has been conjectured in [1] that $HU(4) = \{K_5\}$.
There are many gaps in our knowledge even when we restrict attention to regular graphs. Some questions arise naturally. Here a few of them.

Problem 7.6. *Is* $PU(2k + 1) = \emptyset$ *for* $k \geq 2$*?*

In particular we wonder if $PU(7)$ and $PU(5)$ are empty.

Problem 7.7. *Is* $PU(6)$ *empty?*

Problem 7.8. *Is* K_5 *the only 4–edge–connected graph in* $PU(4)$*?*

In this paper we have also started to investigate relations between pseudo strongly 2–factor isomorphic graphs and a class of graphs called *odd 2–factored snarks.* Next section is devoted to this class of snarks.

8. Odd 2–factored snarks

A *snark* (cf. e.g. [25]) is a bridgeless cubic graph with chromatic index four (by Vizing's theorem the chromatic index of every cubic graph is either three or four so a snark corresponds to the special case of four). In order to avoid trivial cases, snarks are usually assumed to have girth at least five and not to contain a non–trivial 3–edge cut (i.e. they are cyclically 4–edge connected).

Snarks were named after the mysterious and elusive creature in Lewis Caroll's famous poem *The Hunting of The Snark* by Martin Gardner in 1976 [20], but it was P.G. Tait in 1880 that initiated the study of snarks, when he proved that the four colour theorem is equivalent to the statement that *no snark is planar* [41]. The Petersen graph P is the smallest snark and Tutte conjectured that all snarks have Petersen graph minors. This conjecture was proven by Robertson, Seymour and Thomas (cf. [37]). Necessarily, snarks are non–hamiltonian.

The importance of the snarks does not only depend on the four colour theorem. Indeed, there are several important open problems such as the classical cycle double cover conjecture [38, 39], Fulkerson's conjecture [16] and Tutte's 5–flow conjecture [46] for which it is sufficient to prove them for snarks. Thus, minimal counterexamples to these and other problems must reside, if they exist at all, among the family of snarks.

At present, there is no uniform theoretical method for studying snarks and their behaviour. In particular, little is known about the structure of 2–factors in a given snark.

Snarks play also an important role in characterizing regular graphs with some conditions imposed on their 2–factors. Recall that a 2–factor is a 2–regular spanning subgraph of a graph G.

We say that a graph G is *odd* 2*–factored* (cf. [5]) if for each 2–factor F of G each cycle of F is odd.

By definition, *an odd* 2*–factored graph* G *is pseudo* 2*–factor isomorphic.* Note that, odd 2–factoredness is not the same as the *oddness* of a (cubic) graph (cf. e.g.[47]).

Lemma 8.1 ([5])**.** *Let* G *be a cubic* 3*–connected odd* 2*–factored graph then* G *is a snark.*

In [5] we have investigated which snarks are odd 2–factored and we have conjectured that:

Conjecture 8.2 ([5])**.** *A snark is odd* 2*–factored if and only if* G *is the Petersen graph, Blanuša* 2*, or a Flower snark* $J(t)$*, with* $t \geq 5$ *and odd.*

In [4], the authors with R. Rizzi and J. Sheehan, present a general construction of odd 2–factored snarks performing the Isaacs' dot–product [26] on edges with particular properties, called *bold–edges* and *gadget–pairs* respectively, of two snarks L and R.

Construction: Bold–Gadget Dot Product. [4]

We construct (new) odd 2–factored snarks as follows:

- Take two snarks L and R with bold–edges (cf. Definition 8.3) and gadget–pairs (cf. Definition 8.5), respectively;

- Choose a bold–edge xy in L;
- Choose a gadget–pair f, g in R;
- Perform a dot product $L \cdot R$ using these edges;
- Obtain a new odd 2–factored snark (cf. Theorem 8.7).

Note that in what follows the existence of a 2–factor in a snark is guaranteed since they are bridgeless by definition.

Definition 8.3 ([4])**.** Let L be a snark. A *bold–edge* is an edge $e = xy \in L$ such that the following conditions hold:

(i) All 2–factors of $L - x$ and of $L - y$ are odd;

(ii) all 2–factors of L containing xy are odd;

(iii) all 2–factors of L avoiding xy are odd.

Note that not all snarks contain bold–edges (cf. [4, Proposition 4.2] and [4, Lemma 5.1]). Furthermore, conditions (ii) and (iii) are trivially satisfied if L is odd 2–factored.

Lemma 8.4 ([4])**.** *The edges of the Petersen graph P_{10} are all bold–edges.*

Definition 8.5 ([4])**.** Let R be a snark. A pair of independent edges $f = ab$ and $g = cd$ is called a *gadget–pair* if the following conditions hold:

(i) There are no 2–factors of R avoiding both f, g;

(ii) all 2–factors of R containing exactly one element of $\{f, g\}$ are odd;

(iii) all 2–factors of R containing both f and g are odd. Moreover, f and g belong to different cycles in each such factor.

(iv) all 2–factors of $(R - \{f, g\}) \cup \{ac, ad, bc, bd\}$ containing exactly one element of $\{ac, ad, bc, bd\}$, are such that the cycle containing the new edge is even and all other cycles are odd.

Note that, finding gadget–pairs in a snark is not an easy task and, in general, not all snarks contain gadget–pairs (cf. [4, Lemma 5.2]).

Let $H := \{x_1y_1, x_2y_2, x_3y_3\}$ be the two horizontal edges and the vertical edge respectively (in the pentagon–pentagram representation) of P_{10} (cf. Figure 1).

Lemma 8.6 ([4])**.** *Any pair of distinct edges f, g in the set H of P_{10} is a gadget–pair.*

The following theorem allows us to construct new odd 2–factored snarks.

Theorem 8.7 ([4])**.** *Let xy be a bold–edge in a snark L and let $\{ab, cd\}$ be a gadget–pair in a snark R. Then $L \cdot R$ is an odd 2–factored snark.*

In particular, without going into lengthy details (the interested reader might find those in [4]), this method allows us to construct two instances of odd 2–factored snarks of order 26 and 34 isomorphic to those obtained by Brinkmann et al. in [11] through an exhaustive computer search on all snarks of order ≤ 36 that has allowed them to disprove the above conjecture (cf. Conjecture 8.2). Hence, our construction independently also yields counterexample for Conjecture 8.2.

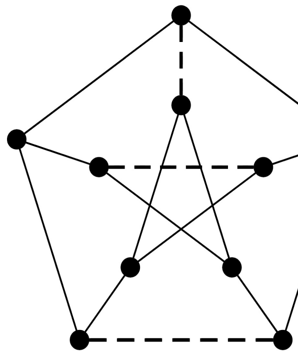


FIGURE 1. Any pair of the dashed edges is a gadget–pair in P_{10}.

To approach the problem of characterizing all odd 2–factored snarks, we consider the possibility of constructing further odd 2–factored snarks with the technique presented above, which relies in finding other snarks with bold–edges and/or gadget–pairs, Therefore, we study the existence of bold–edges and gadget–pairs in the known odd 2–factored snarks. The results obtained so far give rise to the following partial characterization:

Theorem 8.8 ([4]). *Let G be an odd 2–factored snark of cyclic edge–connectivity four that can be constructed from the Petersen graph and the Flower snarks using the bold–gadget dot product construction. Then $G \in \{P_{18}, P_{26}, P_{34}\}$.*

Finally, we pose in [4] a new conjecture about odd 2–factored snarks.

Conjecture 8.9 ([4]). *Let G be a cyclically 5–edge connected odd 2–factored snark. Then G is either the Petersen graph or the Flower snark $J(t)$, for odd $t \geq 5$.*

Remark 8.10. (*i*) A minimal counterexample to Conjecture 8.9 must be a cyclically 5–edge connected snark of order at least 36. Moreover, as highlighted in [11], order 34 is a turning point for several properties of snarks.
(*ii*) It is very likely that, if such counterexample exists, it will arise from the superposition operation by M.Kochol [27] applied to one of the known odd 2–factored snarks.
(*iii*) J. Goedgebeur [22] checked that none of the snarks (in particular those with girth 6 of order 38) that G.Brinkmann and himself generates in [12] is an odd 2–factored snark.

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