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On the palette index of a graph: the case of trees

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Abstract. The palette of a vertex v of a graph G in a proper edge-coloring is the set of colors assigned to the edges which are incident with v. The palette index of G is the minimum number of palettes occurring among all proper edge-colorings of G. After reviewing some results on the palette index of regular graphs, we consider the problem of determining the palette index for non-regular graphs. We begin by considering the family of trees. We give an upper bound for the palette index of a tree in terms of the maximum degree Δ. We show that this bound is best possible by producing, for each $\Delta \geq 3$, a tree T^{Δ} whose palette index reaches the upper bound.

1. Introduction

In this paper we consider finite simple graphs. We refer to [1] for graph theory notation and terminology which are not introduced explicitly here.

The problem of coloring the vertices or edges of a graph has always attracted people's attention. These problems become interesting and challenging when one requires that the coloring satisfies certain conditions. For instance, the vertices or edges of the graph have to be properly colored, that is, adjacent vertices or adjacent edges have to receive distinct colors. Whence the well-known definitions of proper vertex-coloring or proper edge-coloring. We recall that Brooks' theorem [3] and Vizing's theorem [4] settle the problem of a proper vertex-coloring and a proper edge-coloring with as few colors as possible, respectively. For instance, by Vizing's theorem a simple graph with maximum degree Δ admits a proper edge-coloring with Δ or $\Delta + 1$ colors. The existence of a proper vertex-coloring or edge-coloring in a graph G can be formulated in terms of the chromatic parameters $\chi(G)$ or $\chi'(G)$ that are known as the *chromatic number* or the *chromatic index* of G, respectively. Probably, these two parameters are the most popular chromatic parameters, but there are many others. For instance, the circular chromatic index [8], the list chromatic number [7], etc. The chromatic parameters can often be used

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to know something about the structure of a graph. As an example, the chromatic index tells us if a regular graph is decomposable into perfect matchings.

In this paper we consider a chromatic parameter called the *palette index* of a simple graph G. As far as we know, this parameter was introduced in [6] and denoted by $\check{s}(G)$. It can be defined as follows. Let f be a proper edge-coloring of G and let v be a vertex of G. The set of colors assigned by f to the edges incident to v is called the *palette* of v (with respect to f) and is denoted by $P_f(v)$. For every proper edge-coloring of G we can consider the set $\mathcal{P}_f$ of distinct palettes of f, namely, $\mathcal{P}_f = \{P_f(v) : v \in V(G)\}$. The cardinality of $\mathcal{P}_f$ is at most $|V(G)|$. The palette index of G is the minimum number of distinct palettes taken over all proper edge-colorings of G, namely, $\check{s}(G) = \min\{|\mathcal{P}_f| : f \text{ proper edge-coloring of } G\}$. As shown in [6], the palette index of a regular graph is 1 if and only if the graph is class 1 (see [4] for the definition of graphs of class 1 and 2 according to Vizing's Theorem). Moreover, it is different from 2. As remarked in [2], the palette index of a d-regular graph G of class 2 satisfies the inequalities $3 \leq \check{s}(G) \leq d+1$. Whence $\check{s}(G) = 3$ if G is a 2-regular graph of class 2. The behavior of the palette index of a cubic graph is described by the following result.

Theorem 1 ([6])**.** *Let G be a cubic graph. Then*

$$\check{s}(G) = \begin{cases} 1 \quad , & \text{if } G \text{ is class } 1 \\ 3 \quad , & \text{if } G \text{ is class } 2 \text{ and has a perfect matching} \\ 4 \quad , & \text{if } G \text{ is class } 2 \text{ and has no perfect matching} . \end{cases}$$

For the palette index of a 4-regular graph no such complete description is available. We do know that the admissible values are 1, 3, 4 and 5 and for each such value it is possible to find a 4-regular graph whose palette index assumes this value see [2].

The palette index of the complete graph K_n, $n \geq 4$, shows a different behavior, namely it only assumes three values even though the range of admissible values is much wider. More specifically,

Theorem 2 ([6])**.** *Let K_n be the complete graph with n vertices. Then*

$$\check{s}(K_n) = \begin{cases} 1 \quad , & \text{if } n \equiv 0 \pmod 2 \\ 3 \quad , & \text{if } n \equiv 3 \pmod 4 \\ 4 \quad , & \text{if } n \equiv 1 \pmod 4 . \end{cases}$$

In this paper we consider the palette index of a non-regular graph, namely, a tree. In Theorem 3, we give an upper bound for the palette index of a tree and show that there exists a tree whose palette index reaches this bound.

2. The palette index of a tree

We begin this section with a general upper bound for the palette index of a tree.

Theorem 3. *Let T be a tree of maximum degree Δ. Then,*

$$\check{s}(T) \leq \sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil .$$

Proof. Denote by $\{c_0, c_1, \cdots, c_{\Delta-1}\}$ a set of Δ colors. We construct the sets of colors $P^i_j = \{c_{ij}, c_{ij+1}, c_{ij+2}, \cdots, c_{ij+i-1}\}$ of cardinality i, where $i = 1, \cdots, \Delta$, $j = 0, \cdots, \lceil \Delta/i \rceil - 1$ and all indices are taken modulo Δ. Observe that, for any fixed i, the number of sets P^i_j is exactly equal to $\lceil \Delta/i \rceil$. Then, the assertion is proved if we furnish a Δ-edge-coloring of T such that the palette of each vertex is one of the sets P^i_j defined above.

Choose one of the vertices of T, say v, to be the root of the tree. Assign arbitrarily to the edges incident with v the colors of the palette $P_0^{\delta(v)}$, where $\delta(v)$ denotes the degree of v. Start a breadth-first search on the vertices of T. Each time we visit a new vertex u of T, exactly one of the edges incident with u, say ux, is already colored. We consider (one of) the sets $P_j^{\delta(u)}$ containing the color of ux (i.e. this set does exist since for each i the sets P^i_j cover all colors at least once). We assign the colors of $P_j^{\delta(u)}$ different from the color of ux to the remaining edges incident with u. It is clear that we can continue this process until the entire edge-set of T is colored and the assertion follows.

□

Remark. The upper bound of Theorem 3 holds for an arbitrary forest of maximum degree Δ: the procedure described in the proof can be applied separately to the connected components of the forest.

Our next aim is to construct for each $\Delta \geq 3$ a tree T^Δ having maximum degree Δ reaching the upper bound of Theorem 3, thus proving that the previous upper bound is best possible. Note that the path of length 2 is a tree of maximum degree 2 requiring exactly 3 palettes (which is exactly the upper bound for this case). First of all, we construct a forest F^Δ having palette index equal to $\sum_{i=1}^{\Delta} \lceil \Delta/i \rceil$: the tree T^Δ will be obtained by suitably modifying and joining the connected components of F^Δ.

In order to construct F^Δ, we begin by defining Δ-1 subtrees T_r as follows.

For $r = 1$, we set

$$V(T_1) = \{x^1\} \cup \{y^1_i : i = 1, \cdots, \Delta\} ,$$

$$E(T_1) = \{[x^1, y^1_i] : i = 1, \cdots, \Delta\} .$$

For $r = 2, \cdots, \Delta - 1$, we set

$$V(T_r) = \{x^r\} \cup \{y^r_i \, : \, i = 1, \cdots, \Delta\} \cup \{z^r_{i,j} \, : \, i = 1, \cdots, r, \; j = 1, \cdots, r-1\} ,$$

$$E(T_r) = \{[x^r, y^r_i] \, : \, i = 1, \cdots, \Delta\} \cup \bigcup_{i=1}^{r} \{[y^r_i, z^r_{i,j}] \, : \, j = 1, \cdots, r-1\} .$$

The forest F^Δ is the disjoint union of the trees T_r, that is:

$$V(F^\Delta) = \bigcup_{r=1}^{\Delta-1} V(T_r) ,$$

$$E(F^\Delta) = \bigcup_{r=1}^{\Delta-1} E(T_r) .$$

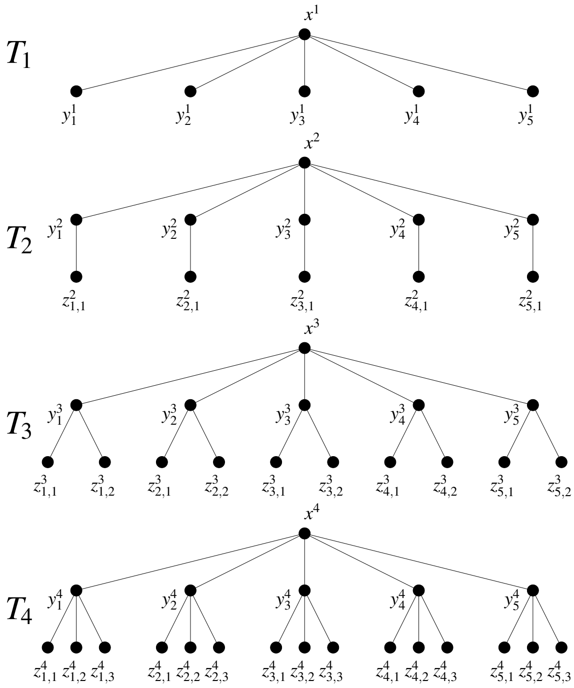

FIGURE 1. The forest F^5.

Proposition 1.

$$\check{s}(F^{\Delta}) = \sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil .$$

Proof. By the remark following Theorem 3 we have

$$\check{s}(F^{\Delta}) \leq \sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil .$$

The assertion follows if we prove the inequality

$$\check{s}(F^{\Delta}) \geq \sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil .$$

The edges of T_1 have Δ distinct colors in every proper edge-coloring of F^{Δ}, since they are adjacent. Hence, there are at least Δ distinct palettes of cardinality 1 in every edge-coloring of F^{Δ}. Furthermore, there are exactly Δ vertices of degree i in T_i, for every $i = 2, \cdots, \Delta - 1$. At most i of them may have the same palette in a proper edge-coloring, otherwise there should be at least two edges incident with x_i sharing one and the same color, a contradiction. Consequently, the number of palettes of cardinality i is at least $\lceil \Delta/i \rceil$ for every $i = 2, \cdots, \Delta - 1$. Finally, there is at least one palette of cardinality Δ occurring at vertices of degree Δ, hence the assertion follows.

□

In order to construct T^{Δ}, we shall substitute the tree T_1 by a new tree T_1', defined as follows:

$$V(T_1') = \{x^1, y_1^1, y_2^1\} \cup \{z_{i,j}^1 \,:\, i = 1,2,\ j = 1, \cdots, \Delta - 1\}$$

$$E(T_1') = \{[x^1, y_i^1] \,:\, i = 1,2\} \cup \bigcup_{i=1}^{2} \{[y_i^1, z_{i,j}^1] \,:\, j = 1, \cdots, \Delta - 1\}\ .$$

The tree T^{Δ} is obtained by taking the trees $T_1', T_2, \cdots, T_{\Delta-1}$ and joining them by introducing new edges.

Here is a complete description of T^{Δ}:

$$V(T^{\Delta}) = V(T_1') \cup \bigcup_{r=2}^{\Delta-1} V(T_r)$$

$$E(T^{\Delta}) = E(T_1') \cup \bigcup_{r=2}^{\Delta-1} E(T_r) \cup \{[x^1, z_{1,1}^2]\} \cup \{[z_{1,1}^r, z_{1,1}^{r+1}] \,:\, r = 2, \cdots, \Delta - 2\}\ .$$

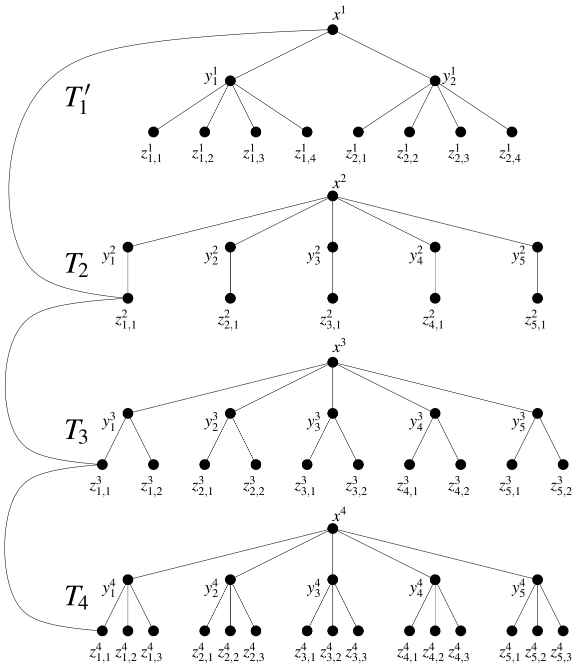

FIGURE 2. The tree T^5.

Proposition 2.

$$\check{s}(T^{\Delta}) = \sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil \ .$$

Proof. Looking at the vertices $y_1^r, \cdots, y_\Delta^r$, $r \geq 2$, we observe at least $\lceil \Delta/r \rceil$ palettes of cardinality r: the argument is the same as in the proof of Proposition 1. Looking at the $\Delta - 1$ vertices $z_{1,1}^1, \cdots, z_{1,\Delta-1}^1$, we observe $\Delta - 1$ distinct palettes of cardinality 1. The total number of palettes that we have counted thus far is one less than the required number: we shall now figure out that we can always single out a further palette. Assume the vertices y_1^1 and y_2^1 (which have degree Δ) have distinct palettes: since we have only counted 1 palette of cardinality Δ thus far, we have just introduced an extra palette. Assume now the vertices y_1^1 and y_2^1 have the same palette: the edges $x^1y_1^1$ and $x^1y_2^1$ have distinct colors, consequently we observe more than $\Delta - 1$ palettes of cardinality 1 on the vertices $z_{1,1}^1, \cdots, z_{1,\Delta-1}^1, z_{2,1}^1, \cdots, z_{2,\Delta-1}^1$, thus producing an extra palette again.

□

3. Final remarks and an open problem

As far as we know, the unique upper bound for the palette index of a general simple graph in terms of Δ is the trivial one. We mean that every admissible subset of the color-set in a proper edge-coloring appears as a palette. More precisely, by Vizing's Theorem a graph of maximum degree Δ admits a $(\Delta + 1)$-edge-coloring, then the palette index is at most $2^{\Delta+1} - 2$. However, there exist classes of graphs for which the palette index is bounded by a constant function of Δ, such as complete graphs. It is not hard to construct a family of graphs for which the palette index grows linearly in Δ. For instance the star of maximum degree Δ.

In the present paper, we show the existence of a family of trees whose palette index is $\sum_{i=1}^{\Delta} \lceil \Delta/i \rceil$: its leading asymptotic behavior is thus $\Delta \ln(\Delta)$. We recall that one of the well-known definitions for the Euler-Mascheroni constant γ is

$$\lim_{\Delta \to +\infty} \left(\sum_{i=1}^{\Delta} \frac{1}{i} - \ln(\Delta) \right) = \gamma \ .$$

Furthermore, the following relation holds (see [5]):

$$\lim_{\Delta \to +\infty} \frac{1}{\Delta} \sum_{i=1}^{\Delta} \left(\left\lceil \frac{\Delta}{i} \right\rceil - \frac{\Delta}{i} \right) = \gamma \ .$$

By using the previous two relations, it is not hard to find out that $\sum_{i=1}^{\Delta} \lceil \Delta/i \rceil$ asymptotically grows as $\Delta \ln(e^{2\gamma} \Delta)$.

In view of previous remarks, we would like to leave the following open problem: find a class of graphs for which the palette index grows exponentially in terms of Δ or prove that it does not exist.

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