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Recent results on the adjacent vertex distinguishing chromatic index of the direct product of graphs

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Abstract. The adjacent vertex distinguishing chromatic index of a graph G is the minimum number of colors in a proper edge coloring of G which distinguishes adjacent vertices. In this paper we collect several recent results involving this parameter and the usual chromatic index in relation to the direct product of graphs.

1. INTRODUCTION

Let $G = (V, E)$ be a finite simple undirected graph. An *edge coloring* of G is a map α from E to a finite set of colors C. The coloring α is *proper* if $\alpha(e_1) \neq \alpha(e_2)$ whenever edges e_1, e_2 are adjacent. The *chromatic index* of G is the minimum number of colors $\chi'(G)$ in a proper edge coloring of G. By the well-known Vizing's Theorem $\chi'(G)$ is either $\Delta(G)$, the maximum degree of G (G is *Class* 1) or $\Delta(G)+1$ (G is *Class* 2) [10]. Note that deciding whether a graph G is Class 1 is an NP-complete problem even for cubic graphs ([14]).

The color set of a vertex $u \in V$ with respect to the coloring α is the set $C_\alpha(u)$, or simply $C(u)$, of colors assigned by α to the edges incident to u.
The coloring α is *adjacent vertex distinguishing* (*avd* for short) if $uv \in E(G)$ implies $C_\alpha(u) \neq C_\alpha(v)$ ([1], [2]). Alternative terminologies for this kind of coloring are *adjacent strong edge coloring* [31] and *neighbor-distinguishing coloring* [8]. The *adjacent vertex distinguishing chromatic index* of the graph G is the minimum number $\chi'_a(G)$ of colors in a proper avd edge coloring of G. Since $\chi'_a(K_1) = 0$ and the graph K_2 does not admit an avd coloring at all, when analyzing the invariant $\chi'_a(G)$ it is sufficient to restrict our attention to connected graphs of order at least 3. This is justified by the obvious fact that if G is a disconnected graph with (non-K_2) components G_i, $1 \leq i \leq q$, then $\chi'_a(G) = \max(\chi'_a(G_i) : 1 \leq i \leq q)$.

In [31] the invariant $\chi'_a(G)$ was introduced and treated for classes of graphs with simple structure such as trees, cycles, complete graphs, complete bipartite graphs;

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in particular $\chi'_a(C_5) = 5$. The results led the authors of the introductory paper [31] to formulate

Conjecture 1.1. *If a connected graph $G \neq C_5$ has at least 3 vertices, then $\chi'_a(G) \leq \Delta(G) + 2$.*

Conjecture 1.1 is known to be true for

- subcubic graphs, bipartite graphs ([1]),
- graphs G with $\mathrm{mad}(G) < 3$ ([29]), where $\mathrm{mad}(G)$ (the parameter called the *maximum average degree* of the graph G) is defined by $\mathrm{mad}(G) := \max(2|E(H)|/|V(H)| : H \subseteq G)$,
- planar graphs G with $\Delta(G) \geq 12$ ([17]).

There are classes of graphs for which $\Delta(G)+1$ is a sharper upper bound for $\chi'_a(G)$.

- graphs satisfying either $\mathrm{mad}(G) < 5/2$ and $\Delta(G) \geq 4$ or $\mathrm{mad}(G) < 7/3$ and $\Delta(G) = 3$ [29],
- bipartite planar graphs with $\Delta(G) \geq 12$ ([8]).

The best general bound so far is given by Hatami [13] who proved that $\chi'_a(G) \leq \Delta(G) + 300$ if $\Delta(G) > 10^{20}$.

The avd chromatic index was discussed also for graphs resulting from binary graph operations. A good information about such operations can be found in a monograph [19] by Imrich and Klavžar. One can mention the Cartesian product ([2, 3]), the direct product ([11], [26], [3]), the strong product [3] and the lexicographic product [3].

The *direct product* of graphs G and H is the graph $G \times H$ with $V(G \times H) := V(G) \times V(H)$ and $E(G \times H) := \{(u,x)(v,y) \,:\, uv \in E(G),\ xy \in E(H)\}$. This product is commutative and associative (up to isomorphisms). This product, also referred to as tensor product, Kronecker product, categorical product and conjunction, has applications in engineering, computer science and related disciplines. The direct product of a bipartite graph with any other graph is bipartite. So every product of a graph G by a path or a cycle is bipartite except when the cycle has odd length and G is not bipartite.

Let $N_G(u)$ be the set of all neighbors and $d_G(u) = |N_G(u)|$ the degree of a vertex $u \in V(G)$; then $N_{G\times H}(u,x) = N_G(u) \times N_H(x)$ and $d_{G\times H}(u,x) = d_G(u)d_H(x)$, where $(u,x) \in V(G \times H)$.

As usual, we denote by C_k, P_k and K_k a cycle, a path and a complete graph on k vertices, respectively. Further, we assume that $V(P_k) = [1,k]$, $E(P_k) = \{\{i, i+1\}, : i \in [1, k-1]\}$ for $k \in [1,\infty)$ and $V(C_k) = [1,k]$, $E(C_k) = \{\{i,(i+1)\} \,:\, i \in [1,k]\}$ for $k \in [3,\infty)$ and the indices are mod k.
For terminologies not defined here we follow [30].

Consider a set $D \subseteq [1, \Delta(G)]$; the graph G is said to be *D-neighbor irregular*, if for any $d \in D$ the set $V_d(G) := \{u \in V(G) \,:\, d_G(u) = d\}$ is independent. In other words, if an edge uv in a D-neighbor irregular graph joins vertices of the same degree d, then $d \in [1, \Delta(G)] \setminus D$.

When working with the avd chromatic index, there are several useful observations following directly from the definitions and from the fact that the color set of a vertex of degree d is of cardinality d.

Proposition 1.2. $\Delta(G) \le \chi'(G) \le \chi'_a(G)$ *for any graph* G.

Proposition 1.3. *If a graph* G *has adjacent vertices of degree* $\Delta(G)$*, then* $\chi'_a(G) \ge \Delta(G) + 1$.

Proposition 1.4. $\chi'_a(G) = \chi'(G)$ *for any* $[1, \Delta(G)]$*-neighbor irregular graph* G.

This paper is organized as follows. In Section 2 we investigate the avd chromatic index of the direct product of a graph by a path. In section 3 we consider the case of the direct product with a regular graph. In Section 4 we study the avd chromatic index of the direct product of a graph by a cycle and we investigate particular strings of sequences. In Section 5 we determine the usual chromatic index of the direct product of a simple graph with the total graph T_2, which is the complete graph K_2 with a loop added to each vertex, and the avd chromatic index of the direct product of a simple graph by a graph with a loop in every vertex. In the last section we describe some cases in which the usual chromatic index turns out to be equal to the avd chromatic index.

2. The direct product of a graph by a path

2.1. **The direct product of a graph by K_2.** In this section we investigate the avd chromatic index of the direct product of a graph G by K_2, the path of length 1; these graphs are extensively studied in [28].

Let $\{u_1, u_2\}$ be the vertex set of K_2. For every vertex v_i of G, we have that $w_i = (v_i, u_1)$ and $w'_i = (v_i, u_2)$ are vertices of $G \times K_2$. Moreover if $v_i v_j$ is an edge of G, then $w_i w'_j$ and $w'_i w_j$ are edges of $G \times K_2$. Clearly,

$$d_{G\times K_2}(v, u_i) = d_G(v) = d_{G\times K_2}(v, u_{3-i}),\ v \in V(G),\ i = 1, 2\ .$$

Theorem 2.1 ([26])**.** *For any graph* G*, we have the inequality*

$$\chi'_a(G \times K_2) \le \chi'_a(G)\ . \tag{1}$$

In particular, if G *is bipartite, then*

$$\chi'_a(G \times K_2) = \chi'_a(G)\ . \tag{2}$$

A stronger result has been proved in [18].

Theorem 2.2. $\chi'_a(G \times K_2) \le \min(\chi'_a(G), \Delta(G) + 2)$ *for every graph* G.

Example. Since $C_n \times K_2 = 2C_n$ when n is even, and $C_n \times K_2 = C_{2n}$ when n is odd, using the avd chromatic index of the cycles obtained in [31], we have

$$\chi'_a(C_n \times K_2) = \begin{cases} 3 & n \equiv 0 \pmod 3 \\ 4 & n \not\equiv 0 \pmod 3\ . \end{cases}$$

Notice that (1) is a strict inequality for $n = 5$, since $\chi'_a(C_5) = 5$ and $\chi'_a(C_5 \times K_2) = 4$.

Another case of an equality is given by the following theorem.

Proposition 2.3 ([26]). *Let G be a graph with at least two adjacent vertices of maximum degree. If $\chi'_a(G) = \Delta(G) + 1$, then we have the identity*

$$\chi'_a(G \times K_2) = \Delta(G) + 1 . \tag{3}$$

An edge coloring β of the graph $G \times K_2$ is said to be *symmetric* provided that $C_\beta(v, u_1) = C_\beta(v, u_2)$ for every $v \in V(G)$.

An edge coloring $\alpha : E(G) \to C$ induces in a natural way the edge coloring $\alpha^\times : E(G \times K_2) \to C$ defined so that

$$\alpha^\times((v, u_1)(w, u_2)) := \alpha(vw) =: \alpha^\times((v, u_2)(w, u_1)) \quad , \quad vw \in E(G) . \tag{4}$$

From the definition it immediately follows:

Proposition 2.4 ([18]). *Let α be an edge coloring of a graph G. Then*

1. *$\alpha^\times$ is a symmetric edge coloring of the graph $G \times K_2$;*
2. *$\alpha^\times$ is proper if α is proper;*
3. *$\alpha^\times$ is avd if α is avd.*

Proposition 2.4 yields the inequality $\chi'(G \times K_2) \leq \chi'(G)$. However, it is possible to prove more:

Theorem 2.5 ([18]). *For any graph G there is a symmetric proper edge coloring of the graph $G \times K_2$ that uses $\Delta(G)$ colors.*

2.2. **The direct product of a graph by a path of length greater than 1.** In this section we study the direct product of simple graph G with a path P_m different from K_2.
Let $V(P_m) = \{u_1, u_2, \cdots, u_m\}$, where $m > 2$, and $V(G) = \{v_1, v_2, \cdots, v_n\}$, $n > 1$. For $i = 1, \cdots, m-1$, we denote by $S(u_i u_{i+1})$ the subgraph of P_m induced by the edge $u_i u_{i+1}$. Notice that the direct product $G \times P_m$ is the union of the $m-1$ edge disjoint subgraphs $H_i = G \times S(u_i u_{i+1})$, $1 \leq i \leq m-1$. The edges of H_i are $(v_t, u_i)(v_j, u_{i+1})$, when v_t, v_j are adjacent vertices of G. Moreover, H_i is bipartite and it is not connected, consisting of two components isomorphic to G if and only if G is bipartite. In any case, the maximum degree of H_i coincides with the maximum degree of G.

Because of Proposition 1.3, if H is a cycle or a path of order at least 3, then $\chi'_a(G \times H)$ can be equal to $\Delta(G \times H) = 2\Delta(G)$ only if $G \times H$ does not have adjacent vertices of degree $2\Delta(G)$. Such a condition is fulfilled only if either $H = P_3$ or G does not have adjacent vertices of degree $\Delta(G)$.

Theorem 2.6 ([18, 26]). *Let G be a graph having maximum degree $\Delta(G)$; then $\chi'_a(G \times P_3) = 2\Delta(G) = \Delta(G \times P_3)$.*

To generalize the result obtained in Theorem 2.6, we will employ the concept of avd d-sequence [11]. For any positive integer $d \in \mathbb{N}$, an *avd d-sequence* of length m is a sequence $(S_1, S_2, \cdots, S_m)$ of d-subsets of a $(2d+1)$-set C satisfying the following properties:

(S1) every set S_i is disjoint from S_{i-1} and S_{i+1},

(S2) the sets S_{i-1} and S_{i+1} are distinct for every $i = 2, \cdots, m-1$. An avd sequence of length m is *cyclic* when the properties (S1) and (S2) are satisfied for every $i = 1, 2, \cdots, m$ (taking the indices modulo m).

An example of an avd d-sequence of length $m > 1$ is given by the sequence $C_{2d+1} = (Q_1, Q_2, \cdots, Q_m)$ where the sets Q_i are obtained by taking d consecutive elements of the set $C = \{1, 2, \cdots, 2d+1\}$ in the following way: $Q_1 = \{1, 2, \cdots, d\}$, $Q_2 = \{d+1, \cdots, 2d\}$, $Q_3 = \{2d+1, 1, 2, \cdots, d-1\}$, and so on.

The *concatenation* of two d-sequences $R = (R_1, R_2, \cdots, R_p)$ and $T = (T_1, T_2, \cdots, T_q)$ is the d-sequence $RT = (R_1, R_2, \cdots, R_p, T_1, T_2, \cdots, T_q)$. If R and T are avd, T_1 is disjoint from R_p and distinct from R_{p-1} and T_2 is distinct from R_p, then also RT is avd. In addition, if T_q is disjoint from R_1 and distinct from R_2, and T_{q-1} is distinct from R_1, then RT is cyclic avd.

Theorem 2.7 ([11])**.** *For $d \geq 3$, there exists a cyclic avd d-sequence of every even length $m > 4$.*

For example, we may find for $d = 3$ the sequences:
$D_6 : 123, 456, 127, 345, 126, 457,$
$D_8 : 123, 456, 127, 345, 267, 145, 236, 457,$
$D_{10} : 123, 456, 127, 345, 267, 145, 367, 245, 136, 457.$

In the following Theorem we prove the existence of the cyclic avd d-sequence of every odd length, slightly different from the proof of the same theorem [11].

Theorem 2.8. *For $d > 2$, there exists a cyclic avd d-sequence of every odd length $m \geq 2d + 1$.*

Proof. Notice that the d-sequence $C_{2d+1} = (Q_1, Q_2, \cdots, Q_m)$, where $m = 2d+1$, turns out to be cyclic.
Now consider the d-sequence C_{2d+3} of length $2d + 3$ obtained from C_{2d+1} by the replacement of Q_{2d} by $R_{2d} = (Q_{2d} \cup \{d+2\}) \setminus \{3\}$ and Q_{2d+1} by $R_{2d+1} = (Q_{2d+1} \cup \{3\}) \setminus \{d + 2\}$ and the addition of the two sets $R_{2d+2} = \{1, 2, 4, \cdots, d + 1\}$ and $R_{2d+3} = \{d + 2, \cdots 2d + 1\}$.
Moreover consider the d-sequence C_{2d+5} obtained from C_{2d+1} by replacing Q_{2d+1} by the set $T_{2d+1} = (Q_{2d+1} \cup \{1\}) \setminus \{2d+1\}$ and the addition of the four d-sets $T_{2d+2} = (Q_1 \cup \{2d+1\}) \setminus \{1\}$, $T_{2d+3} = (Q_2 \cup \{1\}) \setminus \{2d\}$, $T_{2d+4} = (Q_3 \cup \{d, 2d\}) \setminus \{1, 2d+1\}$, $T_{2d+5} = (Q_4 \cup \{2d + 1\}) \setminus \{d\}$.
It is not difficult to prove that C_{2d+3} and C_{2d+5} are cyclic avd. Notice that the first two sets of the sequence C_{2d+1} coincide with the first two sets of the cyclic avd sequences of even length. This allows to concatenate the avd sequences of even length $h \geq 6$ with C_{2d+1}, thus obtaining all the cyclic avd sequences of every odd length $m \geq 2d + 1$.

□

For $d = 3$, an example of avd 3-sequences of length 7, 9, 11 is the following:
$C_7 : 123, 456, 127, 345, 167, 234, 567,$
$C_9 : 123, 456, 127, 345, 167, 245, 367, 124, 567,$
$C_{11} : 123, 456, 127, 345, 167, 234, 156, 237, 145, 236, 457.$

Now, we can prove the following

Theorem 2.9 ([26])**.** *For any bipartite graph G and for any path P_m with $m \geq 4$, we have the inequalities*

$$2\Delta(G) \leq \chi'_a(G \times P_m) \leq 2\Delta(G) + 1 . \tag{5}$$

In theorem 10 of [18], a similar result holds for a graph G not necessarily bipartite.

Theorem 2.10 ([26]). *For any bipartite graph G having at least two adjacent vertices of maximum degree and for any path P_m with $m > 3$, we have the identity*

$$\chi'_a(G \times P_m) = 2\Delta(G) + 1 \ . \tag{6}$$

Proposition 2.11 ([26]). *Let G be a simple bipartite graph and let $m_1, \cdots, m_k \geq 3$ with $k \geq 1$. Then for the direct product $G \times P_{m_1} \times \cdots \times P_{m_k}$ we have the following cases.*

1. *If $m_1 = \cdots = m_k = 3$, then we have the identity*
$$\chi'_a(G \times P_{m_1} \times \cdots \times P_{m_k}) = 2^k \Delta(G) \ .$$
2. *If $\max(m_1, \cdots, m_k) \geq 4$, then we have the inequalities*
$$2^k \Delta(G) \leq \chi'_a(G \times P_{m_1} \times \cdots \times P_{m_k}) = 2^k \Delta(G) + 1 \ .$$
In particular, if G has two adjacent vertices of maximum degree, then we have the identity
$$\chi'_a(G \times P_{m_1} \times \cdots \times P_{m_k}) = 2^k \Delta(G) + 1 \ .$$

3. Regular graphs

Let G be a simple, regular graph of degree $d = \Delta(G)$, having $n > 1$ vertices. Given a simple graph H and an integer $d > 1$, dH denotes the multigraph obtained from H by replacing every edge e by d edges having the same vertices of e.

Proposition 3.1 ([11]). *For a d-regular graph G and an arbitrary graph H, we have*

$$\chi'_a(G \times H) \leq \chi'_a(dH)$$

Note that for $m > 3$, there are adjacent vertices of degree $2d$ and $\chi'_a(G \times P_m) \geq 2d + 1$. In particular, using previous Proposition, the following result is proved in [11]:

Theorem 3.2 ([11]). *Let G be a d-regular graph and $m > 2$ a positive integer. Then*

$$\chi'_a(G \times P_m) = \chi'_a(dP_m) = \begin{cases} 2d & \text{for } m = 3 \\ 2d + 1 & \text{for } m > 3 \end{cases} \tag{7}$$

4. The direct product of a graph by a cycle

In this section we investigate the avd chromatic index of $G \times C_k$, with particular attention to a graph G which is $\{\Delta(G)\}$-neighbor irregular.

4.1. Δ-neighbor irregular graphs.

Theorem 4.1 ([18]). *Suppose that for a graph G and $k \in [4, \infty)$ one of the following assumptions is fulfilled:*

(*i*) *G is $\{\Delta(G)\}$-neighbor irregular and $k \equiv 0 \pmod 4$;*

(*ii*) *$\Delta(G) \equiv 1 \pmod 2$, G is $\{\Delta(G)\}$-neighbor irregular and $k \equiv 2 \pmod 4$;*

(*iii*) $\Delta(G) \equiv 0 \pmod 2$, *G is* $\{\Delta(G)/2, \Delta(G)\}$*-neighbor irregular and* $k \equiv 2 \pmod 4$.

Then $\chi'_a(G \times C_k) = 2\Delta(G) = \Delta(G \times C_k)$.

As a consequence,

Theorem 4.2 ([18]). *If G is a* $\{\Delta(G)\}$*-neighbor irregular graph and* $k \in [4, \infty)$*, then* $\chi'_a(G \times P_k) = 2\Delta(G) = \Delta(G \times P_k)$.

Theorem 4.3 ([18]). *Suppose that* $k \in [3, \infty)$ *and G is a D-neighbor irregular bipartite graph, where either* $\Delta(G)$ *is odd and* $D = \{\Delta(G)\}$ *or* $\Delta(G)$ *is even and* $D = \{\Delta(G)/2, \Delta(G)\}$. *Then* $\chi'_a(G \times C_k) = 2\Delta(G) = \Delta(G \times C_k)$.

In [18] the following Theorem, which confirms Conjecture 1.1 for the graphs $G \times C_k$, has been obtained by using particular sequences of positive integers, named *appropriate*.

Theorem 4.4 ([18]). *If* $\Delta(G) \in [3, \infty)$, $k \in [6, \infty)$*, and either k is even or* $k \geq 2\Delta(G) + 1$*, then* $\chi'_a(G \times C_k) \leq 2\Delta(G) + 1$.

Moreover, if we consider the direct product of a d-regular graph G with a cycle, in [11] the following result was proved using cyclic *avd* d-sequences of even and odd length.

Theorem 4.5 ([11]). *Let G be a d-regular graph, where* $d > 2$*, and a positive integer* $m > 4$*, when even, or* $m \geq 2d + 1$*, when odd. Then*

$$\chi'_a(G \times C_m) = 2d + 1 \; . \tag{8}$$

4.2. **The direct product of two cycles.** Note that Theorem 4.4 does not cover the case $\Delta(G) = 2$. However, if G is a connected graph of maximum degree 2, then G is either a cycle or a path. The direct product of a cycle and a path was analyzed in Theorem 3.2. In the rest of this section we deal with the direct product of two cycles or two paths or a cycle by a path. In [20] decompositions of these graphs into cycles of uniform length are studied.

Note that $\chi'_a(C_m \times C_n)$ is known in the following cases treated in [11]:

- at least one of m, n is even and greater than 4,
- both m, n are odd and greater than 7,
- $m = n \in [3, 4]$.

Other cases are considered in [18].

Theorem 4.6 ([18]). *If* $(m, n) \in [3, \infty) \times [3, \infty)$ *and* $(\{m\} \cup \{n\}) \cap ([3, \infty) \setminus \{3, 4, 7\}) \neq \emptyset$*, then* $\chi'_a(C_m \times C_n) = 5$.

Theorem 4.7 ([18]). *If* $(m, n) \in [3, \infty) \times [3, \infty)$*, then* $5 \leq \chi'_a(C_m \times C_n) \leq 6 = \Delta(C_m \times C_n) + 2$.

Note that there are pairs (m, n), for which the upper bound in Theorem 4.7 applies. Namely, according to [11], $\chi'_a(C_3 \times C_3) = 6 = \chi'_a(C_4 \times C_4)$.

Finally, we turn to the direct product of two paths. Recall that from [11] it is known that $\chi'_a(P_m \times P_n) = 2$ if $(m, n) \in \{(2, 3), (3, 2)\}$ and $\chi'_a(P_m \times P_n) = 3$ if $\min(m, n) = 2$ and $\max(m, n) \geq 4$. By Theorem 2.6 we have $\chi'_a(P_m \times P_n) = 4$ provided that $\min(m, n) = 3$.

Theorem 4.8 ([18])**.** *If* $(m, n) \in [4, \infty) \times [4, \infty)$*, then* $\chi'_a(P_m \times P_n) = 5$.

4.3. **Disarranged strings of sequences.** In this section $R = (a_1, a_2, \cdots, a_n)$ and $S = (b_1, b_2, \cdots, b_n)$ denote n-sequences of distinct elements, sharing exactly $n-1$ elements. We associate with R and S the bijection f defined by the relation $f(a_i) = b_i$, $1 \leq i \leq n$, and represented in two line notation by the $2 \times n$ array

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} . \tag{9}$$

Let u and v be the different elements which belong to the first and the second line respectively. The function f is formed by the linear ordering $l(f) = (u, f(u), f^2(u), \cdots, f^{k-1}(u), v)$, where k is the minimum positive integer such that $f^k(u) = v$, and a permutation $\pi(f)$ on the remaining elements [6]. In [5] a similar function, called widened permutation, is investigated in the context of the theory of species of Joyal.

Definition 4.1. R is said *disarranged* with respect to S if for every set $\{i_1, i_2, \cdots, i_r\} \subseteq [n]$ $\{a_{i_1}, a_{i_2}, \cdots, a_{i_r}\} \neq \{b_{i_1}, b_{i_2}, \cdots, b_{i_r}\}$.

The sequences R and S are called 1-*disarranged* if there exists an index $i \in [n]$ such that $a_i = b_i$ and the sequences, obtained from R and S after deleting a_i and b_i, are disarranged. In this case we say that the pair (R, S) is 1-disarranged.

Now we extend the definition 4.1 to a string of n-sequences.

Definition 4.2. Let $n, m \in \mathbb{N}$; an m-string $(S_1, S_2, \cdots, S_m)$ of n-sequences, is called *disarranged* if:

(A1) S_i is disjoint from S_{i-1} and S_{i+1},

(A2) S_{i-1} and S_{i+1} are disarranged.

for every $i = 2, \cdots, m-1$.

A disarranged m-string of n-sequences is *circular* when the properties (A1) and (A2) are satisfied for every $i = 1, 2, \cdots, m$ (taking the indices modulo m).
An m-string of n-sequences is 1-*disarranged* if there exists at least one index i such that S_{i-1} or S_{i+1} form a 1-disarranged pair.
An m-string of n-sequences is $(0,1)$-disarranged if it is either disarranged or 1-disarranged.

The main result of [4] is the following Theorem.

Theorem 4.9 ([4])**.** *Let m, n be positive integers. For n odd and every $m > 2$ or for n even and $m > 6$ even ($m \neq 14$) or for $m \geq 2n+1$ odd ($m \neq 2n+7$), there exists a circular disarranged m-string of n-sequences. For the remaining cases, there exists a circular 1-disarranged m-string of n-sequences.*

Corollary 4.10. *Let $m \geq 6$ and n be positive integers. Then for either m even or $m \geq 2n+1$ there exists a circular $(0,1)$-disarranged m-string of n-sequences.*

Using previous corollary now we are able to prove Theorem 9 of [18] in a different way.

Theorem 4.11. *For any simple connected graph G and for any cycle C_m, where either $m \geq 6$ even or $m \geq 2\Delta(G)+1$, we have the inequality*

$$\chi'_a(G \times C_m) \leq 2\Delta(G) + 1 .$$

Proof. Denote $(S_1, S_2, \cdots, S_m)$ a circular $(0,1)$-disarranged m-string of n-sequences whose elements belong to a $(2n+1)$-set of colors to be assigned to $H_1, H_2, \cdots, H_m$, where $H_i = G \times S(u_i u_{i+1})$, $1 \leq i \leq m$ and $u_{m+1} = u_1$.
Because H_i is bipartite, its chromatic index holds n. Let γ be a proper edge-coloring of H_1 using the n colors of S_1. By Theorem 2.5 we may assume that γ is symmetric. Moreover we assign a symmetric proper edge coloring of H_j, $2 \leq j \leq m$, by replacing the i-th color of S_1, where $1 \leq i \leq n$, by the i-th color of S_j.
Now, if vz is an edge of G, then the vertices (v, u_1) and (z, u_2) of H_1 are adjacent. We have to prove that the set C_1 of the colors of the edges incident (v, u_1) is distinct from the set C_2 of the colors of the edges incident (z, u_2). Let us assume, by contradiction, that C_1 and C_2 coincide. Let $C_1 = D_1 \cup D_2$ and $C_2 = E_2 \cup E_3$, where D_1 (resp. D_2) is the set of colors assigned to the edges incident (v, u_1) which belong to H_m (resp. H_1) while E_2 (resp. E_3) is the set of colors assigned to the edges incident (z, u_2), which belong to H_1 (resp. H_2). Since D_1 and D_2 are disjoint, as well as E_2 and E_3, D_2 and E_3, and D_1 and E_2, it follows that $D_2 = E_2$ and $D_1 = E_3$. The condition $D_2 = E_2$ implies that the set of the indices of the colors of S_1 used in D_2 coincides with the set of the indices of the colors used in E_2.
Moreover the set of colors related to the vertex vu_2 in H_1 coincides with D_2 because γ is symmetric. By the same motivation the set of colors related to the edges incident to zu_1 in H_1 coincides with E_2. This implies that also the set of the indices of the colors in D_1 and in E_3 coincide. Denote $\{i_1, i_2, \cdots, i_r\}$ the set of these indices, $S_m = \{a_1, a_2, \cdots, a_n\}$ and $S_2 = \{b_1, b_2, \cdots, b_n\}$. Because S_m and S_2 are $(0,1)$-disarranged, then, for $r > 1$ $\{a_{i_1}, a_{i_2}, \cdots, a_{i_r}\} \neq \{b_{i_1}, b_{i_2}, \cdots, b_{i_r}\}$.
Consider the case $r = 1$. If S_m and S_2 are 1-disarranged, we could obtain $D_1 = E_3$. However this assumption implies that v and z have degree 1 and this implies the impossible consequence that G is not connected.

So, the vertices of the edges of H_1 have distinct colors sets. Continuing in this way, we obtain that the same property holds also for H_j, $1 < j \leq m$, thus completing the proof.

□

5. The direct product of a simple graph by a graph with a loop in every vertex

5.1. **Double graphs and generalized double graphs.** In this section we are interested in evaluating the chromatic index and avd chromatic index of $G \times K_k^s$, where $k \in \mathbb{N}, k \geq 1$.
Recall [24] that the *double graph* of a graph G is defined as $\mathcal{D}[G] = G \times T_2$, where T_2 is the total graph on 2 vertices, i.e. the graph obtained from the complete graph K_2 by adding a loop to each vertex.
In [26] all median double graphs are characterized, where a median graph is a connected graph in which every triple of vertices has a unique median, i.e. a unique vertex that lies simultaneously on geodesics between any two of the given vertices.

Theorem 5.1 ([24]). *If G is of class 1 then also $\mathcal{D}[G]$ is of class 1.*

In relation to the chromatic index of the double of a graph G, in [27] it was conjectured that all double graphs are of class 1.

This conjecture was proved to be true in [9]. A property useful to obtain such result concerns a particular edge coloring of the direct product of a simple graph G by K_2 or by T_2.

An edge coloring α of the graph $G \times K_2$ is said to be *reflective* if

$$\alpha((v,u_1)(w,u_2)) = \alpha((v,u_2)(w,u_1)) \quad , \quad vw \in E(G) \; .$$

Moreover an edge coloring β of the graph $G \times T_2$ is said to be *reflective* if

$$\beta((v,u_1)(w,u_2)) = \beta((v,u_2)(w,u_1)) \quad , \quad vw \in E(G) \; .$$

and

$$\beta((v,u_1)(w,u_1)) = \beta((v,u_2)(w,u_2)) \quad , \quad vw \in E(G) \; .$$

Theorem 5.2 ([9])**.** *If G is a graph of class 1, then $G \times T_2$ admits a proper, reflective $2\Delta(G)$-edge coloring.*

Denote $M^2(G)$ the multigraph obtained from G by replacing every edge with a pair of two edges in parallel; in other words every edge $e = v_iv_j$ of G is replaced by different edges e_1 and e_2, with the same vertices v_i, v_j. In other words $M^2(G) = 2G$.

Proposition 5.3 ([9])**.** *There exists a bijection between the set of proper edge-colorings of $M^2(G)$ and the set of proper reflective edge-colorings of $G \times T_2$.*

Using the notion of reflective colorings it is possible to prove the following result.

Theorem 5.4 ([9])**.** *Let G be a simple class 2 graph; then $G \times T_2$ is of class 1.*

For any $k \in \mathbb{N}$, $k \geq 1$, the *generalized double graph* of a graph G is defined as $\mathcal{D}_k[G] = G \times T_k$, where $T_k = K_k^s$ is the total graph [21].

Theorem 5.5 ([26])**.** *If G is a graph of class 1, then also the generalized double graph $\mathcal{D}_k[G] = G \times T_k$ is of class 1.*

In [9] the authors investigated when Theorem 5.5 can be extended to the generalized double graphs.

Proposition 5.6 ([9])**.** *Let $k > 2$ be even and G a simple graph of class 2; then $G \times T_k$ is of class 1.*

Proposition 5.7 ([9])**.** *Let G be a d-regular graph of odd order and class 2. For $k > 1$ odd, $G \times T_k$ is of class 2.*

Note that the generalized double graphs were introduced in [23] as graphs $G[mK_1]$, where $G[H]$ denotes the composition of graphs G and H, also known as the lexicographic product. Recall that the composition of graphs G and H is the graph $G[H]$ with the vertex set $V(G[H]) = V(G) \times V(H)$ and the edge set $E(G[H]) = \{(u,v)(u',v') : \textit{either } (u = u' \text{ and } v \sim v') \textit{ or } u \sim u'\}$.

Proposition 5.8. *Let G be a simple graph and $m \in \mathbb{N}, m \geq 1$. Then $G \times K_m^s$ is isomorphic with $G[mK_1]$.*

Proof. Denote $V(K_m^s) = \{v_1, v_2, \cdots, v_m\}$ and $V(mK_1) = \{w_1, w_2, \cdots, w_m\}$. This implies that $V(G \times K_m^s) = V(G) \times \{v_1, v_2, \cdots, v_m\}$ and $V(G[mK_1]) = V(G) \times \{w_1, w_2, \cdots, w_m\}$ are bijective.

Let u, z be distinct vertices of G.
If (u, v_i) is adjacent to (z, v_j) in $G \times K_m^s$, where $i, j \in \{1, 2, \cdots, m\}$, then u and z (respectively v_i and v_j) are adjacent in G (respectively K_m^s). As a consequence, (u, w_i) is adjacent to (z, w_j) in $G[mK_1]$ because $u \neq z$ and $u \sim z$.
On the other hand, since $V[mK_1]$ is formed by m isolated vertices, if (u, w_i) is adjacent to (z, w_j) in $G[mK_1]$, we must have that u and z are adjacent in G. Because every two vertices of K_m^s are adjacent, we obtain that the vertices (u, v_i) and (z, v_j) are also adjacent in $G \times K_m^s$. This completes the proof.

□

5.2. **The direct product of a simple graph by a graph of order greater than 2 with a loop in every vertex.** In this section we consider the case of the direct product of a simple graph G by the graph H^s, obtained by adding a loop to each vertex of a simple graph H. Note that $G \times H^s$ is still a simple graph and that it can be considered as the edge disjoint union of the subgraphs $G \times H$, $G \times S(w_1w_1), \cdots, G \times S(w_nw_n)$, where $w_1, \cdots, w_n$ are the vertices of H.

Theorem 5.9 ([26]). *For any graph H with no isolated vertices, we have the inequality*

$$\Delta(H) + 1 \leq \chi'_a(K_2 \times H^s) \leq \Delta(H) + 3 .$$

Theorem 5.10 ([26]). *For any connected graph G, different from K_2, and for any connected graph H with at least two vertices, we have the inequality*

$$\chi'_a(G \times H^s) \leq \chi'_a(G) + \Delta(G)\Delta(H) + 1 . \tag{10}$$

In particular, if H is bipartite, then

$$\chi'_a(G \times H^s) \leq \chi'_a(G) + \Delta(G)\Delta(H) . \tag{11}$$

If we restrict our attention to the case of P_m, the following Theorem holds.

Theorem 5.11 ([26]). *For any bipartite graph G, different from K_2, and for any path P_m, with $m \geq 4$, we have the inequalities*

$$3\Delta(G) \leq \chi'_a(G \times P_m^s) \leq 3\Delta(G) + 1 .$$

6. AVD-MINIMAL GRAPHS

In this section we describe some cases in which the usual chromatic index turns out to be equal to the avd chromatic index. We say that a graph G is *avd-minimal* when $\chi'_a(G) = \chi'(G)$.

Examples of avd-minimal graphs are P_3, C_n with $n \equiv 0 \pmod 3$, K_n when $n > 1$ is odd, the Petersen graph $P(5, 2)$.

Proposition 6.1 ([26]). *If G is a graph of class 1, with at least two adjacent vertices of maximum degree, then G is not avd-minimal.*

The *Heawood graph H* [15, p.190], the *Franklin graph F*, and the *Pappus graph P* [15, p.252] are all cubic graphs of class 1, not avd-minimal: $\chi'(H) = \chi'(F) = \chi'(P) = 3$ and $\chi'_a(H) = \chi'_a(F) = \chi'_a(P) = 4$

Proposition 6.2 ([26]). *If G is a bipartite graph with at least two adjacent vertices of maximum degree, then it is not avd-minimal.*

Theorem 6.3 ([26]). *A tree (on at least three vertices) is avd-minimal if and only if it has no adjacent vertices of maximum degree.*

Theorem 6.4 ([26]). *Let G be a graph of class 1. If G is avd-minimal, then also $G \times K_2$ is avd-minimal.*

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