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Some techniques for the construction of hyperpath-designs - a survey

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Abstract. Given an hypergraph $H^{(3)}$, uniform of rank 3, an $H^{(3)}$-*decomposition* of the complete hypergraph $K_v^{(3)}$ is a collection of hypergraphs, all isomorphic to $H^{(3)}$, whose edge-sets partition the edge-set of $K_v^{(3)}$. An $H^{(3)}$-decomposition of $K_v^{(3)}$ is also called an $H^{(3)}$-*design*. In every decomposition, the hypergraphs of the partition are said to be the *blocks* of the system. Every decomposition is said to be *balanced* if the number of blocks containing any given vertex is a constant. In this paper, we give some construction for $P^{(3)}(1,5)$-designs, *balanced* $P^{(3)}(1,5)$-*designs*, $P^{(3)}(2,4)$-designs, *balanced* $P^{(3)}(2,4)$-*designs*, all systems which we will say to belong to the class of the *hyperpath-designs*.

1. Introduction

Let $\lambda \cdot K_v^{(3)} = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank 3, defined in a vertex set $X = \{x_1, x_2, \cdots, x_v\}$, in which every edges has multiplicity λ.
Let $H^{(3)}$ be a subhypergraph of $\lambda K_v^{(3)}$. An $H^{(3)}$-*decomposition* of $\lambda K_v^{(3)}$ is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge set of $\lambda \cdot K_v^{(3)}$ into subsets all of which yield subhypergraphs all isomorphic to $H^{(3)}$. An $H^{(3)}$-decomposition $\Sigma = (X, \mathcal{B})$ of $\lambda K_v^{(3)}$ is also called an $H^{(3)}$-*design* of *order* v and index λ and the classes of the partition $\mathcal{B}$ are said to be the *blocks* of Σ [1].

The concept of $H^{(3)}$-*decomposition* is the natural generalization to uniform hypergraphs of rank 3 of the more classical G-*decomposition* of the complete graph K_v or G-designs [1],[9],[10]. Much work about G-designs has been done in the recent past, with many interesting results and open problems, which can be found in the literature. In what follows, we consider $H^{(3)}$-design, where $H^{(3)}$ is mainly one of the following *path*-hypergraphs or *hyperpaths*.

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- $P^{(3)}(2,4)$: it is the *hyperpath* having four vertices x_1, x_2, x_3, x_4 and edges $\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}$; it will be denoted by $[x_1, (x_2, x_3), x_4]$;

- $P^{(3)}(1,5)$: it is the *hyperpath* having five vertices x, y_1, y_2, y_3, y_4 and edges $\{x, y_1, y_2\}, \{x, y_3, y_4\}$; it will be denoted by $[y_1, y_2, (x), y_3, y_4]$.

The spectrum of these $H^{(3)}$-designs has been determined in [3]. Precisely:

Proposition 1. *A* $P^{(3)}(1,5)$*-design of order* v *exists if and only if:* $v \equiv 0 \ mod\ 2$, *or* $v \equiv 1 \ mod\ 4$, $v \geq 5$.

Proposition 2. *A* $P^{(3)}(2,4)$*-design of order* v *exists if and only if:* $v \equiv 0 \ mod\ 2$, *or* $v \equiv 1 \ mod\ 4$, $v \geq 4$.

Let $H^{(3)}$ be an uniform hypergraph of rank 3, with n vertices. An $H^{(3)}$-design $\Sigma = (X, \mathcal{B})$ is said to be *balanced* if all the vertices of Σ have the same degree $d(x)$. Observe that if $H^{(3)}$ is *regular*, then the correspondent $H^{(3)}$-designs are always balanced, hence the notion of *balanced* $H^{(3)}$-design becomes meaningful only for a non-regular hypergraph $H^{(3)}$.

Example 1.
Let $\Sigma = (X, \mathcal{B})$ be the $P^{(3)}(1,5)(v)$-design of order $v = 5$, defined in $X = \{0, 1, 2, 3, 4\}$, having the blocks:

$$B_1 = [2,3,(0),1,4] \quad , \quad B_2 = [3,4,(1),2,0] \quad , \quad B_3 = [4,0,(2),3,1] \ ,$$

$$B_4 = [0,1,(3),4,2], B_5 = [1,2,(4),0,3] \, .$$

Every vertex of Σ has degree 5. We can verify that Σ is a balanced $P^{(3)}(1,5)(v)$-design of order $v = 5$.

□

Example 2.
Let $\mathcal{C}$ be the collection of the following $P^{(3)}(1,5)$s defined in $X = Z_6$:

$$C_1 = [1,2,(0),3,4] \quad , \quad C_2 = [1,3,(0),4,5] \quad , \quad C_3 = [1,4,(0),2,5] \ ,$$
$$C_4 = [1,5,(0),2,3] \quad , \quad C_5 = [3,5,(0),2,4] \quad , \quad C_6 = [1,3,(2),4,5] \ ,$$
$$C_8 = [2,4,(1),3,5] \quad , \quad C_8 = [1,2,(5),3,4] \quad , \quad C_9 = [2,3,(4),1,5] \ ,$$
$$C_{10} = [2,5,(3),1,4] \ .$$

If Σ=$(X, \mathcal{C})$, then we can verify that Σ is a $P^{(3)}(1,5)$-design of order $v = 6$. Further we can see that the vertex 0 has degree $d(0) = 5$, while the vertex 1 has degree $d(1) = 8$. Therefore, Σ it is *not* a *balanced* design.

□

Example 3.
Let $\Sigma = (X, \mathcal{D})$ be the $P^{(3)}(2,4)(v)$-design of order $v = 4$, defined in $X = \{0, 1, 2, 3\}$, having the blocks:

$$D_1 = [2,(0,1),3] \quad , \quad D_2 = [0,(2,3),1] \, .$$

It is immediate to see that Σ is a *balanced* $P^{(3)}(2,4)$-design of order $v = 4$.

□

Let $\Sigma = (X, \mathcal{B})$ be an $H^{(3)}$-design, where $H^{(3)} = (Y, \mathcal{E}) = [x, y, \cdots, z]$. An *automorphism* defined in Σ is a bijection $\varphi : X \rightarrow X$ such that: 1) B with vertices $x, y, \cdots, z$ belongs to $\mathcal{B}$ if and only if $\varphi(B)$ with vertices $\varphi(x), \varphi(y), \cdots, \varphi(z)$ belongs to $\mathcal{B}$; 2) $\{x, y, z\}$ is an *edge* (triple) of $\mathcal{E}$ if and only if $\{\varphi(x), \varphi(y), \varphi(z)\}$ is an *edge* of $\varphi(E)$.

An $H^{(3)}$-design of order v is *cyclic* if it admits an automorphism that is a permutation consisting of a single cycle of length v.

In this paper we give a survey of constructions concerning $P^{(3)}(2,4)$-designs and $P^{(3)}(1,5)$-designs, also in the case that they are *balanced* and/or *cyclic*.

Observe that, in what follows, for a given non-empty set X of cardinality v odd, we will call *pseudo-factorization* of K_v, defined in X, a partition of the edge-set of K_v in v classes, everyone defining a colouring class in an edge-colouring of K_v by v colours. In the case v even, $F_2(X)$ will indicate any 1-factor belonging to an 1-factorization of K_v, defined in X.

2. $P^{(3)}(2,4)$-designs

Observe that, among all the $H^{(3)}$ subhypergraphs of $K_v^{(3)}$ with two edges, $P^{(3)}(2,4)$s have the minimum number of vertices. It is easy to see that:

Theorem 2.1. *If $\Sigma = (X, \mathcal{B})$ is a $P^{(3)}(2,4)$-design of order v, then:*

(1) $\mathcal{B} = \dfrac{v(v-1)(v-2)}{12}$;

(2) $v \equiv 0 \; mod\, 2$ *or* $v \equiv 1 \; mod\, 4$, $v \geq 4$.

The following constructions permit to determine the spectrum of $P^{(3)}(2,4)$-designs.

CONSTRUCTION $\mathbf{v = 4h \rightarrow v' = 4h+1}$.

Let $\Sigma = (X, \mathcal{B})$ be a $P^{(3)}(4,2)$-design of order $v = 4h$, $h \geq 1$, defined in X.
Further, let $X = \{1, 2, \cdots, 4h\}$, $X' = \{\infty\} \cup X$, where $\infty \in X' - X$.
Define a P_3-design of order $v' = 4h+1$, as follows.
Let $\Gamma = (X, \mathcal{C})$ be a P_3-design of order $v = 4h$. For every block $[a, b, c] \in \mathcal{C}$, consider the hyperpath $P^{(3)}(2,4)$ defined as follows: $[a, (\infty, b), c]$. Then, if:

$$\Pi = \{[a, (\infty, b), c] : [a, b, c] \in \mathcal{C}\} ,$$

and $\mathcal{B}' = \mathcal{B} \cup \Pi$, it is possible to verify that Σ'=$(X', \mathcal{B}')$ is a $P^{(3)}(4,2)$-design of order $v = 4h+1$.

□

CONSTRUCTION $\mathbf{v = 4h+1 \rightarrow v' = 4h+2}$.

Since for every $h \in N, h \geq 1$ there exist P_3-design of order $v = 4h+1$, it is possible to go on exactly as in the previous construction.

□

CONSTRUCTION $\mathbf{v' = 4h, v'' = 4k \rightarrow v = 4h+4k}$.

Let $X_1 = \{x_1, x_2, \cdots, x_{4h}\}$, $h \geq 1$, $X_2 = \{y_1, y_2, \cdots, y_{4k}\}$, $k \geq 1$, $X_1 \cap X_2 = \emptyset$.
Further, let $\Sigma_1 = (X_1, \mathcal{B}_1)$ be a $P^{(3)}(2,4)$-design of order $v' = 4h$, $h \geq 1$, defined in X_1, and let $\Sigma_2 = (X_2, \mathcal{B}_2)$ be a $P^{(3)}(2,4)$-design of order $v'' = 4k$ defined in X_2.

For every pair $x_i, x_j \in X_1$, $x_i \neq x_j$, define:

$$\Pi_{\{x_i,x_j\}} = \{[y', (x_i, x_j), y''] \,:\, \{y', y''\} \in F_2(X_2)\} \,,$$

and, for every pair $y_i, y_j \in X_2$, $y_i \neq y_j$, define:

$$\Pi_{\{y_i,y_j\}} = \{[x', (y_i, y_j), x''] \,:\, \{x', x''\} \in F_2(X_1)\} \,.$$

Then, if:

$$\mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2 \bigcup_{\{x_i,x_j\}\subseteq X_1} \Pi_{\{x_i,x_j\}} \bigcup_{\{y_i,y_j\}\subseteq X_2} \Pi_{\{y_i,y_j\}} \,,$$

it is possible to verify that $\Sigma' = (X', \mathcal{B}')$ is a $P^{(3)}(4,2)$-design of order $v = 4h + 4k$ □

The previous constructions, together with the existence of a $P^{(3)}(2,4)$-design of order 4 (Example 3), prove that:

Theorem 2.2. *There exist $P^{(3)}(2,4)$-designs of order v for every $v \equiv 0$ mod 2 oppure $v \equiv 1$ mod 4, $v \geq 4$.*

3. $P^{(3)}(1,5)$-DESIGNS

There is only a class of $H^{(3)}$, subhypergraphs of $K_v^{(3)}$, with two edges and five vertices: they are the hyperpaths of type $P^{(3)}(1,5)$.

It is easy to see that:

Theorem 3.1. *If $\Sigma = (X, \mathcal{B})$ is a $P^{(3)}(1,5)$-design of order v, then:*

(1) $\mathcal{B} = \dfrac{v(v-1)(v-2)}{12}$;

(2) $v \equiv 0$ *mod* 2 *or* $v \equiv 1$ *mod* 4, $v \geq 5$.

We have:

Theorem 3.2. *There exist $P^{(3)}(1,5)$-designs of order $v = 5$ and of order $v = 6$.*

Proof. See Example 1 and Example 2.

□

Theorem 3.3. *There exist $P^{(3)}(1,5)$-designs of order $v = 8$.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be a $P^{(3)}(1,5)$-design of order $v = 6$, defined in $X = \{1,2,3,4,5,6\}$ and let ∞_1, ∞_2 be two distinct elements not belonging to X. Let $X' = X \cup \{\infty_1, \infty_2\}$. If $\mathcal{C}$ is the family of $P^{(3)}(1,5)$s defined as follows:

$$[1, \infty_2, (\infty_1), 2, 3], [2, \infty_2, (\infty_1), 3, 1], [3, \infty_2, (\infty_1), 1, 2] \,,$$

$$[4, \infty_1, (\infty_2), 1, 2], [5, \infty_1, (\infty_2), 2, 3], [6, \infty_1, (\infty_2), 3, 1] \,,$$

$$[3, 5, (\infty_1), 4, 6], [2, 6, (\infty_1), 4, 5], [1, 4, (\infty_1), 5, 6] \,,$$

$$[3, 5, (\infty_2), 4, 6], [2, 6, (\infty_2), 4, 5], [1, 4, (\infty_2), 5, 6] \,,$$

$$[1, 5, (\infty_1), 4, 2], [1, 6, (\infty_1), 3, 4], [2, 5, (\infty_1), 3, 6] \,,$$

$$[1, 5, (\infty_2), 4, 2], [1, 6, (\infty_2), 3, 4], [2, 5, (\infty_2), 3, 6] \,,$$

then, the system $\Sigma' = (X', \mathcal{B}')$, where $\mathcal{B}' = \mathcal{B} \cup \mathcal{C}$, is a $P^{(3)}(1,5)$-design of order $v' = 8$. Observe that Σ, which could contain a $P^{(3)}(1,5)$-design of order 5, is contained in Σ'.

□

Theorem 3.4. *If there exists a* $P^{(3)}(1,5)$*-design of order* $v = 10$*, then there exist* $P^{(3)}(1,5)$*-designs of order* $v' = 12$*.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be a $P^{(3)}(5,1)$-design of order $v = 10$, defined in $X = \{0,1,2,\cdots,9\}$ and let ∞_1, ∞_2 be two distinct elements not belonging to X. Let $X' = X \cup \{\infty_1, \infty_2\}$. If $\mathcal{C}$ is the family of $P^{(3)}(1,5)$s defined as follows:

$$[1,\infty_2,(\infty_1),2,9],[2,\infty_2,(\infty_1),1,0],[3,\infty_2,(\infty_1),4,7]\ ,$$

$$[4,\infty_2,(\infty_1),5,6],[5,\infty_2,(\infty_1),3,8]\ ,$$

$$[6,\infty_1,(\infty_2),4,7],[7,\infty_1,(\infty_2),1,0],[8,\infty_1,(\infty_2),2,9]\ ,$$

$$[9,\infty_1,(\infty_2),3,8],[10,\infty_1,(\infty_2),5,6]$$

$$[1,2,(\infty_1),3,9],[4,8,(\infty_1),5,7],[6,0,(\infty_1),1,3]\ ,$$

$$[2,0,(\infty_1),4,9],[5,8,(\infty_1),6,7]\ ,$$

$$[1,2,(\infty_2),3,9],[4,8,(\infty_2),5,7],[6,0,(\infty_2),1,3]\ ,$$

$$[2,0,(\infty_2),4,9],[5,8,(\infty_2),6,7]\ ,$$

$$[1,4,(\infty_1),2,3],[5,9,(\infty_1),6,8],[7,0,(\infty_1),1,5]\ ,$$

$$[2,4,(\infty_1),3,0],[6,9,(\infty_1),7,8]\ ,$$

$$[1,4,(\infty_2),2,3],[5,9,(\infty_2),6,8],[7,0,(\infty_2),1,5]\ ,$$

$$[2,4,(\infty_2),3,0],[6,9,(\infty_2),7,8]\ ,$$

$$[1,6,(\infty_1),2,5],[3,4,(\infty_1),7,9],[8,0,(\infty_1),1,7]\ ,$$

$$[2,6,(\infty_1),3,5],[4,0,(\infty_1),8,9]\ ,$$

$$[1,6,(\infty_2),2,5],[3,4,(\infty_2),7,9],[8,0,(\infty_2),1,7]\ ,$$

$$[2,6,(\infty_2),3,5],[4,0,(\infty_2),8,9]\ ,$$

$$[1,8,(\infty_1),2,7],[3,6,(\infty_1),4,5],[9,0,(\infty_1),2,8]\ ,$$

$$[1,9,(\infty_1),3,7],[5,0,(\infty_1),4,6]\ ,$$

$$[1,8,(\infty_2),2,7],[3,6,(\infty_2),4,5],[9,0,(\infty_2),2,8]\ ,$$

$$[1,9,(\infty_2),3,7],[5,0,(\infty_2),4,6]\ ,$$

and $\mathcal{B}' = \mathcal{B} \cup \mathcal{C}$, then the system $\Sigma' = (X', \mathcal{B}')$ is a $P^{(3)}(1,5)$-design of order $v = 12$.
□

CONSTRUCTION $\mathbf{v = 4h \to v' = 4h + 1}$.

Let $\Sigma = (X, \mathcal{B})$ be a $P^{(3)}(1,5)$-design of order $v = 4h$, $h \geq 2$, defined in $X = Z_{4h}$. Let $X'=\{\infty\} \cup X$, where $\infty \in X' - X$.
Further, let $\mathcal{F}=\{F_1, F_2, \cdots, F_{4h-1}\}$ be a factorization defined in X. Since every factor $F_i \in \mathcal{F}$ has cardinality $|F_i| = 2h$, it is possible to define a partition of every F_i into h classes $\{C_{i,1}, C_{i,2}, \cdots, C_{i,h}\}$, where every class is formed by two disjoint pairs. Let

$$\Pi = \{[x', x'', (\infty), y', y''] : \{x', x''\}, \{y', y''\} \in \mathcal{C}_{i,j},\ i = 1, \cdots, 4h-1,\ j = 1, \cdots, h\}.$$

If $\mathcal{B}' = \mathcal{B} \cup \Pi$, it is possible to verify that $\Sigma'=(X', \mathcal{B}')$ is a $P^{(3)}(1,5)$-design of order $v' = 4h + 1$.
□

CONSTRUCTION $\mathbf{v = 4h + 1 \to v' = 4h + 2}$.

Let $\Sigma = (X, \mathcal{B})$ be a $P^{(3)}(1,5)$-design of order $v = 4h + 1$, $h \geq 1$, defined in $X = Z_{4h+1}$. Let $X' = \{\infty\} \cup X$, where $\infty \in X' - X$.
Let $\mathcal{F}^*=\{F_1, F_2, \cdots, F_{4h+1}\}$ be a *pseudo*-factorization defined in X. Since every F_i has cardinality $|F_i| = 2h$, it is possible to go on as in the previous construction. In other words, define a partition of every F_i into h classes of two disjoint pairs, say $\{C_{i,1}, C_{i,2}, \cdots, C_{i,h}\}$, and construct the family $\Pi = \{[x', x'', (\infty), y', y''] : \{x', x''\}, \{y', y''\} \in \mathcal{C}_{i,j},\ i = 1, \cdots, 4h + 1,\ j = 1, \cdots, h\}$. At last, if $\mathcal{B}' = \mathcal{B} \cup \Pi$, then the system $\Sigma' = (X', \mathcal{B}')$ is a $P^{(3)}(1,5)$-design of order $v = 4h + 2$.
□

The previous constructions and results prove that:

Theorem 3.5. *There exist $P^{(3)}(2,4)$-designs or order v for every $v \equiv 0$ mod 2 oppure $v \equiv 1$ mod 4, $v \geq 4$.*

4. The matrix $\mathcal{M}(v)$

In what follows we will use the matrix $\mathcal{M}(v)$, where $v = 3h + 1$ or $v = 3h + 2$, for some positive integer h, defined in $Z_v = \{0, 1, 2, \cdots, v - 1\}$ and constructed as follows.

For the uses and more details about this matrix see [1] and also [5],[7],[8]. This matrix $\mathcal{M}(v)$ is useful to construct *balanced*, and then *cyclic*, $H^{(3)}$-designs.

Let $v \equiv 1, 2$ *mod* 3. $\mathcal{M}(v)$ is a matrix having 3 columns, *associated with* v, such that:

$$\mathcal{M}(v) = \begin{bmatrix} (1,1) & (1,v-2) & (v-2,1) \\ (1,2) & (2,v-3) & (v-3,1) \\ (\cdots) & (\cdots) & (\cdots) \\ (\cdots) & (\cdots) & (\cdots) \\ (1,v-3) & (v-3,2) & (2,1) \\ (2,2) & (2,v-4) & (v-4,2) \\ (\cdots) & (\cdots) & (\cdots) \\ (2,v-5) & (v-5,3) & (3,2) \\ (3,3) & (3,v-6) & (v-6,3) \\ (\cdots) & (\cdots) & (\cdots) \\ (3,v-7) & (v-7,4) & (4,3) \\ (\cdots) & (\cdots) & (\cdots) \\ (\cdots) & (\cdots) & (\cdots) \\ (h,h) & (h,v-2h) & (v-2h,h) \\ (h,v-2h-1) & (v-2h-1,h+1) & (h+1,h) \end{bmatrix}.$$

Observe that:

1) if $v = 3h + 1$, the last row begin with the pair (h, h);

2) if $v = 3h + 2$, the last row begin with the pair $(h, h + 1)$.

We can see that, for any triple $T = \{x, y, z\} \subseteq Z_v$, with $x < y < z$ and $y - x = a, z - y = b$, there exists a row of $\mathcal{M}(v)$ containing the pair (a, b). Further, if we fix any pair (a, b) of $\mathcal{M}(v)$ and write any triple $T = \{x, y, z\}$, with $y - x = a, z - y = b$, i.e. such that its elements have differences a, b, then T can be obtained from $C = (0, a, a + b)$ by translation of blocks: this means that there exists an $i \in Z_v$ such that $x = i, y = a + i, z = y + b$. Thus, if x is added to the elements of C, one obtains T. Therefore, for every $x, y, z \in \{0, 1, 2, \cdots, v - 1\}$, with $x < y < z$, every of the pairs $(y - x, z - y), (z - y, v + x - z), (v + x - z, y - x)$ determines the triple $T = \{x, y, z\}$. For this reason, any two pairs, from the same row, in the matrix $\mathcal{M}$ are said to be *equivalent* among them.

In what follows, fixed $v = 3h + 1$ or $v = 3h + 2$, we will indicate by R_i, for every $i = 1, 2, \cdots, h$, the set of rows of $\mathcal{M}(v)$ having in the first column the pairs:

$$(i, i), (i, i + 1), \cdots, (i, v - 1 - 2i) .$$

If $|R_i| = m_i$, it is possible to calculate the number $m = m_1 + m_2 + \cdots + m_h$ of rows of $\mathcal{M}(v)$.

Theorem 4.1. *Let $v = 3h+1$ or $v = 3h+2$ and let $\mathcal{M}(v)$ be the matrix associated with v. Then:*

1) $m_i = v - 3i$, for every $i = 1, 2, \cdots, h$;

2) $m = \dfrac{h(2v - 3h - 3)}{2}$;

3) $v = 3h + 1 \implies m = \dfrac{h(3h - 1)}{2}$; $v = 3h + 2 \implies m = \dfrac{h(3h + 1)}{2}$.

Proof. It is easy to see that, for every $i = 1, 2, \cdots, h$, one has: $m_i = v - (1 + 2i) - (1 - i) = v - 3i$.

Further, from 1), it follows that:

$$m = m_1 + m_2 + \cdots + m_h = (v-3) + (v-6) + \cdots + (v-3h) =$$
$$= hv - 3(1+2+\cdots+h) = hv - \frac{3h(h+1)}{2} = h \cdot \frac{2v-3(h+1)}{2} .$$

The statement 3) follows directly from 2).

□

It is immediate that *an $H^{(3)}$-design of order $v = 3h+1$ or $v = 3h+2$, $h \geq 1$, constructed by the matrix $\mathcal{M}(v)$, is balanced and also cyclic.*

5. Balanced $P^{(3)}(1,5)$-designs

In this section we see some results about the existence of *balanced* $P^{(3)}(1,5)$-*designs*. If B=$[b,c,(a),d,e]$ is a $P^{(3)}(1,5)$ defined in Z_v, we call *translates* of B all hypergraphs $P^{(3)}(1,5)$ of the form B_i=$[b+i, c+i, (a+i), d+i, e+i]$, for every $i \in Z_v$. We say also that the hypergraph B is a *base-block* having the hypergraphs B_i as *translates*. To have more details about the subject contained in this section see [5].

If $[b,c,(a),d,e]$ is a *path-hypergraph* or *hyperpath* $P^{(3)}(1,5)$ and $\Sigma = (X,\mathcal{B})$ is a $P^{(3)}(1,5)$-design, for every vertex $x \in X$, the parameter C_x is the number of blocks of $\mathcal{B}$ in which x occupies one of the *central* positions a, while L_x is the number of blocks in which x occupies one of the *lateral positions* b,c,d,e. If $d(x)$ is the degree of x, then $d(x) = C_x + L_x$.

At first, we see some necessary conditions.

Theorem 5.1. *If $\Sigma = (X,\mathcal{B})$ is a balanced $P^{(3)}(1,5)$-design of order v, then for every $x \in X$:*

$$d(x) = \frac{5(v-1)(v-2)}{12} \quad ; \quad C_x = \frac{(v-1)(v-2)}{12} \quad ; \quad L_x = \frac{(v-1)(v-2)}{3} .$$

Proof. Let $\Sigma = (X,\mathcal{B})$ be a balanced $P^{(3)}(1,5)$-design of order v.
For every vertex $x \in X$, the degree of x is a constant: $d(x) = D$. Considering that the number of positions that a vertex can occupy in a block of Σ is five, it follows: $5 \cdot |\mathcal{B}| = D \cdot v$, from which:

$$D = \frac{5(v-1)(v-2)}{12} .$$

Further, considering that: 1) every vertex $x \in X$ is contained in $(v-1)(v-2)/2$ triples of X; 2) in any block, the number of triples intersecting in the *center* is 2; 3) in any block, the number of triples containing a lateral vertex is 1; it follows:

$$C_x + L_x = \frac{5(v-1)(v-2)}{12} ;$$
$$2 \cdot C_x + L_x = \frac{(v-1)(v-2)}{2} .$$

Hence:

$$C_x = C = \frac{(v-1)(v-2)}{12} \quad , \quad L_x = L = \frac{(v-1)(v-2)}{3} .$$

which completes the proof.
Observe that it is possible to arrive at the same result considering that the total

number of *central positions* in Σ is $|\mathcal{B}| = v(v-1)(v-2)/12$ and every vertex must occupy these positions C times.

□

Theorem 5.2. *If $\Sigma = (X, \mathcal{B})$ is a balanced $P^{(3)}(1,5)$-design of order v, then:*

$$v \equiv 1 \quad or\ \ 2\ \ or\ \ 5\ \ or\ \ 10,\ mod\,12 \quad , \quad v \geq 5\ .$$

Proof. The statement follows from the previous Theorem, considering that the number $(v-1)(v-2)$ must be a multiple of $3 \cdot 4$ and $v \geq 5$.

□

Therefore, given a balanced $P^{(3)}(1,5)$-design Σ, two parameters C and L are defined: the constant degrees C_x and L_x, respectively, of the vertices x of Σ.

The following Theorems permit to determine completely the spectrum of balanced $P^{(3)}(1,5)$-designs. We will see how they can be proved. The whole proofs can be found in [5].

Theorem 5.3. *For every $v \equiv 1$ mod 12, $v \geq 13$, there exist balanced $P^{(3)}(1,5)$-designs of order v.*

Proof. Observe that, for $v = 12k+1$, $k \geq 1$, we are in the case $v = 3h+1$, for some even number $h = 4k$. Therefore, in the set $M' = \{m_1 = 12k-2, m_3 = 12k-8, \cdots, m_{h-1} = 4\}$ the elements are all even numbers, while in the set $M'' = \{m_2 = 12k-5, m_4 = 12k-11, \cdots, m_h = 1\}$ the elements are all odd numbers and $|M'' = 2k$. This permits to define in $X = Z_v$ the base-blocks, whose translates give the blocks of the $P^{(3)} - (1,5)$designs of order $v = 12k+1$ [5].

□

In what follows, the same technique of the previous Theorem is used, with convenient changes.

Theorem 5.4. *For every $v \equiv 5$ mod 12, $v \geq 5$, there exist balanced $P^{(3)}(1,5)$-designs of order v.*

Proof. Observe that, for $v = 12k+5$, $k \geq 0$, we are in the case $v = 3h+2$, for some *odd* number $h = 4k+1$. As in the previous Theorem, in $M' = \{m_1, m_3, \cdots, m_h = 2\}$ the elements are all even numbers, in $M'' = \{m_2, m_4, \cdots, m_{h-1} = 5\}$ the elements are all odd numbers and $|M''| = 2k$. Therefore, this permits to define in $X = Z_v$ the base-blocks and to construct the blocks of the $P^{(3)}(1,5)$-designs of order $v = 12k+5$ [5].

□

Theorem 5.5. *For every $v \equiv 2$ mod 12, $v \geq 14$, there exist balanced $P^{(3)}(1,5)$-designs of order v.*

Proof. Observe that, for $v = 12k+2$, $k \geq 1$, we are in the case $v = 3h+2$, for some *even* number $h = 4k$. In this case, in $M' = \{m_1, m_3, \cdots, m_{h-1} = 5\}$ the elements are all odd numbers, in $M'' = \{m_2, m_4, \cdots, m_h = 2\}$ the elements are all even numbers and $|M''| = 2k$. This permits to define in $X = Z_v$ the base-blocks, whose translates give the blocks of the $P^{(3)}(1,5)$-designs of order $v = 12k+2$ [5].

□

Theorem 5.6. *For every $v \equiv 10$, mod 12, $v \geq 10$, there exist balanced $P^{(3)}(1,5)$-designs of order v.*

Proof. Observe that, for $v = 12k + 10$, $k \geq 0$, we are in the case $v = 3h + 1$, for some *odd* number $h = 4k + 3$. In this case, in $M' = \{m_1, m_3, \cdots, m_h = 1\}$ the elements are all odd numbers, in $M'' = \{m_2, m_4, \cdots, m_{h-1} = 4\}$ the elements are all even numbers and $|M'| = 2k+2$. Also here, this permits to define in $X = Z_v$ the base-blocks and to construct the blocks of the $P^{(3)}(1,5)$-designs of order $v = 12k+1$ [5].

□

Conclusive result:

Theorem 5.7. *There exist balanced $P^{(3)}(1,5)$-designs of order v if and only if $v \equiv 1$, or 2, or 5, or 10, mod 12, $v \geq 5$.*

Proof. Collecting together all the previous Theorems, the statement follows.

□

6. Balanced $P^{(3)}(2,4)$-designs

In this section we examine the spectrum of *balanced* $P^{(3)}(2,4)$-designs. Let $[a,(b,c),d]$ be an *hyperpath* of type $P^{(3)}(2,4)$. To have more details about the subject contained in this section see [8].

If $\Sigma = (X, \mathcal{B})$ is a $P^{(3)}(2,4)$-design, for every vertex $x \in X$ we will indicate by C_x the number of blocks of $\mathcal{B}$ in which x occupies one of the *central* positions b, c and by L_x the number of blocks in which x occupies one of the *lateral positions* a, d. If $d(x)$ is the degree of x, then of course $d(x) = C_x + L_x$.

Theorem 6.1. *If $\Sigma = (X, \mathcal{B})$ is a balanced $P^{(3)}(2,4)$-design of order v, then for every $x \in X$.*

$$d(x) = \frac{(v-1)(v-2)}{3} \quad ; \quad C_x = L_x = \frac{(v-1)(v-2)}{6} .$$

Proof. Let $\Sigma = (X, \mathcal{B})$ be a balanced $P^{(3)}(2,4)$-design of order v.
Considering that the number of positions that a vertex can occupy in a block of Σ is four, it follows: $4 \cdot |\mathcal{B}| = D \cdot v$. From which: $D = (v-1)(v-2)/3$. Further, since every vertex is contained in $(v-1)(v-2)/2$ triples of X, it follows that:

$$C_x + L_x = \frac{(v-1)(v-2)}{3} \quad ; \quad 2 \cdot C_x + L_x = \frac{(v-1)(v-2)}{2} .$$

Hence: $C_x = L_x = (v-1)(v-2)/6$, which completes the proof.

□

Observe that it is possible to arrive at the same result considering that the total number of *central positions* in Σ is $2 \cdot |\mathcal{B}| = v(v-1)(v-2)/6$ and every vertex must be occupy these positions C times.

□

Theorem 6.2. *If $\Sigma = (X, \mathcal{B})$ is a balanced $P^{(3)}(2,4)$-design of order v, then:*

1) *$v \equiv 2$ or 4, mod 6, for v even, $v \geq 4$;*

2) *$v \equiv 1$ or 5, mod 12, for v odd, $v \geq 5$.*

Proof. The statement follows from the previous Theorem, considering also that the spectrum of $P^{(3)}(2,4)(v)$-designs is v even, $v \geq 4$, or $v \equiv 1$, *mod* 4, $v \geq 5$.

□

Theorem 6.3. *There exist balanced $P^{(3)}(2,4)$-designs of order v, for every $v \equiv 2$ or 4, mod 6, $v \geq 4$.*

Proof. Let $v \equiv 2$ or 4, *mod* 6, for $v \geq 4$. It is well-known that for such a v there exist Steiner quadruple systems. Let $\Sigma = (X, \mathcal{B})$ be an $SQS(v)$. For every block $B = \{x_1, x_2, x_3, x_4\} \in \mathcal{B}$, define the two $P^{(3)}(2,4)$: $P_1 = [x_1, (x_2, x_3), x_4], P_2 = [x_2, (x_1, x_4), x_3]$. The collection of all the $P^{(3)}(2,4)s$ so obtained, generates a $P^{(3)}(2,4)$-design of order v having all the vertices with degree $(v-1)(v-2)/3$.

□

The following Theorems permit to determine completely the spectrum of balanced $P^{(3)}(2,4)$-designs. Also here, we give the main points of the proofs, which can be found with all the details in [8].

- **The case** $v = 12h + 1$

Also here, if B=$[a,(b,c),d]$ is an hypergraph $P^{(3)}(2,4)$ defined in Z_v, its *translates* are all hypergraphs B_i=$[a+i,(b+i,c+i),d+i]$, for every $i \in Z_v$. The hypergraph B will be called *base-block*, having B_i as *translates*.

Theorem 6.4. *There exist balanced $P^{(3)}(2,4)$-designs of order v, for every $v \equiv 1$ mod 12, $v \geq 13$.*

Let $v \equiv 1$ *mod* 12, $v \geq 13$. We write $v = 12h + 1$ and note that $v = 3k + 1$, with $k = 4h$. Let $X = Z_v = \{0, 1, 2, \cdots, v-1\}$.

In general, let $v \equiv 1$ *mod* 12, $v \geq 13$. We write $v = 12h+1$ and note that $v = 3k+1$, with $k = 4h$. Let $X = Z_v = \{0, 1, 2, \cdots, v-1\}$.. Consider $\mathcal{M}(v)$. By this matrix, which has an even number of rows, we can choose conveniently the triples, so to define $h(12h+1)$ base-blocks and construct a $P^{(3)}(2,4)$-design of order $v = 12h+1$, which will result *balanced*.

□

We see a particular case: *Construction of a balanced $P^{(3)}(2,4)$-design of order $v = 13$.*

Base-blocks defined in $X = Z_{13}$:

$$B_1 = [0,(1,2),12]\ ,\ B_2 = [0,(1,3),11]\ ,\ B_3 = [0,(1,4),10]\ ,\ B_4 = [0,(1,5),9]\ ,$$

$$B_5 = [0,(1,6),9]\ ,\ B_6 = [0,(1,7),12]\ ,\ B_7 = [0,(1,8),12]\ ,\ B_8 = [0,(4,5),1]\ ,$$

$$B_9 = [0,(6,8),12]\ ,\ B_{10} = [0,(6,9),12]\ ,\ B_{11} = [0,(2,4),7]\ .$$

If $\mathcal{B}$ is the collection of all the translates of the base-blocks $B_1, B_2, \cdots, B_{11}$, it is possible to verify that $\Sigma = (X, \mathcal{B})$ is a $P^{(3)}(2,4)$-design of order $v = 13$, in which every vertex $x \in X$ belongs to 44 blocks and this implies that Σ is balanced.

□

- **The case** $v = 12h + 5$

Let $v \equiv 5$ *mod* 12, $v \geq 5$. We write $v = 12h + 5$ and note that $v = 3k + 2$, with $k = 4h + 1$.

Let $X = Z_v = \{0, 1, 2, \cdots, v-1\}$.

Let $v \geq 17$. Also here, if consider $\mathcal{M}(v)$, which has an even number of rows, by this matrix we can choose conveniently the triples, so to define $(3h+1)(4h+1)$ base-blocks and construct a $P^{(3)}(2,4)$-design of order $v = 12h+5$, which will result

balanced.
If $v = 5$, the blocks: $[i, (1+i, 2+i), 4+i]$, for every $i = 0, 1, 2, 3, 4$, define a balanced $P^{(3)}(2,4)$-designs of order $v = 5$.

□

7. Systems of index $\lambda = 2$

In this section we examine the existence of hyperpath-design of type $P^{(3)}(2,4)$ and $P^{(3)}(1,5)$ having index $\lambda = 2$.

It is immediate to prove that:

Theorem 7.1. *If $\Sigma = (X, \mathcal{B})$ is a $P^{(3)}(2,4)$-design or a $P^{(3)}(1,5)$-design of order v and index $\lambda = 2$, then:*

(1) $\mathcal{B} = \dfrac{v(v-1)(v-2)}{6}$;

(2) $v \geq 4$ *for* $P^{(3)}(2,4)$*-designs;*

(3) $v \geq 5$ *for* $P^{(3)}(1,5)$*-designs.*

Now, we examine the following constructions.

CONSTRUCTION $\mathbf{v = 4h+3 \rightarrow v' = 4h+7}$, **for** $\mathbf{P^{(3)}(2,4)}$**-designs**.

Let $\Sigma = (X, \mathcal{B})$ be a $P^{(3)}(2,4)$-design of order $v = 4h+3$, $h \geq 1$, and index $\lambda = 2$ defined in $X = Z_{4h+3} = \{1, 2, \cdots, 4h+3\}$. Further, let $Y = \{\alpha, \beta, \gamma, \delta\}$, such that $X \cap Y = \emptyset$, and

$$\mathcal{B}' = \{[\gamma, (\alpha, \beta), \delta]_{(2)}, [\alpha, (\gamma, \delta), \beta]_{(2)}\} ,$$

where the symbol (2) means that the block has multiplicity two, i.e. it is repeated two times in the family $\mathcal{B}'$. Obviously, $\Omega = (X \cup Y, \mathcal{B}')$ is a $P^{(3)}(2,4)$-design of order $v = 4$ and index $\lambda = 2$.

Define a P_3-design of order $v' = 4h+7$ and index 2, as follows.
For every pair of distinct vertices a, b of Y, let

$$\Pi(a,b) : [0, (a,b), 1], [1, (a,b), 2], \cdots, [4h, (a,b), 4h+1], [4h+1, (a,b), 0] ;$$

and for every pair of distinct vertices x, y of X, let

$$\Pi(x,y) : [\alpha, (x,y), \beta], [\beta, (x,y), \gamma], [\gamma, (x,y), \delta], [\delta, (x,y), \alpha] .$$

If

$$\Pi = \bigcup_{a,b \in Y} \Pi(a,b) \quad , \quad \Pi' = \bigcup_{x,y \in X} \Pi(x,y) ,$$

and $X' = X \cup Y$, $\mathcal{B}' = \mathcal{B} \cup \Pi \cup \Pi'$, it is possible to verify that Σ'=$(X', \mathcal{B}')$ is a $P^{(3)}(2,4)$-design of order $v = 4h+7$ and index $\lambda = 2$.

□

CONSTRUCTION $\mathbf{v = 4h+2 \rightarrow v' = 4h+3}$, **for** $\mathbf{P^{(3)}(1,5)}$**-designs**.

Let $\Sigma = (X, \mathcal{B})$ be a $P^{(3)}(1,5)$-design of order $v = 4h+2$, $h \geq 1$, and index $\lambda = 2$ defined in $X = Z_{4h+2}$. Further, let $\infty \notin X$ and $X' = X \cup \{\infty\}$. Define a 1-factorization $\mathcal{F} = \{F_1, F_2, \cdots, F_{4h+1}\}$ of X. For every 1-factor

$$F_i = \{\{x_{i,1}, y_{i,1}\}, \{x_{i,2}, y_{i,2}\}, \cdots, \{x_{i,2h+1}, y_{i,2h+1}\} ,$$

consider the following family $\mathcal{G}(F_i)$ of $P^{(2)}(1,5)$:

$[x_{i,1}, y_{i,1}, (\infty), x_{i,2}, y_{i,2}]$,

$[x_{i,2}, y_{i,2}, (\infty), x_{i,3}, y_{i,3}]$,

....

$[x_{i,2h}, y_{i,2h}, (\infty), x_{i,2h+1}, y_{i,2h+1}]$,

$[x_{i,2h+1}, y_{i,2h+1}, (\infty), x_{i,1}, y_{i,1}]$.

If $\mathcal{B}' = \mathcal{B} \cup \mathcal{G}$, where

$$\mathcal{G} = \bigcup_{i=1}^{4h+1} \mathcal{G}(F_i) \ ,$$

then $\Sigma' = (X', \mathcal{B}')$ is a $P^{(3)}(1,5)$-design of order $v = 4h+3$ and index $\lambda = 2$.

Theorem 7.2. *There exists a $P^{(3)}(2,4)$-design of order $v = 7$ and index $\lambda = 2$.*

Proof. We use the matrix $\mathcal{M}(7)$ defined in Z_7.
Observe that the ordered pairs in the first column are:

$$(1,1), (1,2)(1,3)(1,4)(2,2) \ .$$

This permits to define the following base-blocks:

$$[2,(0,1),6], [3,(0,1),5], [6,(0,3),5], [6,(0,2),4], [5,(0,1),4] \ .$$

If $X = Z_7$ and $\mathcal{B}$ is the family of all the translates of the above base-blocks, then we can verify that $\Sigma = (X, \mathcal{B})$ is a $P^{(3)}(2,4)$-design of order $v = 7$ and index $\lambda = 2$. □

Theorem 7.3. (*1*) *For every $v \geq 4$, there exists a $P^{(3)}(2,4)$-design of order v and index $\lambda = 2$.* (*2*) *For every $v \geq 5$, there exists a $P^{(3)}(1,5)$-design of order v and index $\lambda = 2$.*

Proof. For every v *even* or $v \equiv 1$ *mod* 4, there exist $P^{(3)}(2,4)$-designs and $P^{(3)}(1,5)$-designs, of order v and index 1, with $v \geq 4$ and $v \geq 5$, respectively. Therefore, systems of the same type with index $\lambda = 2$ can be obtained by a repetition of blocks.
Consider the case $v = 4h+3$, for any $h \geq 1$.

By Construction $v' = 4h+2 \longrightarrow v'+1$, for $P^{(3)}(1,5)$-designs, it follows that there are $P^{(3)}(1,5)$-designs of order $v = 4h+3$ and index $\lambda = 2$.

By Construction $v' = 4h+3 \longrightarrow v'+4$, for $P^{(3)}(2,4)$-designs and the previous Theorem, it follows that there are $P^{(3)}(2,4)$-designs of order $v = 4h+7$ and index $\lambda = 2$.

□

8. Systems of index $\lambda \geq 3$

For $\lambda \geq 3$, it is immediate to prove that:

Theorem 8.1. *If $\Sigma = (X, \mathcal{B})$ is a $P^{(3)}(2,4)$-design or a $P^{(3)}(1,5)$-design of order v and index $\lambda \geq 3$, then:*

(1) $\mathcal{B} = \dfrac{\lambda \cdot v(v-1)(v-2)}{12}$;

(2) *if* λ *is odd, then* v *is even, or* $v \equiv 1 \; mod \; 4$, $v \geq 4$;

(3) *if* λ *is even, then* $v \geq 4$ *for* $P^{(3)}(2,4)$*-designs,* $v \geq 5$ *for* $P^{(3)}(1,5)$*-designs.*

For the sufficiency:
- in the case λ *odd*, $\lambda \geq 3$, it is possible to determine the spectrum of these $H^{(3)}$-designs by $P^{(3)}(2,4)$-designs or $P^{(3)}(1,5)$-designs of index *one*, which exist for every v *even, or* $v \equiv 1 \; mod \; 4$, $v \geq 4$, by a repetition of blocks, giving to each block multiplicity λ;

- in the case λ *even*, $\lambda \geq 4$, it is possible to determine the spectrum by $P^{(3)}(2,4)$-designs or $P^{(3)}(1,5)$-designs of index *two*, which exist respectively for every $v \geq 4$ and $v \geq 5$, by a repetition of blocks, giving to each block multiplicity $\lambda/2$.

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