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# Nowhere-zero flows in signed graphs: a survey

Tomáš Kaiser[1], Robert Lukoťka[2] and Edita Rollová[3]

**Abstract**. We survey known results related to nowhere-zero flows and related topics, such as circuit covers and the structure of circuits of signed graphs. We include an overview of several different definitions of signed graph colouring.

## 1. Introduction

Nowhere-zero flows were defined by Tutte [65] as a dual problem to vertex-colouring of (unsigned) planar graphs. Both notions have been extended to signed graphs. The definition of nowhere-zero flows on signed graphs naturally comes from the study of embeddings of graphs in non-orientable surfaces, where nowhere-zero flows emerge as the dual notion to local tensions. There is a close relationship between nowhere-zero flows and circuit covers of graphs as every nowhere-zero flow on a graph $G$ determines a covering of $G$ by circuits. This relationship is maintained for signed graphs, although a signed version of the definition of circuit is required.

The area of nowhere-zero flows on (signed) graphs recently received a lot of attention thanks to the breakthrough result of Thomassen [63] on nowhere-zero 3-flows in unsigned graphs. The purpose of this paper is to capture the current state of knowledge regarding nowhere-zero flows and circuit covers in signed graphs. To cover the latest developments in this active area, we decided to include some papers that are currently still in the review process and only available from the

[1]T. Kaiser, University of West Bohemia, Department of Mathematics, Institute for Theoretical Computer Science (CE-ITI) and European Centre of Excellence NTIS (New Technologies for the Information Society), Univerzitní 8, 306 14, Pilsen, Czech Republic; `kaisert@kma.zcu.cz`

Supported by project GA14-19503S of the Czech Science Foundation.

[2]R. Lukoťka, Comenius University, Department of Computer Science, Faculty of Mathematics, Physics, and Informatics, Mlynská dolina F1, 842 48, Bratislava, Slovakia; `lukotka@dcs.fmph.uniba.sk`

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[3]E. Rollová, University of West Bohemia, European Centre of Excellence NTIS (New Technologies for the Information Society), Univerzitní 8, 306 14, Pilsen, Czech Republic; `rollova@ntis.zcu.cz`

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arXiv. Clearly, until the papers are published, the results have to be taken with caution.

An extensive source of links to material on signed graph is the dynamic survey of Zaslavsky [81]. For the theory of unsigned graphs in general, see [9] or [24]. Nowhere-zero flows and circuit covers in unsigned graphs are the subject of Zhang [82]; see also [83].

Graphs in this survey may contain parallel edges and loops. A circuit is a connected 2-regular graph.

A *signed graph* is a pair $(G, \Sigma)$, where $\Sigma$ is a subset of the edge set of $G$. The edges in $\Sigma$ are *negative*, the other ones are *positive*. An *unbalanced circuit* in $(G, \Sigma)$ is an (unsigned) circuit in $G$ that has an odd number of negative edges. A *balanced circuit* in $(G, \Sigma)$ is an (unsigned) circuit in $G$ that is not unbalanced. A subgraph of $G$ is *unbalanced* if it contains an unbalanced circuit; otherwise, it is *balanced*. A signed graph is *all-negative* (*all-positive*) if all its edges are negative (positive, respectively).

Signed graphs were introduced by Harary [30] as a model for social networks. Zaslavsky [79] then employed the following kind of equivalence on the class of signed graphs. Given a signed graph $(G, \Sigma)$, *switching* at a vertex $v$ is the inversion of the sign of each edge incident with $v$. Thus, the signature of the resulting graph is the symmetric difference of $\Sigma$ with the set of edges incident with $v$. Signed graphs are said to be *equivalent* if one can be obtained from the other by a series of switchings. It is easy to see that equivalent signed graphs have the same sets of unbalanced circuits and the same sets of balanced circuits.

Given a partition $\{A, B\}$ of the vertex set of a graph $G$, we let $[A, B]$ denote the set of all edges with one end in $A$ and one in $B$. Harary [30] characterised balanced graphs:

**Theorem 1.** *A signed graph $(G, \Sigma)$ is balanced if and only if there is a set $X \subseteq V(G)$ such that the set of negative edges is precisely $[X, V(G) - X]$.*

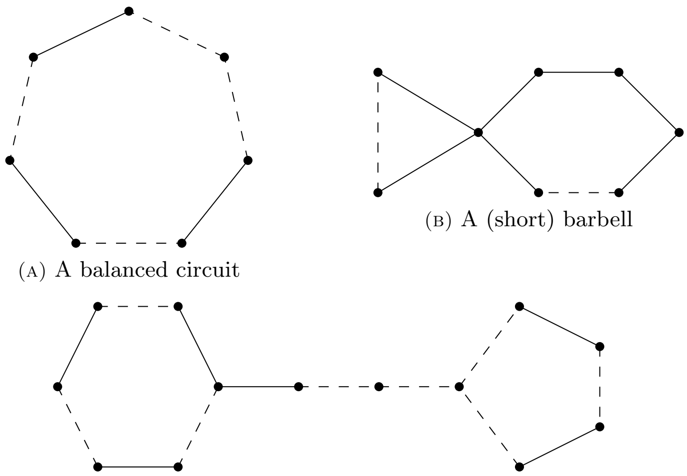
It follows from the characterisation that a signed graph is balanced if it is equivalent to an all-positive signed graph. We say that a graph is *antibalanced* if it is equivalent to an all-negative signed graph.

The role of circuits in unsigned graphs is played by signed circuits (known to be the circuits of the associated signed graphic matroid [78]). A *signed circuit* belongs to one of two types (cf. Figure 1):

- a balanced circuit,
- a *barbell*, i.e., the union of two unbalanced circuits $C_1, C_2$ and a (possibly trivial) path $P$ with endvertices $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$, such that $C_1 - v_1$ is disjoint from $P \cup C_2$ and $C_2 - v_2$ is disjoint from $P \cup C_1$.

If the path in a barbell is trivial, the barbell is sometimes called *short*; otherwise it is *long*. If we need to speak about a circuit in the usual unsigned sense, we call it an *ordinary circuit*.

To obtain an orientation of a signed graph, each edge of $(G, \Sigma)$ is viewed as composed of two half-edges. An *orientation* of an edge $e$ is obtained by giving each of the two half-edges $h, h'$ making up $e$ a direction. We say that $e$ is *consistently oriented* if exactly one of $h, h'$ is directed toward its endvertex. Otherwise, it is

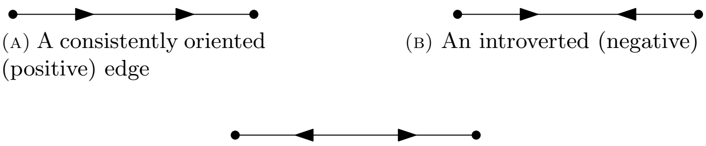


(A) A balanced circuit

(B) A (short) barbell

(C) A (long) barbell

FIGURE 1. Signed circuits. Dashed lines indicate negative edges.



(A) A consistently oriented (positive) edge

(B) An introverted (negative) edge

(C) An extroverted (negative) edge

FIGURE 2. Oriented edges of a signed graph.

*extroverted* if both $h$ and $h'$ point toward their endvertices, and *introverted* if none of them does. (See Figure 2.) We say that an oriented edge $e$ is *incoming* at its endvertex $v$ if its half-edge incident with $v$ is directed toward $v$, and that $e$ is *outgoing* at $v$ otherwise.

An *orientation* (*bidirection*) of $(G, \Sigma)$ is the assignment of an orientation to each edge of $G$ (in the above sense), in such a way that the positive edges are exactly the consistently oriented ones. An oriented signed graph is often called a *bidirected graph.*

## 2. Nowhere-zero flows

2.1. **Introduction.** Let $\Gamma$ be an Abelian group. A $\Gamma$-*flow* in $(G, \Sigma)$ consists of an orientation of $(G, \Sigma)$ and a function $\varphi : E(G) \to \Gamma$ such that the usual conservation law is satisfied – that is, for each vertex $v$, the sum of $\varphi(e)$ over the incoming edges $e$ at $v$ equals the sum of $\varphi(e)$ over the outgoing ones. A $\mathbb{Z}$-flow is said to be a $k$-*flow* (where $k \geq 2$ is an integer) if $|\varphi(e)| < k$ for each edge $e$. A $\Gamma$-flow is *nowhere-zero* if the value 0 is not used at any edge. If a signed graph $(G, \Sigma)$ admits a nowhere-zero $\mathbb{Z}$-flow, then its *flow number* $\Phi(G, \Sigma)$ is defined as the least $k$ such that $(G, \Sigma)$ admits a nowhere-zero $k$-flow. Otherwise, $\Phi(G, \Sigma)$ is defined as $\infty$.

A signed graph is said to be *flow-admissible* if it admits at least one nowhere-zero $\mathbb{Z}$-flow. Bouchet [10] observed the following characterisation in combination with [78].

**Theorem 2.** *A signed graph $(G, \Sigma)$ is flow-admissible if and only if every edge of $(G, \Sigma)$ belongs to a signed circuit of $(G, \Sigma)$.*

Thus, an all-positive signed graph is flow-admissible if and only if it is bridgeless. Nowhere-zero flows on signed graphs are, in fact, a generalization of the same concept on unsigned graphs, because the definition of a flow for all-positive signed graph corresponds to the usual definition of a flow on an unsigned graph.

This useful corollary directly follows from the previous theorem:

**Corollary 3.** *A signed graph with one negative edge is not flow-admissible.*

For 2-edge-connected unbalanced signed graphs the converse is also true [44].

**Theorem 4.** *A 2-edge-connected unbalanced signed graph is flow-admissible except in the case when it contains an edge whose removal leaves a balanced graph.*

Tutte [64] proved that an unsigned graph $G$ admits a nowhere-zero $k$-flow if and only if it admits a nowhere-zero $\mathbb{Z}_k$-flow. He proved, in fact, that the number of nowhere-zero $\Gamma$-flows only depends on $|\Gamma|$ and not on the structure of $\Gamma$. This is not true for signed graphs in general. For instance, an unbalanced circuit admits a nowhere-zero $\mathbb{Z}_2$-flow, but it does not admit any integer flow. It was, however, proved for signed graphs that if $|\Gamma|$ is odd, then the number of nowhere-zero $\Gamma$-flows does not depend on the structure of $\Gamma$ [7]. In the same paper, Beck and Zaslavsky further proved that if $|\Gamma|$ is odd, the number of nowhere-zero $\Gamma$-flows on a signed graph $(G, \Sigma)$ is a polynomial in $|\Gamma|$, independent of the actual group, extending the result of Tutte [65] for unsigned graphs (where arbitrary $|\Gamma|$ is allowed).

2.2. **Bouchet's conjecture.** Nowhere-zero flows in signed graphs were introduced by Edmonds and Johnson [25] for expressing algorithms on matchings, but systematically studied at first by Bouchet in the paper [10]. Bouchet also stated a conjecture that parallels Tutte's 5-Flow Conjecture and occupies a similarly central place in the area of signed graphs:

**Conjecture 5** (Bouchet)**.** *Every flow-admissible signed graph admits a nowhere-zero 6-flow.*

The value 6 would be best possible since Bouchet showed that the Petersen graph with a signature corresponding to its triangular embedding in the projective plane (see Figure 3(A)) admits no nowhere-zero 5-flow. Another signed graph with this property given in Figure 3(B) was found using a computer search by Máčajová [37]; an infinite family of such graphs containing the one in Figure 3(C) was constructed by Schubert and Steffen [57]. (The other graphs in the family are obtained by arranging $k$ unbalanced 2-circuits, $k \geq 5$ odd, in a circular manner as in Figure 3(C).) Interestingly, no other examples of signed graphs with flow number 6 appear to be known (except for signed graphs constructed from the known ones by simple flow reductions).

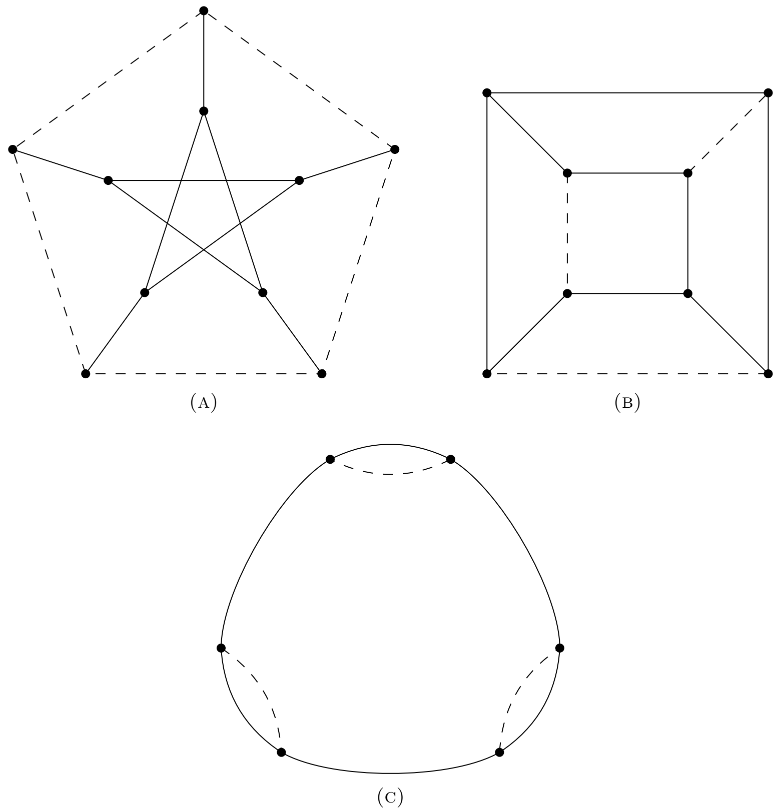


FIGURE 3. Signed graphs without nowhere-zero 5-flows. The example (C) gives rise to an infinite family.

In [10], Bouchet proved that the conjecture holds with the number 6 replaced by 216. The constant was improved by Zýka [85] to 30; according to [22], the same result was obtained by Fouquet (unpublished). Currently, the best general bound is due to DeVos [22]:

**Theorem 6** (DeVos). *Every flow-admissible signed graph admits a nowhere-zero 12-flow.*

2.3. **Higher edge-connectivity.** Assumptions about the connectivity of a graph usually allow for better bounds on its flow number, or at least make the bounds easier to establish. In the unsigned case, for example, it is well-known that any 4-edge-connected graph admits a nowhere-zero 4-flow, so in particular these graphs are not interesting with respect to Tutte's 5-Flow Conjecture.

For signed graphs, connectivity alone is not sufficient, as can be seen by Corollary 3: no signed graph with a single negative edge is flow-admissible. Thus, an assumption of flow-admissibility needs to be included.

Very recently, the following was shown by Cheng et al. [16] for 2-edge-connected signed graphs:

**Theorem 7** (Cheng, Lu, Luo, Zhang). *Every flow-admissible 2-edge-connected signed graph has a nowhere-zero 11-flow.*

The flow number of 3-edge-connected flow-admissible signed graphs was bounded by 25 in [71], which was improved to 15 in [70], and to 9 in [74]. A further improvement, based on the ideas of the proof of the Weak 3-Flow Conjecture [63, 36] was obtained by Wu et al. [72]:

**Theorem 8** (Wu, Ye, Zang and Zhang). *Every flow-admissible 3-edge-connected signed graph admits a nowhere-zero 8-flow.*

Khelladi [35] proved already in 1987 that any flow-admissible 4-edge-connected signed graph has a nowhere-zero 18-flow, and established Bouchet's Conjecture for flow-admissible 3-edge-connected graphs containing no long barbell. Bouchet's Conjecture was also established for 6-edge-connected signed graphs [73]. The best possible result for 4-edge-connected signed graphs was given by Raspaud and Zhu [54] using signed circular flows (for further results regarding signed circular flows, see Section 2.6):

**Theorem 9** (Raspaud and Zhu). *Every flow-admissible 4-edge-connected signed graph admits a nowhere-zero 4-flow.*

2.4. **Signed regular graphs.** Most of the research in the area of signed regular graphs is focused on signed cubic graphs. Máčajová and Škoviera [45] characterised signed cubic graphs with flow number 3 or 4.

Let $(G, \Sigma)$ be a signed graph and let $\{X_1, X_2\}$ be a partition of $V(G)$. If set of positive edges of $G$ is exactly $[X_1, X_2]$, then the partition $\{X_1, X_2\}$ is *antibalanced.* A signed graph $(G, \Sigma)$ has an antibalanced partition if and only if it is antibalanced [30]. For connected antibalanced signed graphs, the antibalanced partition is uniquely determined. The circuit $(Z, \Sigma)$ with antibalanced partition $\{X_1, X_2\}$ is *half-odd* if for some $i = 1, 2$, each component of $Z - X_i$ is either $Z$ itself or a path of odd length.

**Theorem 10** ([45]). *Let $(G, \Sigma)$ be a signed cubic graph. Then the following hold:*

- (*i*) *$(G, \Sigma)$ admits a nowhere-zero 3-flow if and only if it is antibalanced and has a perfect matching,*
- (*ii*) *$(G, \Sigma)$ admits a nowhere-zero 4-flow if and only if an equivalent signed graph contains an antibalanced 2-factor with all components half-odd whose complement is an all-negative perfect matching.*

Group-valued nowhere-zero flows in signed cubic graphs were also investigated in [45], with the following results:

**Theorem 11.** *Let $(G, \Sigma)$ be a signed cubic graph. Then the following hold:*

- (*i*) *$(G, \Sigma)$ has a nowhere-zero $\mathbb{Z}_3$-flow if and only if it is antibalanced,*
- (*ii*) *$(G, \Sigma)$ has a nowhere-zero $\mathbb{Z}_4$-flow if and only if it has an antibalanced 2-factor,*
- (*iii*) *$(G, \Sigma)$ has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow if and only if $G$ is 3-edge-colourable.*

The *integer flow spectrum* $\overline{\mathcal{S}}(G)$ of a graph $G$ is the set of all $t$ such that there is some signature $\Sigma$ of $G$ such that $\Phi(G, \Sigma) = t$. The analogous notion where $\Phi$

is replaced by $\Phi_c$ (circular flow number, for the definition see Section 2.6) is called the *flow spectrum* and denoted by $\mathcal{S}(G)$.

Schubert and Steffen [57] investigated both kinds of flow spectra in signed cubic graphs. They proved the following ($K_2^3$ denotes the graph on two vertices with three parallel edges and no loops):

**Theorem 12.** *If $G$ is a cubic graph different from $K_2^3$, then we have the following equivalences:*

$$G \text{ has a 1-factor} \iff 3 \in \mathcal{S}(G) \iff 3 \in \overline{\mathcal{S}}(G) \iff 4 \in \overline{\mathcal{S}}(G)\ .$$

*Moreover, if $G$ has a 1-factor, then $\{3,4\} \subseteq \overline{\mathcal{S}}(G) \cap \mathcal{S}(G)$.*

Furthermore, there exists a graph with $\mathcal{S}(G) = \{3,4\}$: the cubic graph on 4 vertices that has two double edges; and an infinite family of graphs with $\mathcal{S}(G) = \{3,4,6\}$ [57].

For general $(2t+1)$-regular graphs the following was proved in [57]:

**Theorem 13.** *Let $(G,\Sigma)$ be a signed graph. If $G$ is $(2t+1)$-regular and has a 1-factor, then $3 \in \overline{\mathcal{S}}(G)$.*

A rather different version of a 'flow spectrum' was introduced by Máčajová and Škoviera [45]. Let the *modular flow number* $\Phi_{\text{mod}}(G,\Sigma)$ be the least integer $k$ such that $(G,\Sigma)$ admits a nowhere-zero $\mathbb{Z}_k$-flow. It is easy to see that $\Phi_{\text{mod}}(G,\Sigma) \leq \Phi(G,\Sigma)$. The *modular flow spectrum* $\mathcal{S}_{\text{mod}}(G,\Sigma)$ is the set of all $k$ such that $(G,\Sigma)$ admits a nowhere-zero $\mathbb{Z}_k$-flow.

It is shown in [45] that the modular flow spectrum of a signed cubic graph may contain gaps. Namely, a signed cubic graph is given whose modular flow spectrum equals $\{3,5,6,7,\cdots\}$. On the other hand, it is also shown that bridges are essential in this example:

**Theorem 14.** *Let $(G,\Sigma)$ be a bridgeless cubic signed graph with $\Phi_{\text{mod}}(G,\Sigma) = 3$. Then $(G,\Sigma)$ admits a nowhere-zero $\Gamma$-flow for any Abelian group $\Gamma$ of order at least 3, except possibly for $\mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, $\mathcal{S}_{\text{mod}}(G,\Sigma) = \{k : k \geq 3\}$.*

2.5. **Other classes of signed graphs.** More detailed information about nowhere-zero flows has been obtained for some other classes of signed graphs, namely signed Eulerian, complete and complete bipartite graphs, series-parallel graphs, Kotzig graphs, and signed graphs with two negative edges.

Unsigned Eulerian graphs enjoy the minimum possible flow number of 2. This is not the case with signed Eulerian graphs, for which the following was proved in [73]:

**Theorem 15.** *A connected signed graph has a nowhere-zero 2-flow if and only if it is Eulerian and the number of its negative edges is even.*

Flows in signed Eulerian graphs were further investigated by Máčajová and Škoviera [44] (cf. also the extended abstract [42]). Their results are summarised by Theorem 16 below. Before we state it, we introduce one more notion. A signed Eulerian graph is *triply odd* if it has an edge-decomposition into three Eulerian subgraphs sharing a vertex, each of which contains an odd number of negative edges.

**Theorem 16.** *Let $(G,\Sigma)$ be a connected signed Eulerian graph. Then* (*i*) *if* $(G,\Sigma)$ *is flow-admissible, then it has a nowhere-zero* 4*-flow,*

(*ii*) $\Phi(G,\Sigma) = 3$ *if and only if* $(G,\Sigma)$ *is triply odd.*

Observe that together with Theorem 15 and Corollary 3, this result implies a full characterisation of signed Eulerian graphs of each of the possible flow numbers (2, 3, 4 and $\infty$).

Another result of [44] is a characterisation, for any Abelian group $\Gamma$, of signed Eulerian graphs admitting a nowhere-zero $\Gamma$-flow. Namely:

**Theorem 17.** *Let* $(G,\Sigma)$ *be a signed eulerian graph and let* $\Gamma$ *be a nontrivial Abelian group. The following statements hold true.* (*a*) *If* $\Gamma$ *contains an involution, then* $G$ *admits a nowhere-zero* $\Gamma$*-flow.* (*b*) *If* $\Gamma \cong \mathbb{Z}_3$*, then* $(G,\Sigma)$ *admits a nowhere-zero* $\Gamma$*-flow if and only if* $G$ *is triply odd.* (*c*) *Otherwise,* $G$ *has a nowhere-zero* $\Gamma$*-flow if and only if* $G$ *is flow-admissible.*

Máčajová and Rollová [41] (cf. also the extended abstract [40]) determined the complete characterisation of flow numbers of signed complete and signed complete bipartite graphs:

**Theorem 18.** *Let* $(K_n,\Sigma)$ *be a flow-admissible signed complete graph. Then*

(*i*) $\Phi(K_n,\Sigma) = 2$ *if and only if* $n$ *is odd and* $|\Sigma|$ *is even,*

(*ii*) $\Phi(K_n,\Sigma) = 4$ *if and only if* $(K_n,\Sigma)$ *is equivalent to* $(K_4,\emptyset)$ *or the signed complete graph* $K_5$ *whose negative edges form a* 5*-cycle,*

(*iii*) $\Phi(K_n,\Sigma) = 3$ *otherwise.*

The result of [41] concerning signed complete bipartite graphs involves a special family of graphs. The family $\mathcal{R}$ contains all signed bipartite graphs $(K_{3,n},\Sigma)$ such that $n \geq 3$ and each vertex is incident with at most one negative edge, except possibly for one vertex $v$ of degree $n$, which is incident with at least one and at most $n-2$ negative edges; moreover, the two vertices of degree $n$ other than $v$ are incident with 0 and 1 negative edges, respectively.

**Theorem 19.** *Let* $(K_{m,n},\Sigma)$ *be a flow-admissible signed complete bipartite graph. Then*

(*i*) $\Phi(K_{m,n},\Sigma) = 2$ *if and only if* $m, n, |\Sigma|$ *are even,*

(*ii*) $\Phi(K_{m,n},\Sigma) = 4$ *if and only if either* $(K_{m,n},\Sigma) \in \mathcal{R}$*, or* $m = n = 4$ *and* $|\Sigma|$ *is odd,*

(*iii*) $\Phi(K_{m,n},\Sigma) = 3$ *otherwise.*

Kaiser and Rollová [32] studied nowhere-zero flows in signed series-parallel graphs and showed that Bouchet's Conjecture holds in this class:

**Theorem 20.** *Every flow-admissible signed series-parallel graph admits a nowhere-zero* 6*-flow.*

They pointed out that although every unsigned series-parallel graph admits a nowhere-zero 3-flow and so these graphs are uninteresting from the point of view of the 5-Flow Conjecture, there are signed series-parallel graphs with flow number 6 (such as the one in Figure 3(C) and the other graphs in this family found by Schubert and Steffen in [57]).

Another class of graphs for which Bouchet's Conjecture has been verified (see [57]) is that of *Kotzig graphs*, namely cubic graphs $G$ that admit a 3-edge-colouring such that the removal of the edges of any one colour produces a Hamiltonian circuit of $G$.

A different approach to Bouchet's Conjecture was taken by Rollová et al. in [55]. Consider a flow-admissible signed graph $(G, \Sigma)$ with a minimum signature (that is a representative of the class of signed graphs equivalent to $(G, \Sigma)$ with minimum number of negative edges). If $|\Sigma| = 0$, then the signed graph is all-positive and by Seymour's 6-flow theorem [59], it admits a nowhere-zero 6-flow. By Corollary 3, $|\Sigma| \neq 1$. Thus, the least number of negative edges for which Bouchet's Conjecture is open is 2. In [55] it was proved that the flow number of $(G, \Sigma)$ with $|\Sigma| = 2$ is at most 7, and if Tutte's 5-flow conjecture holds, then it is 6. Moreover, Bouchet's Conjecture is true for cubic $(G, \Sigma)$ with $|\Sigma| = 2$ if $G$ contains a bridge or if $G$ is 3-edge-colourable or if $G$ is a critical snark (that is a graph that does not admits a nowhere-zero 4-flow such that $G - e$ admits a nowhere-zero 4-flow for any edge $e$ of $G$). Furthermore, if $G$ is bipartite, then $(G, \Sigma)$ admits a nowhere-zero 4-flow.

2.6. **Circular flows.** Raspaud and Zhu [54] generalised the concept of unsigned circular flow to signed graphs. For $t \geq 2$ real, a *circular $t$-flow* in $(G, \Sigma)$ is an $\mathbb{R}$-flow $(D, \varphi)$ such that for each edge $e$,

$$1 \leq |\varphi(e)| \leq t - 1\ .$$

If there is $t$ such that $(G, \Sigma)$ has a circular $t$-flow, then its *circular flow number* $\Phi_c(G, \Sigma)$ is the infimum of all such $t$. The circular flow number is infinite if $(G, \Sigma)$ admits no circular $t$-flow for any $t$.

It is natural to ask whether the relation between $\Phi$ and $\Phi_c$ for signed graphs is as close as in the unsigned case, where it is known that $\Phi(G) = \lceil \Phi_c(G) \rceil$ for any unsigned graph $G$ – in other words, the difference between $\Phi$ and $\Phi_c$ is less than 1. Raspaud and Zhu [54] established the following upper bound on $\Phi$:

**Theorem 21** (Raspaud and Zhu)**.** *For any flow-admissible signed graph $(G, \Sigma)$,*

$$\Phi(G, \Sigma) \leq 2\lceil \Phi_c(G, \Sigma) \rceil - 1\ .$$

Furthermore, it was conjectured in [54] that $\Phi(G, \Sigma) - \Phi_c(G, \Sigma) < 1$ just as in the unsigned case. However, the conjecture was disproved by Schubert and Steffen [57] who constructed signed graphs $(G, \Sigma)$ with the difference $\Phi(G, \Sigma) - \Phi_c(G, \Sigma)$ arbitrarily close to 2. (See Figure 4 for one element of a family mentioned in [57].) This result was strengthened by Máčajová and Steffen [46] who proved that the difference $\Phi(G, \Sigma) - \Phi_c(G, \Sigma)$ can be arbitrarily close to 3. Furthermore, the result of [46] shows that Theorem 21 cannot be improved in general.

Bounds for the circular flow number of sufficiently edge-connected signed graphs have been obtained by Raspaud and Zhu [54]:

**Theorem 22** (Raspaud and Zhu)**.** *Let $(G, \Sigma)$ be a flow-admissible signed graph. If $G$ is 6-edge-connected, then $\Phi_c(G, \Sigma) < 4$.*

Thomassen [63] showed that with increasing edge-connectivity of an (unsigned) graph, the circular flow number approaches 2. Zhu [84] studied the existence of circular $t$-flows in signed graphs for $t$ close to 2 and found that the case of signed graphs is somewhat different. It is easy to see [84] that a graph that has exactly $2k + 1$ negative edges has circular flow number at least $2 + 1/k$. Therefore it is

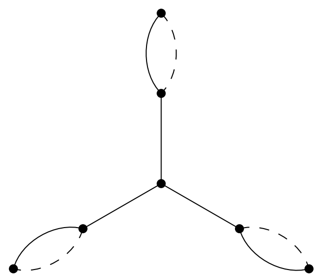


FIGURE 4. One member of a family of signed graphs for which the difference $\Phi - \Phi_c$ tends to 2.

natural to introduce the notion of *essentially* $(2k+1)$*-imbalanced* signed graph, defined as a signed graph such that every equivalent signed graph has either at least $2k+1$ negative edges, or an even number of negative edges. An analogue to the result of Thomassen [63], shows that if a signed graph is sufficiently edge-connected and sufficiently essentially imbalanced, then the circular flow number tends to 2 [84].

**Theorem 23.** *Let $k$ be a positive integer. If a signed graph is essentially $(2k+1)$-imbalanced and $(12k-1)$-edge-connected, then its circular flow number is at most $2+1/k$.*

Another interesting aspect of circular flows is, that there exist gaps among the values of circular flow number of regular graphs [62] that separate bipartite graphs from non-bipartite graphs. The gaps remain even in signed case [57]:

**Theorem 24.** *Let $t \geq 1$ be an integer and $(G, \Sigma)$ be a signed $(2t+1)$-regular graph. If $\Phi_c(G, \Sigma) = r$, then $r = 2 + 1/t$ or $r \geq 2 + 2/(2t-1)$.*

2.7. **Zero-sum flows.** Let $G$ be a graph. A *zero-sum flow* in $G$ (also called 'unoriented flow') is a function $f : E(G) \to \mathbb{R} - \{0\}$ such that for each vertex $v$, the values $f(e)$ of all edges $e$ incident with $v$ sum to zero. For a positive integer $k$, a zero-sum flow $f$ is a *zero-sum $k$-flow* if for each edge $e$, $f(e)$ is an integer and $1 \leq |f(e)| \leq k-1$. The *zero-sum flow number* of a graph $G$, which will be denoted by $\Phi_z(G)$, is defined similarly to the flow number, namely as the least $k$ such that $G$ admits a zero-sum $k$-flow (possibly $\infty$). The following was conjectured in [3]:

**Conjecture 25** (Zero-Sum Conjecture)**.** *If a graph $G$ admits a zero-sum flow, then it admits a zero-sum 6-flow.*

Zero-sum flows are tightly connected with nowhere-zero flows on signed graphs. Indeed, a zero-sum $k$-flow in $G$ is nothing but a nowhere-zero $k$-flow in the all-negative graph $(G, E(G))$. Therefore, if Bouchet's 6-flow Conjecture is true, so is the Zero-Sum Conjecture. Surprisingly, the converse statement is also true [1], since we can find a nowhere-zero $k$-flow on $(G, \Sigma)$ by finding a zero-sum $k$-flow of graph $G'$ obtained from $(G, \Sigma)$ by subdividing all positive edges with one vertex of degree 2. Thus to prove Bouchet's 6-flow Conjecture, it is enough to prove the Zero-Sum conjecture for subcubic graphs.

Much of the work regarding zero-sum flows pertains to regular graphs. While not all 2-regular graphs admit a zero-sum flow, Akbari et al. [3] proved that any $r$-regular graph with $r \geq 4$ even has a zero-sum 3-flow, and any cubic graph has a zero-sum 5-flow. This led to the following conjecture, stated in [1]:

**Conjecture 26.** *Any $r$-regular graph with $r \geq 3$ has a zero-sum 5-flow.*

It was shown in [1] that Conjecture 26 holds for odd $r$ divisible by 3, and that $\Phi_z(G) \leq 7$ for general $r$. This was improved in [4] where the conjecture was proved for all $r$ except $r = 5$. Furthermore, a mention in [56] indicates that the case of 5-regular graphs has also been settled in the affirmative, namely in [75]. If so, then Conjecture 26 is proved.

Wang and Hu [67] gave exact values of $\Phi_z(G)$ for certain $r$-regular graphs $G$:

- $\Phi_z(G) = 2$ if either $r \equiv 0 \pmod 4$, or $r \equiv 2 \pmod 4$ and $|E(G)|$ is even,
- $\Phi_z(G) = 3$ if one of the following holds:
  - $r \equiv 2 \pmod 4$ and $|E(G)|$ is odd,
  - $r \geq 3$ is odd and $G$ has a perfect matching,
  - $r \geq 7$ is odd and $G$ is bridgeless.

The zero-sum flow number is determined in [67] for the Cartesian product of an $r$-regular graph $G$ with an $s$-vertex path $P_s$ $(r, s \geq 2)$; namely, $\Phi_z(G \times P_s) = 2$ if $r$ is odd and $s = 2$, and $\Phi_z(G \times P_s) = 3$ in the other cases.

Further results concerning the zero-sum flow number of (not necessarily regular) graphs include:

- $\Phi_z(G) \leq 6$ if $G$ is bipartite bridgeless [3],
- $\Phi_z(G) \leq 12$ if $G$ is hamiltonian [2].

As for particular graph classes, exact values of the zero-sum flow number are known for graphs of the following types:

- wheels and fans [68],
- all induced subgraphs of the hexagonal grid [69],
- certain subgraphs of the triangular grid [68].

Results on the computational complexity of deciding the existence of a zero-sum flow were obtained in [21]. For positive integers $k \leq \ell$, let us define a *$(k, \ell)$-graph* to be an unsigned graph with minimum degree at least $k$ and maximum degree at most $\ell$. It is known [3] that Conjecture 25 is equivalent to its restriction to $(2, 3)$-graphs (that is, subdivisions of cubic graphs).

Let us summarise the findings of [21]:

**Theorem 27.**

(*i*) *There is a polynomial-time algorithm to determine whether a given $(2, 4)$-graph with $O(\log n)$ vertices of degree 4 has a zero-sum 3-flow,*

(*ii*) *it is NP-complete to decide whether a given $(3, 4)$-graph admits a zero-sum 3-flow,* (*iii*) *it is NP-complete to decide whether a given* $(2,3)$*-graph admits a zero-sum* $4$*-flow.*

In [56], zero-sum flows are viewed in the more general setting of matrices. A *zero-sum k-flow* of a real matrix $M$ is an integer-valued vector in its nullspace with nonzero entries of absolute value less than $k$. From this point of view, a zero-sum flow in a graph is a zero-sum flow of its vertex-edge incidence matrix. The main result of [56] proves the existence of a zero-sum $k$-flow of the incidence matrix for linear subspaces of dimension 1 and $m$, respectively, of the vector space $\mathbb{F}_q^n$ (where $q$ is a prime power, $0 \leq m \leq n$ and $k$ is a suitable integer depending on $q, m$ and $n$).

2.8. **Structural results.** Zaslavsky [79] proved that any signed graph $(G, \Sigma)$ is associated with a matroid, called the *frame matroid* of $(G, \Sigma)$ and denoted by $M(G, \Sigma)$. (For background on matroid theory, see, e.g., [52].) One way to define this matroid is to specify its circuits, and these are precisely the signed circuits introduced in Section 1. Properties of $M(G, \Sigma)$ are studied in [79] and subsequently in a number of papers. See, for instance, [61] (decomposition theorems), [28, 53] (minors), [60] and the references therein.

Let us call a $\mathbb{Z}$-flow in a (signed or unsigned) graph *nonnegative* if all its values are nonnegative. A nonnegative $\mathbb{Z}$-flow is *irreducible* if it cannot be expressed as a sum of two nonnegative $\mathbb{Z}$-flows. Tutte [66] showed that in an unsigned graph, irreducible nonnegative $\mathbb{Z}$-flows are precisely the *circuit flows* (elementary flows along circuits). Equivalently, any nonnegative $\mathbb{Z}$-flow is a sum of circuit flows.

For signed graphs, a characterisation of irreducible flows appears in the (apparently unpublished) manuscript [15] and in [14]. Let us say that a signed graph $(H, \Theta)$ is *essential* if it satisfies the following properties:

- the degree of any vertex of $H$ is 2 or 3,
- each vertex is contained in at most one circuit of $H$,
- for any circuit $C$ of $H$, $|E(C) \cap \Theta|$ has the same parity as the number of bridges of $H$ incident with $C$.

(See Figure 5 for an example.)

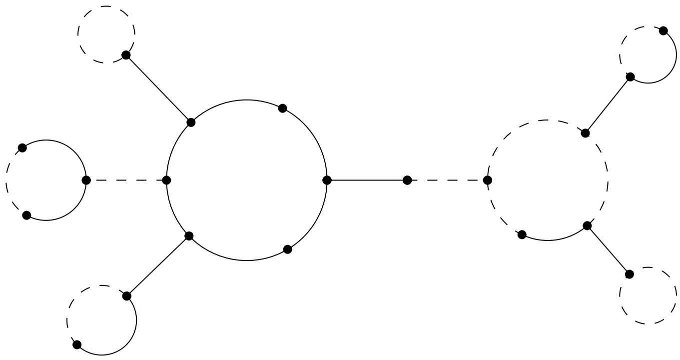
Let $C$ be a circuit in $H$ and let $\Theta' = \Theta \cap E(C)$. An orientation $D$ of $(C, \Theta')$ is *consistent* at a vertex $v$ of $C$ if $v$ is incident with precisely one incoming half-edge. Otherwise, $D$ is *inconsistent* at $v$. A vertex is a *source* (*sink*) in an orientation of a signed graph if all incident half-edges are outgoing (incoming, respectively).

Given an essential signed graph $(H, \Theta)$, an *essential flow* in $(H, \Theta)$ is obtained as follows:

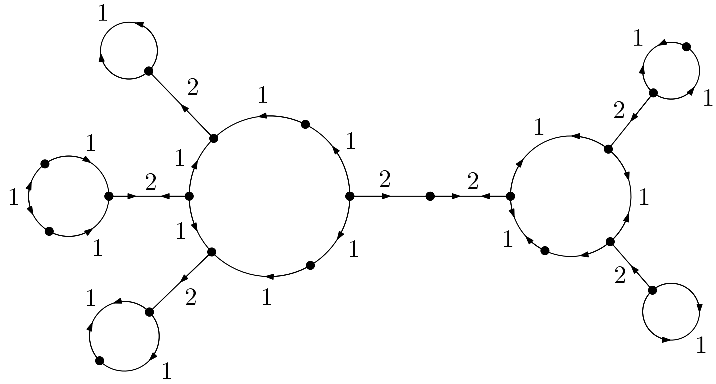
- choose an orientation of $(H, \Theta)$ with no sources nor sinks, such that the induced orientation of any circuit $C$ of $H$ is inconsistent at any vertex incident with a bridge of $H$ (we call this an *essential orientation of* $(H, \Theta)$),
- assign flow value 2 to each bridge of $H$ and value 1 to the other edges.

The main result of [15] reads as follows:

**Theorem 28.** *Irreducible flows in a signed graph* $(G, \Sigma)$ *are exactly the essential flows on essential subgraphs of* $(G, \Sigma)$.



(A)



(B)

FIGURE 5. (A) An essential signed graph $(H, \Theta)$. (B) An essential flow in $(H, \Theta)$.

Independently, Máčajová and Škoviera [43] proved the closely related result that nonnegative flows in signed graphs can be obtained as sums of elementary flows along signed circuits at the cost of allowing half-integral values. Let $C$ be a signed circuit in $(G, \Sigma)$. An orientation of $C$ will be called *proper* if it is an essential orientation of the essential subgraph $C$ as defined above. The *characteristic flow* of $C$ is the function $\chi_C : E(G) \to \mathbb{Q}$ with values

$$\chi_C(e) = \begin{cases} 1 & \text{if } C \text{ is balanced and } e \in E(C)\,, \\ \dfrac{1}{2} & \text{if } C \text{ is unbalanced and } e \text{ is contained in an ordinary circuit of } C\,, \\ 1 & \text{if } C \text{ is unbalanced and } e \text{ is contained in the connecting path of } C\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

**Theorem 29.** *Let $(D, \varphi)$ be a $\mathbb{Z}$-flow on a signed graph $(G, \Sigma)$ such that all values of $\varphi$ are nonnegative. Then $D$ contains proper orientations of signed circuits*

*$C_1, \cdots, C_m$ and positive integers $\alpha_i$ $(i = 1, \cdots, m)$ such that*

$$\varphi = \sum_{i=1}^{m} \alpha_i \chi_{C_i} .$$

Chen and Wang [13] defined cuts in signed graphs and introduced the circuit and bond lattices of signed graphs.

## 3. Circuit covers

The Shortest Cycle Cover Conjecture of Alon and Tarsi [5] asserts that the edge set of any (unsigned) bridgeless graph with $m$ edges can be covered with circuits of total length at most $7m/5$. For background on the conjecture and its relation to other problems see, e.g., [83, Chapter 14]. The best general upper bound to the Shortest Cycle Cover Conjecture is due to Bermond et al. [8], and Alon and Tarsi [5], who independently proved that any bridgeless graph admits a circuit cover of total length at most $5m/3$.

The signed version of the problem was introduced by Máčajová et al. [38]. In a signed graph $(G, \Sigma)$, it is natural to seek a cover of the entire edge set by signed circuits (a *signed circuit cover*) of small length. Here, the length of a signed circuit is the number of its edges, and the *length* of the cover is defined as the sum of lengths of its elements. Similarly as in the case of flows, for a signed circuit cover of $G$ to exist, each edge has to be contained in a signed circuit, which is equivalent to $(G, \Sigma)$ being flow-admissible (see Theorem 2). The authors of [38] obtained the following bounds:

**Theorem 30.** *Let $(G, \Sigma)$ be a flow-admissible signed graph with $m$ edges. Then there is a signed circuit cover of $(G, \Sigma)$ of length at most $11m$. Moreover, if $G$ is bridgeless, then it admits a signed circuit cover of length $9m$.*

An application of Theorem 29 given in [43] yields the following result:

**Theorem 31.** *Let $(G, \Sigma)$ be a signed graph with $m$ edges. If $(G, \Sigma)$ admits a nowhere-zero $k$-flow, then it has a signed circuit cover of length at most $2(k-1)m$.*

Currently the best bound for this problem is due to Cheng et al. [17]. Let the *negativeness* of a signed graph be the minimum number of negative edges in any equivalent signed graph.

**Theorem 32.** *Let $(G, \Sigma)$ be a flow-admissible signed graph with $n$ vertices, $m$ edges and negativeness $\varepsilon > 0$. Then the following hold:*

(*i*) *$(G, \Sigma)$ has a signed circuit cover of length at most*

$$m + 3n + \min\left\{\frac{2m}{3} + \frac{4\varepsilon}{3} - 7, n + 2\varepsilon - 8\right\} \leq \frac{14m}{3} - \frac{5\varepsilon}{3} - 4 ,$$

(*ii*) *if $G$ is bridgeless and $\varepsilon$ is even, then $(G, \Sigma)$ admits a signed circuit cover of length at most*

$$m + 2n + \min\left\{\frac{2m}{3} + \frac{\varepsilon}{3} - 4, n + \varepsilon - 5\right\} .$$

## 4. Relation to colouring

As indicated in Section 1, nowhere-zero flows on signed graphs are dual to local tensions on non-orientable surfaces (see [23]). Let $G$ be a directed graph embedded in a non-orientable surface and $\phi : E(G) \to A$ a function. Given an orientation of a face of $G$, the *height* of a facial walk along this face is the sum of values of $\phi$ on the edges of the face directed consistently with the orientation minus the sum of values on the edges directed inconsistently. The function $\phi$ is a *local tension* if the height of every facial walk is 0. Since the surface is non-orientable, the dual graph $G^*$ of a directed graph $G$ is bidirected, and the facial walk condition for local tensions of $G$ translates to the conservation law at each vertex of $G^*$. This duality can be used to show that the signed Petersen graph of Figure 3(A) does not admit any nowhere-zero 5-flow, because its dual in the projective plane is $K_6$, which does not admit any 5-local-tension.

The flow-colouring duality does not hold for signed graphs for any of the known definitions of signed colourings. However, given the importance of the duality in the unsigned case, it will perhaps be useful to include an overview of the essential definitions of signed colourings and the appropriate literature for completeness. The first definition is due to Zaslavsky [78].

**Signed colouring according to Zaslavsky.** If $(G, \Sigma)$ is a signed graph, a *signed colouring* of $(G, \Sigma)$ in $k$ colours (or in $2k + 1$ *signed colours*) is a mapping

$$f : V(G) \to \{-k, -k+1, \cdots, -1, 0, 1, \cdots, k-1, k\}\ .$$

A colouring is zero-free if it never uses the value 0. The signed colouring is *proper* if $f(u) \neq f(v) \cdot \Sigma(uv)$ holds for every edge $uv$. Zaslavsky further defined the chromatic polynomial $\xi_G(2k + 1)$ of a signed graph G to be the function counting the number of proper signed colourings of $G$ in $k$ colours. The balanced chromatic polynomial $\xi^b_G(2k)$ is the function which counts the zero-free proper signed colourings in $k$ colours. Then the chromatic number $\gamma(G)$ of $G$, according to Zaslavsky, is the smallest nonnegative integer $k$ for which $\xi_G(2k + 1) > 0$. Furthermore, the strict chromatic number $\gamma^*(G)$ of $G$ is the smallest nonnegative integer $k$ such that $\xi^b_G(2k) > 0$. Neither of Zaslavsky's definitions of chromatic number is a direct extension of the usual chromatic number of an unsigned graph. Note that an unbalanced circuit on four vertices $UC_4$ admits a proper signed colouring in 3 signed colours, i.e. in 1 colour. Thus $\xi_{UC_4}(3) > 0$. Since $UC_4$ does not admit a proper signed colouring in 1 signed colour, $\gamma(UC_4) = 1$. Both signed colourings of Zaslavsky were further investigated by himself in the series of papers [77, 80, 76], and recently by Beck et al. [6], Davis [20] and Schweser and Stiebitz [58].

**Signed colouring according to Máčajová, Raspaud and Škoviera.** A modification of Zaslavsky's definition of a signed colouring gave rise to a definition of signed colouring introduced by Máčajová, Raspaud and Škoviera in [39]. For each $n \geq 1$, let $M_n \subseteq \mathbb{Z}$ be the set $M_n = \{\pm 1, \pm 2, \cdots, \pm k\}$ if $n = 2k$, and $M_n = \{0, \pm 1, \pm 2, \cdots, \pm k\}$ if $n = 2k + 1$. A proper colouring of a signed graph $G$ that uses colours from $M_n$ is an *n-colouring*. Thus, an $n$-colouring of a signed graph uses at most $n$ distinct colours. Note that if $G$ admits an $n$-colouring, then it also admits an $m$-colouring for each $m \geq n$. The smallest $n$ such that $G$ admits an $n$-colouring is the *signed chromatic number* of $G$ and is denoted by $\xi(G)$. It is easy to see that the signed chromatic number of a balanced signed graph coincides with the chromatic number of its underlying unsigned graph, and hence this definition of chromatic number differs from Zaslavsky's. Moreover, $\xi(G) = \gamma(G) + \gamma^*(G)$. Note that $\xi(UC_4) = 3$. This recently introduced definition has been studied by Jin et al. [31] and by Fleiner and Wiener [26]. The chromatic spectrum related to this colouring was studied by Kang and Steffen [33].

**Signed colourings via signed circular colourings according to Kang and Steffen.** In the recent paper [34], Kang and Steffen defined circular colourings for signed graphs to be an extension of circular colourings of graphs as follows. Let $k$ and $d$ be positive integers such that $k \geq 2d$, and let $\mathbb{Z}_k$ be the cyclic group of integers modulo $k$. A $(k, d)$-*colouring* of a signed graph $(G, \Sigma)$ is a mapping $c\colon V(G) \to \mathbb{Z}_k$ such that $d \leq |c(v) - \Sigma(e)c(w)|_k$ holds for each edge $e = vw$. (Here, $|x|_k$ stands for the minimum of $\{x, k - x\}$ taken with respect to the usual ordering of $\{0, \cdots, k-1\}$.) The circular chromatic number $\chi_c((G, \Sigma))$ is $\inf\{k/d;\ (G, \Sigma)$ has a $(k, d)$-colouring$\}$. Then the minimum $k$ such that $(G, \Sigma)$ has a $(k, 1)$-colouring is the *signed chromatic number* of $(G, \Sigma)$ and is denoted by $\chi((G, \Sigma))$. The signed chromatic number of Kang and Steffen is different from the chromatic number of Zaslavsky as well as from the signed chromatic number of Máčajová, Raspaud and Škoviera, since $\chi(UC_4) = 2$. The chromatic spectrum related to this colouring was studied by Kang and Steffen [33].

**Signed colourings via signed homomorphisms according to Naserasr, Rollová and Sopena.** It is well known that the chromatic number of a graph can be defined in terms of graph homomorphisms. This is the case also for signed graphs and signed homomorphisms. The notion of homomorphism of signed graphs was introduced by Guenin [29], and later systematically studied by Naserasr et al. [50] (cf. also the extended abstract [48]). Given two signed graphs $(G, \Sigma_1)$ and $(H, \Sigma_2)$, we say there is a *homomorphism* of $(G, \Sigma_1)$ to $(H, \Sigma_2)$ if there exists a signed graph $(G, \Sigma_1')$ equivalent to $(G, \Sigma_1)$, a signed graph $(H, \Sigma_2')$ equivalent to $(H, \Sigma_2)$, and a mapping $\Phi : V(G) \to V(H)$ such that every edge of $(G, \Sigma_1')$ is mapped to an edge of $(H, \Sigma_2')$ of the same sign. The *signed chromatic number* of a signed graph $(G, \Sigma)$ is the smallest order of a signed graph to which $(G, \Sigma)$ admits a homomorphism.

Note that the signed chromatic number of a balanced signed graph is the same as the chromatic number of an unsigned graph. Since the signed chromatic number of $UC_4$ in the homomorphism sense equals 4, this definition of the signed chromatic number differs from the three notions defined above. However, as observed by Brewster and Raspaud [11] in 2012, there is a connection between all three types of signed colouring on the one hand and signed homomorphisms on the other. Let $A$ be one of the following sets:

- $A = \{-k, \cdots, -1, 0, 1, \cdots, k\}$ in the case of the colouring defined by Zaslavsky, as well as the colouring of Máčajová et al. for $n = 2k + 1$,
- $A = \{-k, \cdots, -1, 1, \cdots, k\}$ in the case of the zero-free colouring of Zaslavsky, and the colouring of by Máčajová et al. for $n = 2k$,
- $A = \mathbb{Z}_k$ in the case of the colouring according to Kang and Steffen.

A signed graph $(G, \Sigma)$ admits a proper signed colouring with colours from $A$ if and only if there is a homomorphism of $(G, \Sigma)$ to a signed multigraph $\mathcal{A}$, where

$V(\mathcal{A}) = A$ and for every $u, v \in A$ there is a positive edge $uv$ if $u \neq v$, and there is a negative edge $uv$ (or a negative loop $uv$ in case $u = v$) if $u \neq -v$.

Signed homomorphisms were investigated by Naserasr et al. [47] (cf. also the extended abstract [49]), Foucaud and Naserasr [27], Brewster et al. [12], Ochem et al. [51], and recently by Das et al. in [19] and Das et al. in [18].

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